2-local triple derivations on von Neumann algebras

GPOTS 2016
University of Illinois

Bernard Russo
(Joint with Karimbergen Kudaybergenov, Timur Oikhberg, Antonio Peralta)

University of California, Irvine

May 23-27, 2016

Outline

1. Local and 2-local derivations on von Neumann algebras
2. Nonassociative structures on C*-algebras
3. Local and 2-local triple derivations on von Neumann algebras
4. Why study triple derivations?
5. Other contexts

1A Local derivations on von Neumann algebras

A **derivation** is a linear map D from an algebra A to a two sided A-module M satisfying the Leibniz identity: $D(ab) = a \cdot D(b) + D(a) \cdot b$ for all $a, b \in A$.

It is **inner** if there is an element $m \in M$ such that $D(a) = m \cdot a - a \cdot m$.

Kaplansky 1953 Amer. J. Math.,
Kadison, Sakai 1966 Ann. Math.

Every derivation of a von Neumann algebra (into itself) is inner.

A **local derivation** from an algebra into a module is a linear mapping T whose value at each point a in the algebra coincides with the value of some derivation D_a at that point.

Kadison 1990 J. Alg., Johnson 2001 Trans. AMS

Every local derivation of a C*-algebra (into a Banach module) is a derivation.

Larson and **Sourour** showed in 1990 that a local derivation on the algebra of all bounded linear operators on a Banach space is a derivation. (+non self-adjoint)

Example 1 (attributed to C. U. Jensen)

Let $\mathbb{C}(x)$ denote the algebra of all rational functions (quotients of polynomials). There exists a local derivation of $\mathbb{C}(x)$ which is not a derivation.

Exercise 1

The derivations of $\mathbb{C}(x)$ are the mappings of the form $f \mapsto gf'$ for some g in $\mathbb{C}(x)$, where f' is the usual derivative of f.

Exercise 2

The local derivations of $\mathbb{C}(x)$ are the mappings which annihilate the constants.

Exercise 3

Write $\mathbb{C}(x) = S + T$ where S is the 2-dimensional space generated by 1 and x. Define $\alpha : \mathbb{C}(x) \to \mathbb{C}(x)$ by $\alpha(a + b) = b$. Then α is a local derivation which is not a derivation.

Example 2 (attributed to I. Kaplansky 1990)

There exists a local derivation of $\mathbb{C}(x)/[x^3]$ which is not a derivation (dim $= 3$) .

Example 1 (attributed to C. U. Jensen)

Let $\mathbb{C}(x)$ denote the algebra of all rational functions (quotients of polynomials). There exists a local derivation of $\mathbb{C}(x)$ which is not a derivation.

Exercise 1

The derivations of $\mathbb{C}(x)$ are the mappings of the form $f \mapsto gf'$ for some g in $\mathbb{C}(x)$, where f' is the usual derivative of f.

Exercise 2

The local derivations of $\mathbb{C}(x)$ are the mappings which annihilate the constants.

Exercise 3

Write $\mathbb{C}(x) = S + T$ where S is the 2-dimensional space generated by 1 and x. Define $\alpha : \mathbb{C}(x) \to \mathbb{C}(x)$ by $\alpha(a + b) = b$. Then α is a local derivation which is not a derivation.

Example 2 (attributed to I. Kaplansky 1990)

There exists a local derivation of $\mathbb{C}(x)/[x^3]$ which is not a derivation (dim $= 3$) .

Example 1 (attributed to C. U. Jensen)

Let $\mathbb{C}(x)$ denote the algebra of all rational functions (quotients of polynomials). There exists a local derivation of $\mathbb{C}(x)$ which is not a derivation.

Exercise 1

The derivations of $\mathbb{C}(x)$ are the mappings of the form $f \mapsto gf'$ for some g in $\mathbb{C}(x)$, where f' is the usual derivative of f.

Exercise 2

The local derivations of $\mathbb{C}(x)$ are the mappings which annihilate the constants.

