Jordan cohomology for operator algebras

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Outline

- Jordan Derivations
- Jordan Weak*-Amenability
- Jordan 2-cocycles

NOTE: Jordan can mean Jordan algebra or Jordan triple (depending on my fancy)

Building on earlier work of Kadison, Sakai proved that very derivation $\delta : M \to M$ of a von Neumann algebra into itself is inner (1966).

$$\delta(ab) = a\delta(b) + \delta(a)b \quad , \quad \delta(x) = \text{ad}a(x) = ax - xa$$

Thus the first Hochschild cohomology group $H^1(M, M)$ vanishes for any von Neumann algebra M.

Building on earlier work of Bunce and Paschke, Haagerup showed in 1983 that every derivation $\delta : M \to M_*$ of a von Neumann algebra into its predual is inner, and as a consquence that every C*-algebra is weakly amenable. .

$$\delta(ab) = a.\delta(b) + \delta(a).b \quad , \quad \delta(x) = \text{ad}\varphi(x) = \varphi.x - x.\varphi$$
$$\varphi.x(y) = \varphi(xy) \quad , \quad x.\varphi(y) = \varphi(yx)$$

Thus the first Hochschild cohomology group $H^1(M, M_*)$ vanishes for any von Neumann algebra M.

PROPOSITION 1 (special case of Upmeier 1980)

Let M be any von Neumann algebra. Then every Jordan derivation of M is an inner Jordan derivation. Thus the first "Jordan cohomology group" $H^1_J(M, M)$ vanishes for any von Neumann algebra M.

Earlier History

FD SS char 0: **Jacobson** 1949, 1951; char $\neq 2$: **Harris** 1959

Definition

When x_0 is an element in a Jordan Banach A-module, X, over a Jordan Banach algebra [a], A, for each $b \in A$, the mapping $\delta_{x_0,b} = [L(b), L(x_0)] : A \to X$,

$$\delta_{x_0,b}(a) := (x_0 \circ a) \circ b - (b \circ a) \circ x_0, \ (a \in A),$$

is a Jordan derivation. Finite sums of derivations of this form are called **inner Jordan derivations**.

[a] For purposes of this talk, Jordan algebra means an associative algebra with the product $a \circ b = (ab + ba)/2$, so for a Jordan derivation $D(a^2) = 2a \circ D(a)$ is enough.

Commutators in von Neumann algebras

Pearcy-Topping '69; Fack-delaHarpe '80

If M is a finite von Neumann algebra, then every element of M of central trace zero is a finite sum of commutators

Halmos '52,'54; Brown-Pearcy-Topping '68; Halpern '69

If M is properly infinite (no finite central projections), then every element of M is a finite sum of commutators

Thus for any von Neumann algebra, we have $M = Z(M) + [M, M]$, where $Z(M)$ is the center of M and $[M, M]$ is the set of finite sums of commutators in M.

PROOF of PROPOSITION 1

Suppose δ is a Jordan derivation of M. Then δ is an associative derivation (Sinclair) and by Kadison-Sakai, $\delta(x) = ax - xa$ where $a = z + \sum[x_i, y_i]$ with $z \in Z(M)$ and $x_i, y_i \in M$. Since $\mathrm{ad}\,[x, y] = 4[L(x), L(y)]$, δ is an inner Jordan derivation. Q.E.D.

PROPOSITION 2 (special case of Ho-Martinez-Peralta-Russo 2002)

Every Jordan triple derivation of M is an inner triple derivation. Thus $H^1_t(M,M)=0$

Earlier History

FD SS char 0: **Meyberg** 1972: Jordan Pair: **Loos** 1977, **Kühn-Rosendahl** 1978

Definition

Let E be a Jordan triple[a] and let X be a triple E-module. For each $b\in E$ and each $x_0\in X$, the mapping $\delta = L(b,x_0)-L(x_0,b) : E\to X$, defined by

$$\delta(a)=\{b,x_0,a\}-\{x_0,b,a\}\quad (a\in E), \tag{1}$$

is a triple derivation from E into X. Finite sums of derivations of the form $\delta(b,x_0)$ are called **inner triple derivations**.

