PACIFIC SUMMER UNSOLVED MATHEMATICS SEMINAR FULLERTON COLLEGE

DERIVATIONS

Introduction to non-associative algebra
OR
Playing havoc with the product rule?

PART II—TRIPLE SYSTEMS

JULY 21, 2011

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PART I
ALGEBRAS
FEBRUARY 8, 2011

PART II
TRIPLE SYSTEMS
JULY 21, 2011

PART III
INTERPLAY BETWEEN ALGEBRAS AND TRIPLES
FALL 2011—TBA

PART IV
MODULES AND COHOMOLOGY
SPRING 2012—TBA

REVIEW OF PART I—ALGEBRAS

AXIOMATIC APPROACH

AN ALGEBRA IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH TWO BINARY OPERATIONS, CALLED ADDITION AND MULTIPLICATION

ADDITION IS DENOTED BY

$$a + b$$

AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE

$$a + b = b + a, \quad (a + b) + c = a + (b + c)$$

MULTIPLICATION IS DENOTED BY

$$ab$$

AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION

$$(a + b)c = ac + bc, \quad a(b + c) = ab + ac$$

AN ALGEBRA IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)

(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

Table 2

ALGEBRAS

commutative algebras

$$ab = ba$$

associative algebras

$$a(bc) = (ab)c$$

Lie algebras

$$a^2 = 0$$

$$(ab)c + (bc)a + (ca)b = 0$$

Jordan algebras

$$ab = ba$$

$$a(a^2b) = a^2(ab)$$

DERIVATIONS ON THE SET OF MATRICES

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER

MATRIX ADDITION

$$A + B$$

AND

MATRIX MULTIPLICATION

$$A \times B$$

WHICH IS ASSOCIATIVE BUT NOT COMMUTATIVE.

DEFINITION 2

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \times B) = \delta(A) \times B + A \times \delta(B)$$

.

PROPOSITION 2

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION (WHICH CAN BE NON-ZERO)

THEOREM 2

(1942 Hochschild)

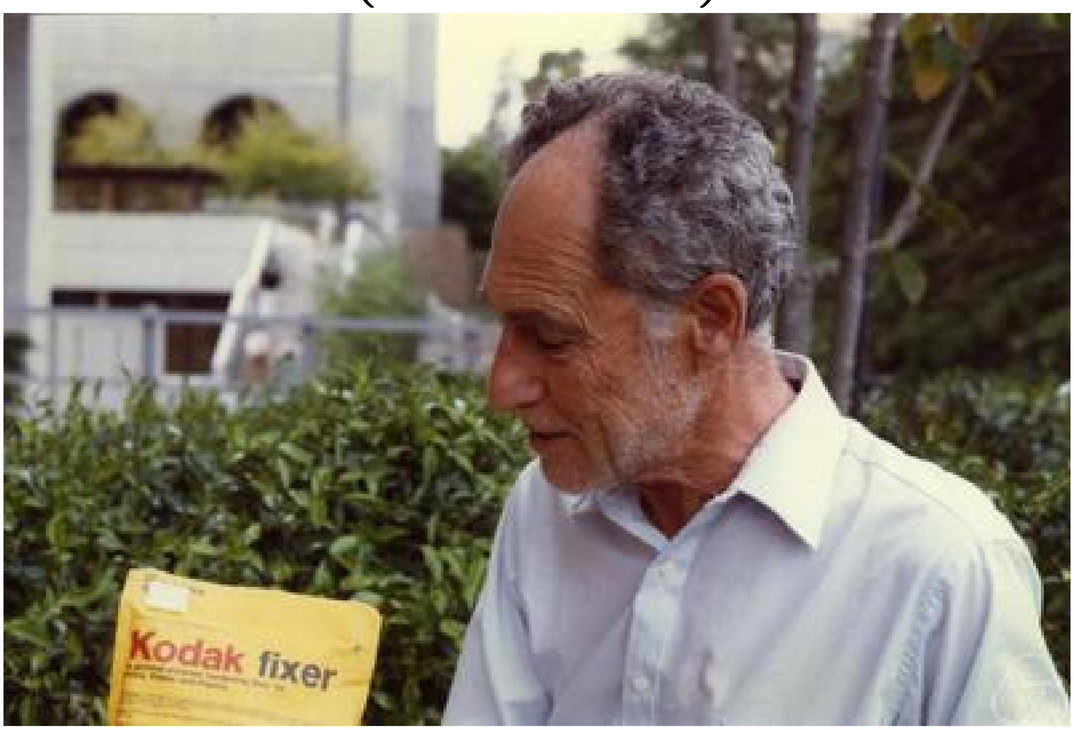
EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

Gerhard Hochschild (1915–2010)

(Photo 1968)

Gerhard Paul Hochschild was an American mathematician who worked on Lie groups, algebraic groups, homological algebra and algebraic number theory.