Exercise 3

Write $\mathbb{C}(x) = S + T$ where S is the 2-dimensional space generated by 1 and x. Define $\alpha : \mathbb{C}(x) \to \mathbb{C}(x)$ by $\alpha(a + b) = b$. Then α is a local derivation which is not a derivation.

Example 2 (attributed to I. Kaplansky 1990)

There exists a local derivation of $\mathbb{C}(x)/[x^3]$ which is not a derivation (dim $= 3$) .

1B 2-local derivations on von Neumann algebras

A **2-local derivation** from an algebra into a module is a mapping T (not necessarily linear) whose values at each pair of points a, b in the algebra coincides with the values of some derivation $D_{a,b}$ at those two points.

Semrl 1997, Ayupov-Kudaybergenov 2015

Every 2-local derivation of a von Neumann algebra A is a derivation.

- Semrl 1997 (Proc. AMS) $A = B(H)$, H separable infinite dimensional
- Kim and Kim 2004 (Proc. AMS) $A = B(H)$, H finite dimensional
- Ayupov-Kudaybergenov 2012 (J. Math. Anal. Appl) $A = B(H)$, H arbitrary
- Ayupov-Kudaybergenov 2015 (Positivity) A arbitrary

Example: Zhang-Li 2006 (Acta Math. Sinica)

Consider the algebra of all upper-triangular complex 2×2-matrices

$$A = \left\{ x = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ 0 & \lambda_{22} \end{pmatrix} : \lambda_{ij} \in \mathbb{C} \right\}.$$

Define an operator Δ on A by

$$\Delta(x) = \begin{cases} 0, & \text{if } \lambda_{11} \neq \lambda_{22}, \\ \begin{pmatrix} 0 & 2\lambda_{12} \\ 0 & 0 \end{pmatrix}, & \text{if } \lambda_{11} = \lambda_{22}. \end{cases}$$

Then Δ is a 2-local derivation, which is not a derivation.

2A Nonassociative structures on C*-algebras

Let A be a C*-algebra (or any associative *-algebra). A becomes

- A **Lie algebra** in the binary product $[a, b] = ab - ba$
- A **Jordan algebra** in the binary product $a \circ b = (ab + ba)/2$ (prototype of JB^*-algebra)
- A **Jordan triple system** in the triple product $\{abc\} := (ab^*c + cb^*a)/2$ (prototype of JB^*-triple)

Jordan algebra emphasizes the order structure of A

Jordan triple system emphasizes the metric geometric and holomorphic structure

- surjective isometry = triple isomorphism
- open unit ball is bounded symmetric domain

Lesser siblings:

- **associative triple systems** ab^*c (Ternary ring of operators, or TRO)
- **Lie triple systems** $[a, [b, c]]$

2B Nonassociative derivations on C*-algebras

Derivations

- **Lie derivation** $D([a,b]) = [a, D(b)] + [D(a), b]$
- **Jordan derivation** $D(a \circ b) = a \circ D(b) + D(a) \circ b$
  - simplifies to $D(a^2) = 2 \circ D(a)$
- **(Jordan) triple derivation** $D\{abc\} = \{D(a)bc\} + \{aD(b)c\} + \{abD(c)\}$

$$D(ab^*c)+D(cb^*a) = (Da)b^*c+cb^*(Da)+a(Db)^*c+c(Db)^*a+ab^*(Dc)+(Dc)b^*a$$

- simplifies to $D(ab^*a) = (Da)b^*a + a(Db)^*a + ab^*(Da)$

Inner derivations

- Inner Lie derivation: $x \mapsto [a, x]$ for some $a \in A$
- Inner Jordan derivation: $x \mapsto a \circ (b \circ x) - b \circ (a \circ x)$ for some $a, b \in A$
- Inner triple derivation: $x \mapsto \{abx\} - \{bax\}$ for some $a, b \in A$

Outer derivation

Any linear map into the center which vanishes on commutators is a Lie derivation

2C Some positive results on non associative derivations

Sinclair 1970 (Proc. AMS)
Johnson 1996 (Math. Proc. Camb. Ph. Soc.)