[a] For purposes of this talk, a Jordan triple is an associative *-algebra with the triple product $\{a,b,c\}=(ab^*c+cb^*a)/2$ and a triple derivation satisfies $\delta\{a,b,c\}=\{\delta a,b,c\}+\{a,\delta b,c\}+\{a,b,\delta c\}$

LEMMA

Let A be a unital Banach *-algebra equipped with the ternary product given by $\{a, b, c\} = \frac{1}{2}\,(ab^*c + cb^*a)$ and the Jordan product $a \circ b = (ab + ba)/2$.

- Let D be an inner derivation, that is, $D = \operatorname{ad} a$ for some a in A. Then D is a *-derivation whenever $a^* = -a$. Conversely, if D is a *-derivation, then $a^* = -a + z$ for some z in the center of A.
- Every triple derivation is the sum of a Jordan *-derivation and an inner triple derivation.

PROOF of PROPOSITION 2

Suppose δ is a self-adjoint Jordan derivation of M. Then δ is an associative derivation (Sinclair) and by Kadison-Sakai and the Lemma, $\delta(x) = ax - xa$ where $a^* + a = z$ is a self adjoint element of the center of M. Since $M = Z(M) + [M, M]$, where $Z(M)$ denotes the center of M, we can therefore write

$$a = z' + \sum_j [b_j + ic_j, b_j' + ic_j'],$$

where b_j, b_j', c_j, c_j' are self adjoint elements of M and $z' \in Z(M)$.

It follows that

$$0 = a^* + a - z = (z')^* + z' - z + 2i \sum_j ([c_j, b_j'] + [b_j, c_j'])$$

so that $\sum_j ([c_j, b_j'] + [b_j, c_j'])$ belongs to the center of M. We now have

$$\delta = \mathrm{ad}\, a = \mathrm{ad} \sum_j ([b_j, b_j'] - [c_j, c_j']) \tag{2}$$

We have just seen that a self adjoint Jordan derivation δ of M has the form (2). A direct calculation shows that δ is equal to the inner triple derivation

$$\sum_j \left(L(b_j, 2b_j') - L(2b_j', b_j) - L(c_j, 2c_j') + L(2c_j', c_j) \right).$$

Thus, every triple derivation is inner.

Commutators in the predual of a von Neumann algebra

Theorem 1 Let M be a von Neumann algebra.

(a) If every Jordan derivation of M into M_* is approximated in norm by inner Jordan derivations, then M is finite.

(b) Conversely, if M is a finite von Neumann algebra acting on a separable Hilbert space or if M is a finite factor, then every Jordan derivation of M into M_* is approximated in norm by inner Jordan derivations.

Corollary (Cohomological characterization of finiteness)

If M acts on a separable Hilbert space, or if M is a factor, then M is finite if and only if every Jordan derivation of M into M_* is approximated in norm by inner Jordan derivations.

Theorem 1 and its corollary hold with Jordan derivation replaced by Jordan triple derivation

Theorem 2 Let M be an infinite factor

The complex vector space of Jordan derivations of M into M_*, modulo the norm closure of the inner Jordan derivations, has dimension 1.

Corollary (Zero-One Law)

If M is a factor, the linear space of Jordan derivations into the predual, modulo the norm closure of the inner Jordan derivations, has dimension 0 or 1: It is zero if the factor is finite; and it is 1 if the factor is infinite.

Theorem 2 and its corollary hold with Jordan derivation replaced by Jordan triple derivation

Theorems 1 and 2 and Propositions 3 and 4 which follow are joint work with Robert Pluta

Summary: If M is a factor,

M is infinite $\Leftrightarrow \dfrac{\text{Jordan Derivations into } M_*}{\textbf{Norm closure} \text{ of inner } \textbf{Jordan} \text{ derivations into } M_*} \sim \mathbb{C}$

M is finite $\Leftrightarrow \dfrac{\text{Jordan Derivations into } M_*}{\textbf{Norm closure} \text{ of inner } \textbf{Jordan} \text{ derivations into } M_*} = 0$

M is infinite $\Leftrightarrow \dfrac{\text{Jordan triple Derivations into } M_*}{\textbf{Norm closure} \text{ of inner } \textbf{triple} \text{ derivations into } M_*} \sim \mathbb{R}$

M is finite $\Leftrightarrow \dfrac{\text{Jordan triple Derivations into } M_*}{\textbf{Norm closure} \text{ of inner } \textbf{triple} \text{ derivations into } M_*} = 0$

Which von Neumann algebras are Jordan weak*-amenable? That is, every Jordan derivation into the predual is inner

Short answer: commutative, finite dimensional. Any others?