(Photo 1976)



(Photo 1981)

THE BRACKET PRODUCT ON THE SET OF MATRICES

THE BRACKET PRODUCT ON THE SET $M_n(\mathbf{R})$ OF MATRICES IS DEFINED BY

$$[X, Y] = X \times Y - Y \times X$$

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND BRACKET MULTIPLICATION, WHICH IS NOT ASSOCIATIVE AND NOT COMMUTATIVE.

DEFINITION 3

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO BRACKET MULTIPLICATION

IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta([A,B]) = [\delta(A),B] + [A,\delta(B)]$$

.

PROPOSITION 3

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = [A,X] = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION

THEOREM 3

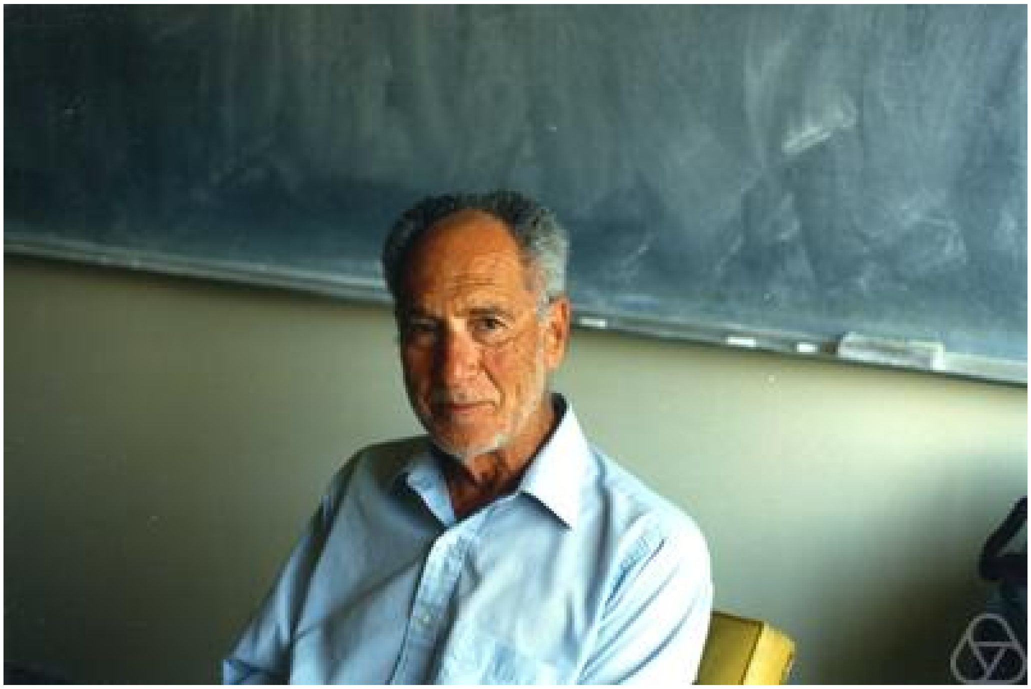
(1942 Hochschild, Zassenhaus)

EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

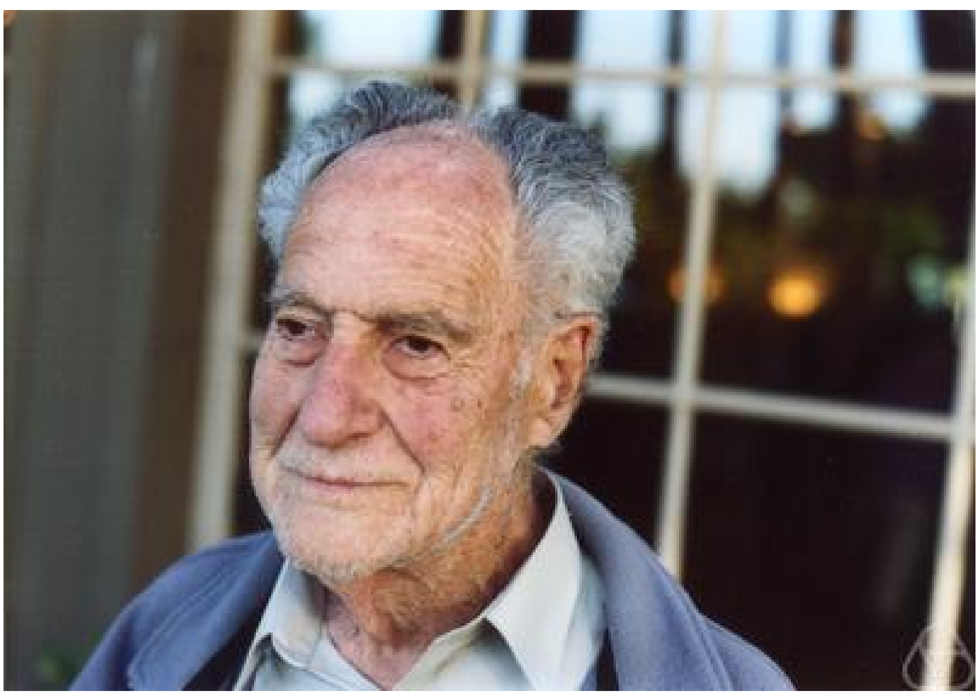
Hans Zassenhaus (1912–1991)

Hans Julius Zassenhaus was a German mathematician, known for work in many parts of abstract algebra, and as a pioneer of computer algebra.

Gerhard Hochschild (1915–2010)



(Photo 1986)



(Photo 2003)

THE CIRCLE PRODUCT ON THE SET OF MATRICES

THE CIRCLE PRODUCT ON THE SET $M_n(\mathbf{R})$ OF MATRICES IS DEFINED BY

$$X \circ Y = (X \times Y + Y \times X)/2$$

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND CIRCLE MULTIPLICATION, WHICH IS COMMUTATIVE BUT NOT ASSOCIATIVE.

DEFINITION 4

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION

IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$

PROPOSITION 4

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION

THEOREM 4

(1972-Sinclair)

EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

REMARK

(1937-Jacobson)

THE ABOVE PROPOSITION AND THEOREM NEED TO BE MODIFIED FOR THE SUBALGEBRA (WITH RESPECT TO CIRCLE MULTIPLICATION) OF SYMMETRIC MATRICES.

Alan M. Sinclair (retired)

Nathan Jacobson (1910–1999)

Nathan Jacobson was an American mathematician who was recognized as one of the leading algebraists of his generation, and he was also famous for writing more than a dozen standard monographs.

Table 1

$M_n(\mathbf{R})$ (ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a,b] = ab - ba$	$a \circ b = ab + ba$
Th. 2	Th.3	Th.4
$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$

GRADUS AD PARNASSUM
PART I—ALGEBRAS

1. Prove Proposition 2
2. Prove Proposition 3
3. Prove Proposition 4
4. Let A, B are two fixed matrices in $M_n(\mathbf{R})$. Show that the linear process

   $$\delta_{A,B}(X) = A \circ (B \circ X) - B \circ (A \circ X)$$

   is a derivation of $M_n(\mathbf{R})$ with respect to circle multiplication.
   (cf. Remark following Theorem 4)
5. Show that $M_n(\mathbf{R})$ is a Lie algebra with respect to bracket multiplication. In other words, show that the two axioms for Lie algebras in Table 2 are satisfied if ab denotes $[a, b] = ab - ba$ (a and b denote matrices and ab denotes matrix multiplication)

6. Show that $M_n(\mathbf{R})$ is a Jordan algebra with respect to circle multiplication. In other words, show that the two axioms for Jordan algebras in Table 2 are satisfied if ab denotes $a \circ b = ab + ba$ (a and b denote matrices and ab denotes matrix multiplication—forget about dividing by 2)
7. (Extra credit)
   Let us write $\delta_{a,b}$ for the linear process $\delta_{a,b}(x) = a(bx) - b(ax)$ in a Jordan algebra. Show that $\delta_{a,b}$ is a derivation of the Jordan algebra by following the outline below. (cf. Homework problem 4 above.)