Every Jordan derivation of a C*-algebra into a module is a derivation.

Upmeier 1980 (Math. Scand.)

Every Jordan derivation of a von Neumann algebra is an inner Jordan derivation.

Ho-Martinez-Peralta-R 2002 (J. Lon. Math. Soc.)

- Every triple derivation on a von Neumann algebra is an inner triple derivation.
- $B(H, K)$ has outer derivations if and only if $\dim H < \dim K = \infty$

Miers 1973 (Duke M. J.)
Johnson 1996 (Math. Proc. Camb. Ph. Soc.)

Every Lie derivation of a C*-algebra into a module is an inner Lie derivation plus a center valued linear map which vanishes on commutators.
(automatic continuity: Alaminos-Bresar-Villena 2004 MPCPS)

3A Local triple derivations on C*-algebras

A **local triple derivation** from a triple system into itself is a linear mapping whose value at each point coincides with the value of some triple derivation at that point.

Mackey 2012, Burgos-Polo-Peralta 2014 (Bull. Lon. Math. Soc.)

Every local triple derivation on a C*-algebra is a triple derivation.

More precisely,

- Mackey 2012: continuous local triple derivation on a JBW*-triple .
- Burgos-Polo-Peralta 2014: local triple derivation on a JB*-triple

Example (Variant of C. U. Jensen's example)

Let $\mathbb{C}(x)$ denote the algebra of all rational functions (quotients of polynomials). There exists a local triple derivation of $\mathbb{C}(x)$ which is not a triple derivation.

Exercise 1′

The triple derivations of $\mathbb{C}(x)$ are the mappings of the form $\delta_{u,v}$, where $\delta_{u,v}f = uf' + ivf$, for f in $\mathbb{C}(x)$ and u, v fixed self adjoint elements of $\mathbb{C}(x)$.

Exercise 2′

The local triple derivations of $\mathbb{C}(x)$ are the mappings which take 1 to iv for some fixed self adjoint v in $\mathbb{C}(x)$..

Exercise 3′

The linear map $f \mapsto i(xf)'$ is a local triple derivation which is not a triple derivation.

Example (Variant of C. U. Jensen's example)

Let $\mathbb{C}(x)$ denote the algebra of all rational functions (quotients of polynomials). There exists a local triple derivation of $\mathbb{C}(x)$ which is not a triple derivation.

Exercise $1'$

The triple derivations of $\mathbb{C}(x)$ are the mappings of the form $\delta_{u,v}$, where $\delta_{u,v}f = uf' + ivf$, for f in $\mathbb{C}(x)$ and u, v fixed self adjoint elements of $\mathbb{C}(x)$.

Exercise $2'$

The local triple derivations of $\mathbb{C}(x)$ are the mappings which take 1 to iv for some fixed self adjoint v in $\mathbb{C}(x)$..

Exercise $3'$

The linear map $f \mapsto i(xf)'$ is a local triple derivation which is not a triple derivation.

3B 2-local triple derivations on von Neumann algebras

A **2-local triple derivation** from a triple system into itself is a mapping (not necessarily linear) whose values at each pair of points coincides with the values of some triple derivation at those two points.

Kudaybergenov-Oikhberg-Peralta-R 2014 (Ill. J. Math.)

Every 2-local triple derivation on a von Neumann algebra (considered as a Jordan triple system) is a triple derivation.

Ingredients

- The 2-local triple derivation is weak*-completely additive on projections
- Dorofeev-Shertsnev boundedness theorem
- Mackey-Gleason-Bunce-Wright theorem

Negative example? What about C*-algebras?

