DERIVATIONS
An introduction to non associative algebra
(or, Playing havoc with the product rule)

Series 2—Part 5
The linking algebra of a triple system

Colloquium
Fullerton College

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HISTORY

Series 1

- PART I FEBRUARY 8, 2011 **ALGEBRAS; DERIVATIONS**
- PART II JULY 21, 2011 **TRIPLE SYSTEMS; DERIVATIONS**
- PART III FEBRUARY 28, 2012 **MODULES; DERIVATIONS**
- PART IV JULY 26, 2012 **COHOMOLOGY (ASSOCIATIVE ALGEBRAS)**
- PART V OCTOBER 25, 2012 **THE SECOND COHOMOLOGY GROUP**
- PART VI MARCH 7, 2013 **COHOMOLOGY (LIE ALGEBRAS)**
- PART VII JULY 25, 2013 **COHOMOLOGY (JORDAN ALGEBRAS)**
- PART VIII SEPTEMBER 17, 2013 **VANISHING THEOREMS IN DIMENSIONS 1 AND 2 (ASSOCIATIVE ALGEBRAS)**
- PART IX FEBRUARY 18, 2014 **VANISHING THEOREMS IN DIMENSIONS 1 AND 2 (JORDAN ALGEBRAS)**

Series 2

- PART I JULY 24, 2014 **THE REMARKABLE CONNECTION BETWEEN JORDAN ALGEBRAS AND LIE ALGEBRAS (Two theorems relating different types of derivations)**
- PART II NOVEMBER 18, 2014 **THE REMARKABLE CONNECTION BETWEEN JORDAN ALGEBRAS AND LIE ALGEBRAS (Two theorems embedding triple systems in Lie algebras)**
- (digression) FEBRUARY 24, 2015 **GENETIC ALGEBRAS**
- PART III JULY 15, 2015 **LOCAL DERIVATIONS**
- (Fall 2015 missed due to the flu)
- PART IV FEBRUARY 23, 2016 **2-LOCAL DERIVATIONS**
- PART V (today) JUNE 28, 2016

  THE LINKING ALGEBRA OF A TRIPLE SYSTEM

"Slides" for all series 1 and series 2 talks available at

http://www.math.uci.edu/INSERT a "$\sim$" HERE brusso/undergraduate.html

The Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

DIFFERENTIATION IS A LINEAR PROCESS

$$(f+g)' = f' + g'$$

$$(cf)' = cf'$$

Product Rule

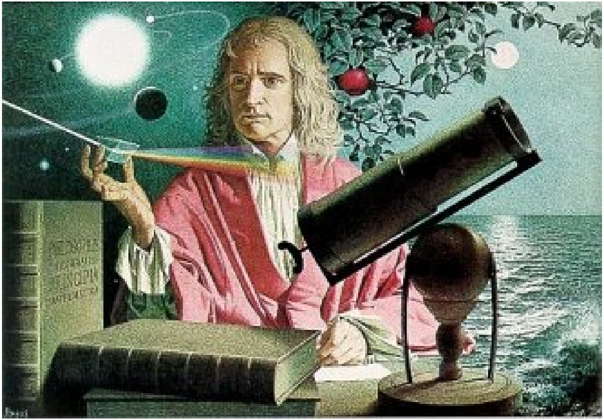
THE SET OF DIFFERENTIABLE FUNCTIONS FORMS AN **ALGEBRA**

$$(fg)' = fg' + f'g$$

If f and g are differentiable, so are $f+g$, fg and cf where c is a constant.

HEROS OF CALCULUS

#1 Sir Isaac Newton (1642-1727)



Isaac Newton was an English physicist, mathematician, astronomer, natural philosopher, alchemist, and theologian, and is considered by many scholars and members of the general public to be one of the most influential people in human history.