   (a) In the Jordan algebra axiom

   $$u(u^2v) = u^2(uv),$$

   replace u by $u + w$ to obtain the two equations

   $$2u((uw)v) + w(u^2v) = 2(uw)(uv) + u^2(wv) \tag{1}$$

   and (**correcting the misprint in part I**)

   $$u(w^2v) + 2w((uw)v) = w^2(uv) + 2(uw)(\mathbf{w}v).$$

(Hint: Consider the “degree” of w on each side of the equation resulting from the substitution)

(b) In (1), interchange v and w and subtract the resulting equation from (1) to obtain the equation

$$2u(\delta_{v,w}(u)) = \delta_{v,w}(u^2). \qquad (2)$$

(c) In (2), replace u by $x+y$ to obtain the equation

$$\delta_{v,w}(xy) = y\delta_{v,w}(x) + x\delta_{v,w}(y),$$

which is the desired result.

END OF REVIEW OF PART I

BEGINNING OF PART II

IN THESE TALKS, I AM MOSTLY INTERESTED IN NONASSOCIATIVE ALGEBRAS (PART I) AND NONASSOCIATIVE TRIPLE SYSTEMS (PART II), ALTHOUGH THEY MAY OR MAY NOT BE COMMUTATIVE.

(ASSOCIATIVE AND COMMUTATIVE HAVE TO BE INTERPRETED APPROPRIATELY FOR THE TRIPLE SYSTEMS CONSIDERED WHICH ARE NOT ACTUALLY ALGEBRAS)

DERIVATIONS ON RECTANGULAR MATRICES

MULTIPLICATION DOES NOT MAKE SENSE ON $M_{m,n}(\mathbf{R})$ if $m \neq n$.

NOT TO WORRY!

WE CAN FORM A TRIPLE PRODUCT

$$X \times Y^t \times Z$$

(TRIPLE MATRIX MULTIPLICATION)

COMMUTATIVE AND ASSOCIATIVE DON'T MAKE SENSE HERE. RIGHT?

WRONG!!

$$(X \times Y^t \times Z) \times A^t \times B = X \times Y^t \times (Z \times A^t \times B)$$

DEFINITION 5

A DERIVATION ON $M_{m,n}(\mathbf{R})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION

IS A LINEAR PROCESS δ WHICH SATISFIES THE (TRIPLE) PRODUCT RULE

$$\delta(A \times B^t \times C) =$$
$$\delta(A) \times B^t \times C + A \times \delta(B)^t \times C + A \times B^t \times \delta(C)$$

PROPOSITION 5

FOR TWO MATRICES A, B in $M_{m,n}(\mathbf{R})$,

DEFINE $\delta_{A,B}(X) =$

$$A \times B^t \times X + X \times B^t \times A - B \times A^t \times X - X \times A^t \times B$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION

THEOREM 8*

EVERY DERIVATION ON $M_{m,n}(\mathbf{R})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

REMARK

THESE RESULTS HOLD TRUE AND ARE OF INTEREST FOR THE CASE $m = n$.

*Theorems 5,6,7 are in part I

TRIPLE BRACKET MULTIPLICATION

LET'S GO BACK FOR A MOMENT TO SQUARE MATRICES AND THE BRACKET MULTIPLICATION.

MOTIVATED BY THE LAST REMARK, WE DEFINE THE TRIPLE BRACKET MULTIPLICATION TO BE $[[X, Y], Z]$

DEFINITION 6

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION

IS A LINEAR PROCESS δ WHICH SATISFIES THE TRIPLE PRODUCT RULE

$$\delta([[A, B], C]) =$$
$$[[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$$

PROPOSITION 6

FIX TWO MATRICES A, B IN $M_n(\mathbf{R})$ AND DEFINE $\delta_{A,B}(X) = [[A, B], X]$
THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION.

THEOREM 9

EVERY DERIVATION OF $M_n(\mathbf{R})$ WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION IS A SUM OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

TRIPLE CIRCLE MULTIPLICATION

LET'S RETURN TO RECTANGULAR MATRICES AND FORM THE TRIPLE CIRCLE MULTIPLICATION

$$(A \times B^t \times C + C \times B^t \times A)/2$$

For sanity's sake, let us write this as

$$\{A, B, C\} = (A \times B^t \times C + C \times B^t \times A)/2$$

DEFINITION 7

A <u>DERIVATION</u> ON $M_{m,n}(\mathbf{R})$ WITH <u>RESPECT TO</u> <u>TRIPLE CIRCLE MULTIPLICATION</u>

IS A LINEAR PROCESS δ WHICH SATISFIES THE TRIPLE PRODUCT RULE

$$\delta(\{A,B,C\}) = \{\delta(A), B, C\} + \{A, \delta(B), C\} + \{A, B, \delta(C)\}$$

PROPOSITION 7

FIX TWO MATRICES A, B IN $M_{m,n}(\mathbf{R})$ AND DEFINE

$$\delta_{A,B}(X) = \{A, B, X\} - \{B, A, X\}$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION.