4. Why study triple derivations?

4A **The Tits-Kantor-Koecher (TKK) Lie algebra**

Given two elements a, b in a Jordan triple V (for example, a von Neumann algebra under $\{abc\} = (ab^*c + cb^*a)/2$), we define the *box operator* $a\Box b : V \to V$ by

$$a\Box b(\cdot) = \{a, b, \cdot\}.$$

Since $[a\Box b, c\Box d] = [\{abc\}, d] - [c, \{bad\}]$,

$$V_0 = \{h = \sum a_j\Box b_j : a_j, b_j \in V\}$$

is a Lie algebra with involution $(a\Box b)^\natural = b\Box a$.

The TKK Lie algebra $\mathfrak{L}(V)$ of V is $\mathfrak{L}(V) = V \oplus V_0 \oplus V$, with Lie product

$$[(x, h, y), (u, k, v)] = (hu - kx,\ [h, k] + x\Box v - u\Box y,\ k^\natural y - h^\natural v),$$

and involution $\theta : \mathfrak{L}(V) \to \mathfrak{L}(V), \quad \theta(x, h, y) = (y, -h^\natural, x)$

Let $\omega : V \to V$ be a triple derivation. Then $\mathfrak{L}(\omega) : \mathfrak{L}(V) \to \mathfrak{L}(V)$ defined by

$$\mathfrak{L}(\omega)(x, a \,\square\, b, y) = (\omega(a), \omega(a) \,\square\, b + a \,\square\, \omega(b), \omega(y))$$

is a θ-invariant Lie derivation, that is, $\mathfrak{L}(\omega) \circ \theta = \theta \circ \mathfrak{L}(\omega)$, and ω is an inner triple derivation if and only if $\mathfrak{L}(\omega)$ is an inner Lie derivation. Hence,

Theorem (Chu-R 2016, Cont. Math.)

If V is any von Neumann algebra, then every (Lie) derivation of $\mathfrak{L}(V)$ is inner.

Proposition (Chu-R 2016, Cont. Math.)

If V is a finite von Neumann algebra, then $\mathfrak{L}(V) = [M_2(V), M_2(V)]$
(Lie isomorphism)

In particular, if $V = M_n(\mathbb{C})$, $\mathfrak{L}(V) = sl(2n, \mathbb{C})$ (classical Lie algebra of type A)

Compare this with Miers's work quoted earlier

Theorem. Miers Duke 1973

Every Lie derivation $D : [M, M] \to M$ where M is a von Neumann algebra, extends to an associative derivation of M, so $D[x, y] = [a, [x, y]]$ for some $a \in M$ and all $x, y \in M$.

Corollary

Every Lie derivation $D : M \to M$ where M is a von Neumann algebra, has the form $D(x) = [a, x] + \lambda(x)$ for some $a \in M$ and all $x, y \in M$, where λ is a center valued linear map which annihilates commutators,

Theorem. Miers PAMS 1978

Every Lie triple derivation $D : M \to M$ where M is a von Neumann algebra without central abelian projections, has the form $D(x) = [a, x] + \lambda(x)$ for some $a \in M$ and all $x, y \in M$, where λ is a center valued linear map which annihilates commutators.

Definition of Lie triple derivation

$$D([x[y, z]]) = [Dx[y, z]] + [x, [Dy, z]] + [x, [y, Dz]]$$

Choi-Effros 1977 (J. Funct. Anal.)

The range of a unital completely contractive projection on a C*-algebra A is isometrically isomorphic to a C*-algebra

Effros-Stormer 1979 (Trans. Amer. Math. Soc.)

The range of a unital contractive projection on a C*-algebra A is isometrically isomorphic to a Jordan C*-algebra, that is, a JC*-algebra

Friedman-R 1985 (J. Funct. Anal.)

The range of a contractive projection on a C*-algebra A is isometric to a subspace of A^{**} closed under $ab^*c + cb^*a$, that is, a JC*-triple.

Kaup 1984 (Math. Scand.)