HEROS OF CALCULUS

#2 Gottfried Wilhelm Leibniz (1646-1716)

Gottfried Wilhelm Leibniz was a German mathematician and philosopher. He developed the infinitesimal calculus independently of Isaac Newton, and Leibniz's mathematical notation has been widely used ever since it was published.

LEIBNIZ RULE

$$(fg)' = f'g + fg'$$

****

$$(fgh)' = f'gh + fg'h + fgh'$$

(Will be very important for us)

****

$$(f_1 f_2 \cdots f_n)' = f_1' f_2 \cdots f_n + \cdots + f_1 f_2 \cdots f_n'$$

The chain rule,

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

plays no role in this talk. Neither does the quotient rule

$$(f/g)' = \frac{gf' - fg'}{g^2}$$

RECTANGULAR MATRICES

$$M_{p,q} = \text{ all } p \text{ by } q \text{ real matrices}$$

$$\mathbf{a} = [a_{ij}]_{p\times q} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \cdots & \cdots & \cdots & \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix} \quad (a_{ij} \in \mathbb{R})$$

Matrix Multiplication $M_{p,q} \times M_{q,r} \subset M_{p,r}$

$$\mathbf{ab} = [a_{ij}]_{p\times q}[b_{kl}]_{q\times r} = [c_{ij}]_{p\times r} \text{ where } c_{ij} = \sum_{k=1}^{q} a_{ik}b_{kj}$$

Example

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = [a_{ij}]_{3\times 2}[b_{ij}]_{2\times 1} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} \\ a_{31}b_{11} + a_{32}b_{21} \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} = [c_{ij}]_{3\times 1}$$

SQUARE MATRICES

$$M_p = M_{p,p} = \text{ all } p \text{ by } p \text{ real matrices}$$

$$\mathbf{a} = [a_{ij}]_{p\times p} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \cdots & \cdots & \cdots & \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{bmatrix} \quad (a_{ij} \in \mathbb{R})$$

Matrix Multiplication $M_p \times M_p \subset M_p$

$$\mathbf{ab} = [a_{ij}]_{p\times p}[b_{kl}]_{p\times p} = [c_{ij}]_{p\times p} \text{ where } c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

Examples $p = 1, 2$

- $M_1 = \{[a_{11}] : a_{11} \in \mathbb{R}\}$ (Behaves exactly as $\mathbb{R}$)
- $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = [a_{ij}]_{2\times 2}[b_{ij}]_{2\times 2} = \begin{bmatrix} a_{11}b_{11} + a_{11}b_{12} & a_{21}b_{11} + a_{22}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

Important special cases

$M_{p,q}$ is a linear space

$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$, $c[a_{ij}] = [ca_{ij}]$

$M_{1,p} = \mathbb{R}^p$ (row vectors)

$M_{p,1} = \mathbb{R}^p$ (column vectors)

M_p is an algebra

$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$, $c[a_{ij}] = [ca_{ij}]$, $[a_{ij}] \times [b_{ij}] = [\sum_{k=1}^{n} a_{ik} b_{kj}]$

$M_{p,q}$ is a triple system

You need the transpose of a matrix; details forthcoming.

Review of Algebras—Axiomatic approach

AN ALGEBRA IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE OVER A FIELD) WITH TWO BINARY OPERATIONS, CALLED ADDITION AND MULTIPLICATION—we are downplaying multiplication by scalars (=numbers=field elements)

ADDITION IS DENOTED BY $a + b$ AND IS REQUIRED TO BE
COMMUTATIVE $\quad a + b = b + a$
AND ASSOCIATIVE $\quad (a + b) + c = a + (b + c)$

MULTIPLICATION IS DENOTED BY ab AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION
$(a + b)c = ac + bc, \quad a(b + c) = ab + ac$

AN ALGEBRA IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)
(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

Table 1 (FASHIONABLE) ALGEBRAS

commutative algebras $ab = ba$
(Real numbers, Complex numbers, Continuous functions)