Proposition 3 Let M be a finite von Neumann algebra.

(a) If M acts on a separable Hilbert space or is a factor (hence admits a faithful normal finite trace tr), and if $\text{tr}^{-1}(0) = [M_*, M]$, then M is Jordan weak*-amenable. (Extended trace)

(b) If M is a factor and M is Jordan weak*-amenable, then $\text{tr}^{-1}(0) = [M_*, M]$.

Corollary

No factor of type II_1 is Jordan weak*-amenable

Proposition 3 and its corollary hold with Jordan derivation replaced by Jordan triple derivation

If M is a finite von Neumann algebra of type I_n, with $n < \infty$, we can assume

$$M = L^\infty(\Omega, \mu, M_n(\mathbb{C})) = M_n(L^\infty(\Omega, \mu)),$$

$$M_* = L^1(\Omega, \mu, M_n(\mathbb{C})_*)) = M_n(L^1(\Omega, \mu))$$

and

$$Z(M) = L^\infty(\Omega, \mu)1.$$

It is known that the center valued trace on M is given by

$$\mathrm{TR}\,(x) = \frac{1}{n}(\sum_1^n x_{ii})1 \quad , \quad \text{for } x = [x_{ij}] \in M$$

We thus define, for a finite von Neumann algebra of type I_n which has a faithful normal finite trace tr,

$$\mathrm{TR}\,(\psi) = \frac{1}{n}(\sum_1^n \psi_{ii})\mathrm{tr} \quad , \quad \text{for } \psi = [\psi_{ij}] \in M_*.$$

(a) If $\mathrm{TR}(\psi) = 0$, then ψ vanishes on the center $Z(M)$ of M.

(b) $\psi^* = -\psi$ on $Z(M)$ if and only if $\mathrm{tr}\,(\psi(\omega))$ is purely imaginary for almost every ω.

$$
\begin{aligned}
\psi(x) &= \int_\Omega \langle \psi(\omega), x(\omega)\rangle \, d\mu(\omega) = \int_\Omega \mathrm{tr}\,(\psi(\omega)x(\omega))\, d\mu(\omega) \\
&= \int_\Omega \mathrm{tr}\,([\sum_k \psi_{ik}(\omega)x_{kj}(\omega)])\, d\mu(\omega) \\
&= \int_\Omega (\sum_i \sum_k \psi_{ik}(\omega)x_{ki}(\omega))\, d\mu(\omega) \\
&= \int_\Omega (\sum_k \psi_{kk}(\omega)x_{kk}(\omega))\, d\mu(\omega) \\
&= \int_\Omega (\sum_k \psi_{kk}(\omega)) f(\omega)\, d\mu(\omega) = 0
\end{aligned}
$$

proving (a). As for (b), use $\psi(f \cdot 1) = \int_\Omega f(\omega)\mathrm{tr}\,(\psi(\omega))\, d\mu(\omega)$.

Proposition 4

Let M be a finite von Neumann algebra of type I_n with $n < \infty$, which admits a faithful normal finite trace tr (equivalently, M is countably decomposable $= \sigma$-finite). Then M is Jordan weak*-amenable if and only if

$$\mathrm{TR}^{-1}(0) = [M_*, M].$$

Corollary

Let M be a finite von Neumann algebra of type I_n admitting a faithful normal finite trace tr. If $\mathrm{tr}^{-1}(0) = [M_*, M]$, then M is Jordan weak*-amenable.