The range of a contractive projection on a JB*-triple is isometrically isomorphic to a JB*-triple. (Stacho: nonlinear projections on bounded symmetric domains)

5. Other contexts

5A **JB*-triples**

Burgos-Polo-Peralta 2014 (Bull. Lon. Math. Soc.)

Every local triple derivation on a JB*-triple is a triple derivation.

JB*-triples include C*-algebras, JC*-triples (=J* algebras of Harris), Jordan C*-algebras (=JB*-algebras), Cartan factors

Cartan factors: $B(H,K), A(H), S(H)$,spin factors + 2 exceptional

Gelfand-Naimark Theorem. Friedman-R 1986 (Duke Math. J.)

Every JB*-triple is isomorphic to a subtriple of a direct sum of Cartan factors.

- second dual is JB*-triple (Dineen 1984): contractive projection (Kaup 1984) and local reflexivity
- atomic decomposition of JBW*-triples (Friedman-R 1985): $(\oplus C_\alpha) \oplus \{$no extreme functionals$\}$

JBW*-triples (=JB*-triples with predual)

Hamhalter-Kudaybergenov-Peralta-R 2016 (J. Math. Phys.)

Every 2-local triple derivation on a continuous JBW*-triple is a triple derivation.

Structure of JBW*-triples.
Horn 1987 (Math. Zeit.) Horn-Neher 1988 (Trans. AMS)

$$\underbrace{\oplus_\alpha L^\infty(\Omega_\alpha, C_\alpha)}_{Type\ I} \oplus \underbrace{pM \oplus H(N,\beta)}_{Continuous},$$

where each C_α is a Cartan factor, M and N are continuous von Neumann algebras, p is a projection in M, and β is a *-antiautomorphism of N of order 2 with fixed points $H(N,\beta)$.

Ingredients of the Proof

pM

- p properly infinite: reduces to local derivation
- p finite:
  - $D = L_a + R_b + D_1,\ D_1|_{pMp} = 0, a, b$ skew hermitian
  - $\tau(D(x)y^*) = -\tau(xD(y)^*)$

$H(M, \beta)$

- Jordan version of Dorofeev boundedness theorem (Matveichuk 1995)
- a completely additive complex measure on $P(H(M, \beta))$ is bounded
- $D|_{P(H(M,\beta))}$ is completely additive

Ayupov-Kudaybergenov 2016 (Lin. Alg. Appl.)

A local Lie derivation on finite dimensional semisimple Lie algebra is Lie derivation.

Ayupov-Kudaybergenov-Rakhimov 2015 (Lin. Alg. Appl.)

A 2-local Lie derivation on finite dimensional semisimple Lie algebra is derivation.

Examples

- Let L be a finite-dimensional nilpotent Lie algebra with $\dim L \geq 2$. Then L admits a 2-local derivation which is not a derivation.
- Let L be a finite-dimensional filiform Lie algebra with $\dim L \geq 3$. Then L admits a local derivation which is not a derivation.

A Lie algebra L is called filiform if $\dim L^k = n - k - 1$ for $1 \leq k \leq n-1$, where $L^0 = L$, $L^k = [L^{k-1}, L]$, $k \geq 1$

The case of general nilpotent finite-dimensional Lie algebras is still open.

5C **Derivations on algebras of measurable operators**

Theorem

Every derivation on the algebra of locally measurable operators affiliated with a properly infinite von Neumann algebra is continuous in the local measure topology.

Proof

- Type I_∞ **Albeverio-Ayupov-Kudaybergenov** J. Funct. Anal. 2009
- Type III **Ayupov-Kudaybergenov**
  Inf. Dim. Anal. Quant. Prob. Rel. Top. 2010
- Type II_∞ **Ber-Chilin-Sukochev** Int. Eq. Op. Th. 2013

Ber-Chilin-Sukochev 2004 (Math. Notes) 2006 (Extracta Math.)

If M is a commutative von Neumann algebra, then the algebra of measurable operators (= measurable functions) admits a non-inner derivation if and only if the projection lattice of M is atomic.