****
* **associative algebras** $a(bc) = (ab)c$ *
* (Matrix multiplication) *
****

Lie algebras $a^2 = 0$, $(ab)c + (bc)a + (ca)b = 0$
(Bracket multiplication on associative algebras: $[x, y] = xy - yx$)

Jordan algebras $ab = ba$, $a(a^2b) = a^2(ab)$
(Circle multiplication on associative algebras: $x \circ y = (xy + yx)/2$)

DERIVATIONS ON MATRIX ALGEBRAS

THE SET $M_n(\mathbb{R})$ of n by n MATRICES IS AN ALGEBRA UNDER
MATRIX ADDITION $A+B$
AND **MATRIX MULTIPLICATION** $A \times B$ (or AB)
WHICH IS ASSOCIATIVE BUT NOT COMMUTATIVE.

For the Record: (square matrices)

$A+B=[a_{ij}]+[b_{ij}]=[a_{ij}+b_{ij}]$ $\qquad$ $A\times B=[a_{ij}]\times[b_{ij}]=[\sum_{k=1}^n a_{ik}b_{kj}]$

DEFINITION

A DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS A LINEAR PROCESS δ: $\quad \delta(A+B)=\delta(A)+\delta(B)$
WHICH SATISFIES THE PRODUCT RULE

$$\delta(A\times B)=\delta(A)\times B+A\times\delta(B)$$

or

$$\delta(AB)=\delta(A)B+A\delta(B)$$

Compare with $\qquad (fg)'=f'g+fg'$

PROPOSITION

FIX A MATRIX A in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A \qquad (= AX - XA)$$

THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION (WHICH IS CALLED AN **INNER DERIVATION**)

THEOREM 1

Finite dimensional 1920-1940: Noether,Wedderburn,Hochschild,Jacobson,...
Infinite dimensional 1950-1970: Kaplansky,Kadison,Sakai,...

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS INNER, THAT IS, OF THE FORM δ_A FOR SOME A IN $M_n(\mathbb{R})$.

We gave a proof of this theorem for $n = 2$ in part 8 of series 1.

Here is that proof

Matrix units

Let $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

LEMMA

- $E_{11} + E_{22} = I$
- $E_{ij}^t = E_{ji}$
- $E_{ij} E_{kl} = \delta_{jk} E_{il}$

THEOREM 1 (restated)

Let $\delta : M_2 \to M_2$ be a derivation: δ is linear and $\delta(AB) = A\delta(B) + \delta(A)B$. Then there exists a matrix K such that $\delta(X) = XK - KX$ for X in M_2.

PROOF OF THEOREM 1

$$\begin{aligned} 0 &= \delta(1) = \delta(E_{11}+E_{22}) = \delta(E_{11}) + \delta(E_{22}) \\ &= \delta(E_{11}E_{11}) + \delta(E_{21}E_{12}) \\ &= E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} + E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} \\ &= E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) + \delta(E_{11})E_{11} + \delta(E_{21})E_{12}. \end{aligned}$$

Let $K = E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) = -\delta(E_{11})E_{11} - \delta(E_{21})E_{12}$. Then

- $KE_{11} = -\delta(E_{11})E_{11} \quad , \quad E_{11}K = E_{11}\delta(E_{11})$
- $KE_{12} = -\delta(E_{11})E_{12} \quad , \quad E_{12}K = E_{11}\delta(E_{12})$
- $KE_{21} = -\delta(E_{21})E_{11} \quad , \quad E_{21}K = E_{21}\delta(E_{11})$
- $KE_{22} = -\delta(E_{21})E_{12} \quad , \quad E_{22}K = E_{21}\delta(E_{12})$

- $E_{11}K - KE_{11} = E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} = \delta(E_{11}E_{11}) = \delta(E_{11})$
- $E_{12}K - KE_{12} = E_{11}\delta(E_{12}) + \delta(E_{11})E_{12} = \delta(E_{11}E_{12}) = \delta(E_{12})$
- $E_{21}K - KE_{21} = E_{21}\delta(E_{11}) + \delta(E_{21})E_{11} = \delta(E_{21}E_{11}) = \delta(E_{21})$
- $E_{22}K - KE_{22} = E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} = \delta(E_{21}E_{12}) = \delta(E_{22})$ Q.E.D.