Proposition 4 and its corollary hold with Jordan derivation replaced by Jordan triple derivation

Problem

Is a finite von Neumann algebra of type I Jordan weak*-amenable? or triple weak*-amenable? If M admits a faithful normal finite trace, is

$$\mathrm{TR}^{-1}(0) = [M_*, M]?$$

Jordan 2-cocycles

Let M be a von Neumann algebra. A **Hochschild 2-cocycle** is a bilinear map $f : M \times M \to M$ satisfying

$$af(b,c) - f(ab,c) + f(a,bc) - f(a,b)c = 0 \tag{3}$$

EXAMPLE: **Hochschild 2-coboundary**

$$f(a,b) = a\mu(b) - \mu(ab) + \mu(a)b \quad , \quad \mu : M \to M \text{ linear}$$

A **Jordan 2-cocycle** is a bilinear map $f : M \times M \to M$ satisfying

$$f(a,b) = f(b,a) \text{ (symmetric)}$$

$$\begin{aligned} f(a^2, a\circ b) + f(a,b)\circ a^2 + f(a,a)\circ(a\circ b) \\ -f(a^2\circ b, a) - f(a^2,b)\circ a - (f(a,a)\circ b)\circ a = 0 \end{aligned} \tag{4}$$

EXAMPLE: **Jordan 2-coboundary**

$$f(a,b) = a\circ\mu(b) - \mu(a\circ b) + \mu(a)\circ b, \quad , \quad \mu : M \to M \text{ linear}$$

$$H^1(M,M) = \frac{\text{1-cocycles}}{\text{1-coboundaries}} = \frac{\text{derivations}}{\text{inner derivations}}$$

$$H^1_J(M,M) = \frac{\text{Jordan 1-cocycles}}{\text{Jordan 1-coboundaries}} = \frac{\text{Jordan derivations}}{\text{inner Jordan derivations}}$$

$$H^2(M,M) = \frac{\text{2-cocycles}}{\text{2-coboundaries}} \quad , \quad H^2_J(M,M) = \frac{\text{Jordan 2-cocycles}}{\text{Jordan 2-coboundaries}}$$

For almost all von Neumann algebras, $H^2(M,M) = 0$. How about $H^2_J(M,M)$?

FD char 0: **Albert** 1947, **Penico** 1951; char $\neq 2$: **Taft** 1957

Two elegant approaches: Jordan classification; Lie algebras
One inelegant approach: solving linear equations

Linear algebra approach-Level 1

Let h be a Hochschild 1-cocycle, that is, a linear map $h : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfying $h(ab) - ah(b) - h(a)b = 0$. To show that there is an element $x \in M_n(\mathbb{C})$ such that $h(a) = xa - ax$, it is enough to prove this with $a \in \{e_{ij}\}$. With

$$x = \sum_{p,q} x_{pq} e_{pq}. \tag{5}$$

and γ_{ijpq} defined by

$$h(e_{ij}) = \sum_{p,q} \gamma_{ijpq} e_{pq}, \tag{6}$$

we arrive at the system of linear vector equations

$$\sum_{p,q} \gamma_{ijpq} e_{pq} = \sum_{p,q} \delta_{qi} x_{pq} e_{pj} - \sum_{p,q} \delta_{jp} x_{pq} e_{iq}. \tag{7}$$

with n^2 unknowns x_{ij}. Then any solution of (7) proves the result.

Linear algebra approach-Level 2

Let h be a Hochschild 2-cocycle, that is, a bilinear map $h : M_n(\mathbb{C}) \times M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfying $ah(b,c) - h(ab,c) + h(a,bc) - h(a,b)c = 0$.
To show that there is a linear transformation $\mu : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that $h(a,b) = \mu(ab) - a\mu(b) - \mu(a)b$, it is enough to prove that this holds with $a, b \in \{e_{ij}\}$, that is

$$h(e_{ij}, e_{kl}) = \delta_{jk}\mu(e_{il}) - e_{ij}\mu(e_{kl}) - \mu(e_{ij})e_{kl}. \tag{8}$$

With $\mu(e_{ij}) = \sum_{k,l} \mu_{ijkl} e_{kl}$ and γ_{ijklpq} defined by $h(e_{ij}, e_{kl}) = \sum_{p,q} \gamma_{ijklpq} e_{pq}$, we arrive at the system of n^6 linear equations

$$\sum_{p,q} \gamma_{ijklpq} e_{pq} = \sum_{p,q} \delta_{jk}\mu_{ilpq} - \sum_{p,q} \delta_{jp}\mu_{klpq} e_{iq} - \sum_{p,q} \delta_{qk}\mu_{ijpq} e_{pl}, \tag{9}$$

with n^4 unknowns μ_{ijkl}. Then any solution of (9) proves (8).