Ber-Chilin-Sukochev Proc. Lon. Math. Soc. 2014

- **Theorem** Every continuous derivation on the algebra of locally measurable operators is inner.
- **Corollary** Every derivation on the algebra of locally measurable operators affiliated with a properly infinite von Neumann algebra is inner.

Proof

- Type I_∞ **Albeverio-Ayupov-Kudaybergenov** Siberian Adv. Math. 2008, Positivity 2008, J. Funct. Anal. 2009, Extracta Math. 2009
- Type III **Ayupov-Kudaybergenov** J. Op. Thy. 2012, J. Math. Anal. Appl. 2013
- Type II_∞ **Ber-Chilin-Sukochev** Proc. Lon. Math. Soc. 2014

Continuity for the II_1 case remains open.

5D **Local and 2-local derivations on algebras of measurable operators**

Albeverio-Ayupov-Kudaybergenov-Nurjanov Comm. Contem. Math. 2011

- If M is a von Neumann algebra with a faithful normal semifinite trace τ, then every local derivation on the τ-measurable operators, which is continuous in the τ-measure topology, is a derivation, inner if M is of Type I.
- If M is finite of type I, all local derivations are derivations.

Example

If M is a commutative von Neumann algebra, then the algebra of measurable operators (= measurable functions) admits a local derivation which is not a derivation if and only if the projection lattice of M is atomic.

Ayupov-Kudaybergenov-Alauadinov Ann. Funct. Anal. 2013, Lin. Alg. Appl. 2013

• If M is a von Neumann algebra of Type I_∞, then every 2-local derivation on the algebra of locally measurable operators is a derivation.
• If M is a finite von Neumann algebra of Type I without abelian direct summands, then every 2-local derivation on the algebra of locally measurable operators (= measurable operators) is a derivation.

Example

If M is a commutative von Neumann algebra, then the algebra of measurable operators (= measurable functions) admits a 2-local derivation which is not a derivation if and only if the projection lattice of M is atomic.

5E **Murray-von Neumann algebras**

A closed densely defined operator T on a Hilbert space H is affiliated with a von Neumann algebra R when $UT = TU$ for each unitary operator U in R', the commutant of R. If operators S and T are affiliated with R, then $S + T$ and ST are densely defined, preclosed and their closures are affiliated with R. Such algebras are referred to as **Murray-von Neumann algebras**.

If R is a finite von Neumann algebra, we denote by $A_f(R)$ its associated Murray-von Neumann algebra. It is natural to conjecture that every derivation of $A_f(R)$ should be inner.

Kadison-Liu Proc. Natl. Acad. Sci., Math. Scand. 2014

- Extended derivations of $A_f(R)$ (those that map R into R) are inner.
- Each derivation of $A_f(R)$ with R a factor of type II_1 that maps $A_f(R)$ into R is 0. (noncommutative unbounded version of the Singer-Wermer conjecture). This is extended to the general von Neumann algebra of type II_1.

An application of local derivation

Ayupov-Kudaybergenov (ArXiv 1604.07147 April 25,2016)

Let $S(M)$ be the algebra of measurable operators affiliated with a von Neumann algebra M. The following conditions are equivalent:
(a) M is abelian;
(b) For every derivation D on $S(M)$ its square D^2 is a local derivation

Example

Consider the algebra A of all upper-triangular complex 2×2-matrices.

- All derivations of A are inner. (Coelho-Milies 1993 Lin. Alg. Appl.)
- If $a = [a_{ij}]$ and $D = \mathrm{ad}\, a$, then D is a derivation and D^2 is the inner derivation ad b, where

$$b = \begin{pmatrix} a_{11}^2 + a_{22}^2 & (a_{11} - a_{22})a_{12} \\ 0 & 2a_{11}a_{22} \end{pmatrix}$$

- So the square of every derivation is a derivation, hence a local derivation
- A is not commutative.