Transpose (to the rescue)

If $\mathbf{a} = [a_{ij}] \in M_{p,q}$ then $\mathbf{a}^t = [a^t_{ij}] \in M_{q,p}$ where $a^t_{ij} = a_{ji}$

$$\mathbf{a} = [a_{ij}]_{p\times q} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \cdots & \cdots & \cdots & \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix}$$

$$\mathbf{a}^t = [a^t_{ij}]_{q\times p} = \begin{bmatrix} a^t_{11} & a^t_{12} & \cdots & a^t_{1p} \\ a^t_{21} & a^t_{22} & \cdots & a^t_{2p} \\ \cdots & \cdots & \cdots & \\ a^t_{q1} & a^t_{q2} & \cdots & a^t_{qp} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{p1} \\ a_{12} & a_{22} & \cdots & a_{p2} \\ \cdots & \cdots & \cdots & \\ a_{1q} & a_{2q} & \cdots & a_{pq} \end{bmatrix}$$

$(ab)^t = b^t a^t$

Proof: If $\mathbf{a} = [a_{ij}]$, $\mathbf{b} = [b_{ij}]$ and $\mathbf{c} = \mathbf{ab} = [c_{ij}]$, so $c_{ij} = \sum_{k=1}^q a_{ik} b_{kj}$,

$$\mathbf{b}^t\mathbf{a}^t = [b^t_{ij}][a^t_{ij}] = [\textstyle\sum_{k=1}^q b^t_{ik} a^t_{kj}] \qquad \text{and}$$

$$(\mathbf{ab})^t = [c^t_{ij}] = [\textstyle\sum_{k=1}^q a_{jk} b_{ki}] = [\sum_{k=1}^q a^t_{kj} b^t_{ik}] \quad Q.E.D.$$

DERIVATIONS ON RECTANGULAR MATRICES

MULTIPLICATION DOES NOT MAKE SENSE ON $M_{m,n}(\mathbb{R})$ if $m \neq n$.
NOT TO WORRY! WE CAN FORM A TRIPLE PRODUCT $X \times Y^t \times Z$
(TRIPLE MATRIX MULTIPLICATION)

If X, Y are m by n with $m \neq n$, then $X \times Y$ is not defined.
However, $X \times Y^t$ is defined (m by m) and $Y^t \times X$ is defined (n by n).
So $X \times Y^t \times Z = (X \times Y^t) \times Z = X \times (Y^t \times Z)$ is m by n

For the Record (square matrices):

$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$, $[a_{ij}] \times [b_{ij}] = [\sum_{k=1}^n a_{ik} b_{kj}]$, $[a_{ij}]^t = [a_{ji}]$

For the Record (rectangular matrices):

$[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$, $[a_{ij}]_{m\times p} \times [b_{ij}]_{p\times n} = [\sum_{k=1}^p a_{ik} b_{kj}]_{m\times n}$

$[a_{ij}]^t_{m\times n} = [a_{ji}]_{n\times m}$ Note: $(M_{m,n})^t = M_{n,m}$

DEFINITION

A DERIVATION ON $M_{m,n}(\mathbb{R})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION IS A LINEAR PROCESS δ WHICH SATISFIES THE (TRIPLE) PRODUCT RULE

$$\delta(A \times B^t \times C) =$$

$$\delta(A) \times B^t \times C + A \times \delta(B)^t \times C + A \times B^t \times \delta(C)$$

or

$$\delta(AB^tC) = \delta(A)B^tC + A\delta(B)^tC + AB^t\delta(C)$$

Compare with

$$(fgh)' = f'gh + fg'h + fgh'$$

PROPOSITION

FOR TWO SKEW SYMMETRIC (square) MATRICES $A \in M_m(\mathbb{R})$, $B \in M_n(\mathbb{R})$, THAT IS $A^t = -A$, $B^t = -B$, DEFINE $\delta_{A,B}(X) = A \times X + X \times B$. THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION

THEOREM 2

Finite dimensional 1960-1980: Meyberg, Carlsson, Lister, ...
Infinite dimensional 1995-2015: Ho,Peralta,Martinez,Russo,Zalar,Pitts,Pluta,...

EVERY DERIVATION ON $M_{m,n}(\mathbb{R})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION IS OF THE FORM $\delta_{A,B}$.

The main point of today's talk is to give a proof of Theorem 2 using Theorem 1.

REMARK

THIS THEOREM HOLDS TRUE AND IS OF INTEREST FOR THE CASE $m = n$.

Some notation

Let X denote $M_{p,q}$ and let $x, y, z, \ldots$ denote elements of X. We define

$$XX^t = \{x_1y_1^t + x_2y_2^t + \cdots + x_ny_n^t : x_i, y_i \in X, n = 1, 2 \ldots\}$$

and

$$X^tX = \{x_1^ty_1 + x_2^ty_2 + \cdots + x_n^ty_n : x_i, y_i \in X, n = 1, 2 \ldots\}$$

$$XX^t = M_{p,q}M_{q,p} \text{ is a subalgebra of } M_p$$

and

$$X^tX = M_{q,p}M_{p,q} \text{ is a subalgebra of } M_q$$

$$\text{Let } A = \begin{bmatrix} XX^t & X \\ X^t & X^tX \end{bmatrix} \qquad \left(\text{Note: } A \subset \left[\begin{smallmatrix} M_p & M_{p,q} \\ M_{q,p} & M_q \end{smallmatrix}\right] \subset M_{p+q}\right)$$

$$= \left\{ \left[\begin{smallmatrix} \sum_{i=1}^n x_iy_i^t & x \\ y^t & \sum_{j=1}^m z_j^tw_j \end{smallmatrix}\right] : x, y, x_i, y_i, z_j, w_j \in X, n = 1, 2, \ldots, m = 1, 2, \ldots \right\}$$

Proposition

Let $X = M_{p,q}$ and let $D : X \to X$ be a triple matrix derivation of X. If $A = \begin{pmatrix} XX^t & X \\ X^t & X^tX \end{pmatrix} \subset \begin{pmatrix} M_p & M_{p,q} \\ M_{q,p} & M_q \end{pmatrix} \subset M_{p+q}$, then A is an algebra and the map $\delta : A \to A$ given, for $x, y, x_i, y_i, z_j, w_j \in X$, by

$$\begin{bmatrix} \sum_i x_i y_i^t & x \\ y^t & \sum_j z_j^t w_j \end{bmatrix} \mapsto \begin{bmatrix} \sum_i (x_i(Dy_i)^t + (Dx_i)y_i^t) & Dx \\ (Dy)^t & \sum_j (z_j^t(Dw_j) + (Dz_j)^t w_j) \end{bmatrix}$$

is well defined and a derivation of A, which extends D (when X is embedded in A via $x \mapsto \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$). If $a = \begin{pmatrix} \alpha & x \\ y^t & \beta \end{pmatrix} \in A$ then $\delta(a^t) = \delta(a)^t$ where $a^t = \begin{pmatrix} \alpha^t & y \\ x^t & \beta^t \end{pmatrix}$.