Linear algebra approach-Level 3

Let $M = M_2(L^\infty(\Omega))$ be a finite von Neumann algebra of type I_n with $n = 2$. Let f be a Jordan 2-cocycle, that is, a symmetric bilinear map $f : M \times M \to M$ with

$$f(a^2, ab) + f(a, b)a^2 + f(a, a)(ab) - f(a^2 b, a) - f(a^2, b)a - (f(a, a)b)a = 0. \quad (10)$$

(To save space, ab denotes the Jordan product in the associative algebra M)

To show that there is a linear transformation $\mu : M \to M$ such that

$$f(a, b) = \mu(ab) - a\mu(b) - \mu(a)b, \quad (11)$$

it is enough to prove, for $a, b \in Z(M)$,

$$f(ae_{ij}, be_{kl}) = \delta_{jk}\mu(abe_{il}) - ae_{ij}\mu(be_{kl}) - \mu(ae_{ij})be_{kl}. \quad (12)$$

With $\mu(ae_{ij}) = \sum_{k,l} \mu_{ijkl}(a)e_{kl}$ and $\gamma_{ijklpq}(a,b) \in Z(M)$ defined by

$$f(ae_{ij}, be_{kl}) = \sum_{p,q} \gamma_{ijklpq}(a,b)e_{pq}, \tag{13}$$

we arrive at the system of n^6 linear vector equations with $3n^4$ unknowns $\mu_{ijkl}(ab), \mu_{ijkl}(a), \mu_{ijkl}(b)$

$$\begin{aligned} 2\sum_{p,q} \gamma_{ijklpq}(a,b)e_{pq} &= \delta_{jk} \sum_{p,q} \mu_{ilpq}(ab)e_{pq} \\ &- \sum_{p,q} a(\delta_{jp}\mu_{klpq}(b)e_{iq} + \delta_{iq}\mu_{klpq}(b)e_{pj}) \\ &- \sum_{p,q} b(\delta_{qk}\mu_{ijpq}(a)e_{pl} + \delta_{lp}\mu_{ijpq}(a)e_{kq}). \end{aligned} \tag{14}$$

Then any solution of (14) proves (12) and hence (11).

Some properties of Jordan 2-cocycles

Proposition

Every symmetric Hochschild 2-cocycle is a Jordan 2-cocycle.

Recall that every Jordan derivation on a semisimple Banach algebra is a derivation (Sinclair). If every Jordan 2-cocycle was a Hochschild 2-cocycle, GAME OVER

Proposition Let M be a von Neumann algebra.

(a) Let $f : M \times M \to M$ be defined by $f(a,b) = a \circ b$. Then f is a Jordan 2-cocycle with values in M, which is not a Hochschild 2-cocycle unless M is commutative.

(b) If M is finite with trace tr, then $f : M \times M \to M_*$ defined by $f(a,b)(x) = \mathrm{tr}((a \circ b)x)$ is a Jordan 2-cocycle with values in M_* which is not a Hochschild 2-cocycle unless M is commutative.

Proposition

Let f be a Jordan 2-cocycle on the von Neumann algebra M. Then $f(1,x) = xf(1,1)$ for every $x \in M$ and $f(1,1)$ belongs to the center of M.

Recall the definition of Jordan 2-cocycle

$$f(a^2, a \circ b) + f(a, b) \circ a^2 + f(a, a) \circ (a \circ b) \qquad (15)$$
$$-f(a^2 \circ b, a) - f(a^2, b) \circ a - (f(a, a) \circ b) \circ a = 0$$

Proof of part (a) of the second proposition

Let $f(a, b) = a \circ b$. The equation (15) reduces to

$$a^2 \circ (a \circ b) + (a \circ b) \circ a^2 + a^2 \circ (a \circ b) - (a^2 \circ b) \circ a - (a^2 \circ b) \circ a - (a^2 \circ b) \circ a,$$

which is zero by the Jordan axiom, so f is a Jordan 2-cocycle.
If this f were a Hochschild 2-cocycle, we would have

$$c(a \circ b) - (ca) \circ b + c \circ (ab) - (c \circ a)b = 0,$$

which reduces to $[[c, b], a] = 0$ and therefore $[M, M] \subset Z(M)$ (the center of M). Since $M = Z(M) + [M, M]$, M is commutative. This proves (a).

Merci!

Fin