- A is an algebra: $a = \begin{pmatrix} \alpha & x \\ y^t & \beta \end{pmatrix}$, $b = \begin{pmatrix} \alpha_1 & x_1 \\ y_1^t & \beta_1 \end{pmatrix}$, $ab = \begin{pmatrix} \alpha\alpha_1 + xy_1^t & \alpha x_1 + x\beta_1 \\ y^t\alpha_1 + \beta y_1^t & y^t x_1 + \beta\beta_1 \end{pmatrix}$
- δ is well-defined: $\sum_i x_i y_i^t = 0 \Rightarrow \sum_i (Dx_i)y_i^t + x_i(Dy_i)^t) = 0$ (see next page)
- δ is linear: $\delta(a+b) = \delta(a) + \delta(b)$; $\delta(ca) = c\delta(a)$
- $\delta(a^t) = \delta(a)^t$ (see next page)
- $\delta(ab) = a\delta(b) + \delta(a)b$ (Not difficult, but messy)
- $\delta \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & Dx \\ 0 & 0 \end{pmatrix}$

If $\sum_i x_i y_i^t = 0$, then for every $z \in X$,

$$\begin{aligned} 0 &= D(\sum_i x_i y_i^t z) \\ &= \sum_i ((Dx_i) y_i^t z + x_i (Dy_i)^t z + x_i y_i^t (Dz)) \\ &= (\sum_i (Dx_i) y_i^t + x_i (Dy_i)^t) z, \end{aligned}$$

so δ is well defined (this argument is incomplete—see the next page)

If $a = \begin{pmatrix} \sum_i x_i y_i^t & x \\ y^t & \sum_j z_j^t w_j \end{pmatrix}$, then $\delta(a) = \begin{pmatrix} \sum_i (x_i (Dy_i)^t + (Dx_i) y_i^t) & Dx \\ (Dy)^t & \sum_j (z_j^t (Dw_j) + (Dz_j)^t w_j) \end{pmatrix}$, and

$$\begin{aligned} \delta(a^t) &= \delta \begin{pmatrix} \sum_i y_i x_i^t & y \\ x^t & \sum_j w_j^t z_j \end{pmatrix} \\ &= \begin{pmatrix} \sum_i (y_i (Dx_i)^t + (Dy_i) x_i^t) & Dy \\ (Dx)^t & \sum_j (w_j^t (Dz_j) + (Dw_j)^t z_j) \end{pmatrix} \\ &= \delta(a)^t. \end{aligned}$$

Correction

Let α denote $\sum_i (Dx_i)y_i^t + x_i(Dy_i)^t)$. We know that $\alpha z{=}0$ for every $z \in X$ and we must conclude $\alpha = 0$. This is true but it is not so simple. You need to know about the norms of vectors and norms of linear transformations.

If you believe the above, you can also believe that if we let β denote $\sum_j (z_j^t(Dw_j) + (Dz_j)^t w_j)$ and if we know that $z\beta = 0$ for every $z \in X$, we must conclude that $\beta = 0$.

If you accept these two statements (they are true), it is clear that the map δ is well defined, since if $a = \begin{pmatrix} \sum_i x_i y_i^t & x \\ y^t & \sum_j z_j^t w_j \end{pmatrix} = 0$, then $\sum_i x_i y_i^t = 0,\ \sum_j z_j^t w_j = 0, x = 0, y = 0$; so

$$\delta(a) = \begin{bmatrix} \sum_i (x_i(Dy_i)^t + (Dx_i)y_i^t) & Dx \\ (Dy)^t & \sum_j (z_j^t(Dw_j) + (Dz_j)^t w_j) \end{bmatrix} = 0$$

Theorem 2 (restated)

For every triple matrix derivation D of $M_{p,q}$, there exist $\alpha \in M_p$ and $\beta \in M_q$ such that $\alpha^t = -\alpha,\ \beta^t = -\beta$, and $Dx = \alpha x + x\beta$ for every $x \in M_{p,q}$.

Proof

Let δ be as in the Proposition. Then there exists[a] $b \in A$ such that $\delta(a) = ba - ab$ for every $a \in A$. We have $ba^t - a^t b = \delta(a^t) = \delta(a)^t = a^t b^t - b^t a^t$ so that $(b + b^t)a^t = a^t(b + b^t)$ for every $a \in A$. Let $z = b + b^t$. Then $z = z^t$ and $za = az$ for every $a \in A$.

[a]We only proved this if $A = M_n$ but it is true for any subalgebra A of M_n such that $A^t = A$ (**Artin-Wedderburn Theorem**; more about this later).

Let $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then for any $a = \left(\begin{smallmatrix} \alpha & x \\ y^t & \beta \end{smallmatrix}\right) \in A$, $eae = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\delta(eae) \in eAe$. In particular, $\delta(e) = \delta(eee) \in eAe$, so $\delta(e) = e\delta(e)e$. On the other hand, $\delta(e) = \delta(ee) = e\delta(e) + \delta(e)e$, so $e\delta(e) = e\delta(e) + e\delta(e)e$, thus $e\delta(e)e = 0$ and so $\delta(e) = 0$.

If we write $b = \begin{pmatrix} \alpha_1 & x_1 \\ y_1^t & \beta_1 \end{pmatrix}$, then $0 = \delta(e) = be - eb$ shows that $x_1 = 0$ and $y_1 = 0$.

Let $c = (b - b^t)/2$. Then $c^t = -c$ and $\delta(a) = ca - ac$ for all $a \in A$, and

$$c = \begin{pmatrix} (\alpha_1 - \alpha_1^t)/2 & 0 \\ 0 & (\beta_1 - \beta_1^t)/2 \end{pmatrix} = \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix}$$

where $\alpha_2^t = -\alpha_2$ and $\beta_2^t = -\beta_2$.

Finally

$$\begin{aligned} \begin{pmatrix} 0 & Dx \\ 0 & 0 \end{pmatrix} &= \delta \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \alpha_2 x \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & x\beta_2 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \alpha_2 x - x\beta_2 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Q. E. D.

Example of a subalgebra

$$A = \left\{ \left[\begin{array}{cc|ccc|cc} x_{11} & x_{12} & 0 & 0 & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & y_{11} & y_{12} & y_{13} & 0 & 0 \\ 0 & 0 & y_{21} & y_{22} & y_{23} & 0 & 0 \\ 0 & 0 & y_{31} & y_{32} & y_{33} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] : x_{ij}, y_{ij} \in \mathbb{R} \right\} \subset M_7$$

$$\text{that is, } A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & O_{2\times 3} & O_{2\times 2} \\ O_{3\times 2} & A_2 & O_{3\times 2} \\ O_{2\times 2} & 0_{2\times 3} & O_{2\times 2} \end{bmatrix}$$

$$A_1 = M_2 \quad , \quad A_2 = M_3, \quad O_{m\times n} = [a_{ij}]_{m\times n}, \quad a_{ij} = 0.$$

Joseph Henry Maclagan Wedderburn (1882–1948)

Scottish mathematician, who taught at Princeton University for most of his career. A significant algebraist, he proved that a finite division algebra is a field, and part of the Artin–Wedderburn theorem on simple algebras. He also worked on group theory and matrix algebra.

Theorem of Artin-Wedderburn

Every subalgebra A of M_n such that $A^t = A$ is of the form[a]

$$\left[\begin{array}{cccccc|c} A_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & A_k & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & O \end{array}\right] \subset M_n$$

where $A_j = M_{n_j}$, $m = n_1 + n_2 + \cdots n_k \leq n$, and $O = O_{(n-m)\times(n-m)}$

[a] more precisely, "is isomorphic to"

Exercises

• Every derivation of such A is inner. (Hint: Show $\delta(A_j) \subset A_j$ for each j)

• In the proof of Theorem 2, we assumed that $e \in A$. Show that the proof of Theorem 2 can be revised so as to cover the case when $e \notin A$.

(Hint: Extend δ to the algebra $A + \mathbb{R}e$)