THEOREM 10

EVERY DERIVATION OF $M_{m,n}(\mathbf{R})$ WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

IT IS TIME FOR ANOTHER SUMMARY OF THE PRECEDING

Table 3

$M_{m,n}(\mathbf{R})$ (TRIPLE SYSTEMS)

triple matrix	triple bracket	triple circle
ab^tc	$[[a,b],c]$	$ab^tc + cb^ta$
Th. 8	Th.9	Th.10
$\delta_{a,b}(x)$ $=$ ab^tx $+xb^ta$ $-ba^tx$ $-xa^tb$	$\delta_{a,b}(x)$ $=$ abx $+xba$ $-bax$ $-xab$	$\delta_{a,b}(x)$ $=$ ab^tx $+xb^ta$ $-ba^tx$ $-xa^tb$
(sums)	(sums) $(m=n)$	(sums)

LET'S PUT ALL THIS NONSENSE TOGETHER

Table 1 $M_n(\mathbf{R})$ (ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a,b] = ab - ba$	$a \circ b = ab + ba$
Th. 2	Th.3	Th.4
$\delta_a(x)$ $=$ $ax - xa$	$\delta_a(x)$ $=$ $ax - xa$	$\delta_a(x)$ $=$ $ax - xa$

Table 3 $M_{m,n}(\mathbf{R})$ (TRIPLE SYSTEMS)

triple matrix	triple bracket	triple circle
ab^tc	$[[a,b],c]$	$ab^tc + cb^ta$
Th. 8	Th.9	Th.10
$\delta_{a,b}(x)$ $=$ ab^tx $+xb^ta$ $-ba^tx$ $-xa^tb$	$\delta_{a,b}(x)$ $=$ abx $+xba$ $-bax$ $-xab$	$\delta_{a,b}(x)$ $=$ ab^tx $+xb^ta$ $-ba^tx$ $-xa^tb$
(sums)	(sums) $(m = n)$	(sums)

HEY! IT IS NOT SO NONSENSICAL!

AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

AN TRIPLE SYSTEM IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH ONE BINARY OPERATION, CALLED ADDITION AND ONE TERNARY OPERATION CALLED TRIPLE MULTIPLICATION

ADDITION IS DENOTED BY

$$a + b$$

AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE

$$a + b = b + a, \quad (a + b) + c = a + (b + c)$$

TRIPLE MULTIPLICATION IS DENOTED

$$abc$$

AND IS REQUIRED TO BE LINEAR IN EACH VARIABLE

$$(a + b)cd = acd + bcd$$
$$a(b + c)d = abd + acd$$
$$ab(c + d) = abc + abd$$

SIMPLE BUT IMPORTANT EXAMPLES OF TRIPLE SYSTEMS CAN BE FORMED FROM ANY ALGEBRA

IF ab DENOTES THE ALGEBRA PRODUCT, JUST DEFINE A TRIPLE MULTIPLICATION TO BE $(ab)c$

LET'S SEE HOW THIS WORKS IN THE ALGEBRAS WE INTRODUCED IN PART I

$\mathcal{C}, \mathcal{D}$; $fgh = (fg)h$

$(M_n(\mathbf{R}), \times)$; $abc = a \times b \times c$ or $a \times b^t \times c$

$(M_n(\mathbf{R}), [,])$; $abc = [[a, b], c]$

$(M_n(\mathbf{R}), \circ)$; $abc = (a \circ b) \circ c$ (**NO GO!**)

A TRIPLE SYSTEM IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)

(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

IN THE TRIPLE CONTEXT THIS MEANS THE FOLLOWING

ASSOCIATIVE

$$ab(cde) = (abc)de = a(bcd)e$$

OR $ab(cde) = (abc)de = a(dcb)e$

COMMUTATIVE: $abc = cba$

THE TRIPLE SYSTEMS $\mathcal{C}$, $\mathcal{D}$ AND $(M_n(\mathbf{R}), \times)$ ARE EXAMPLES OF ASSOCIATIVE TRIPLE SYSTEMS.

$\mathcal{C}$ AND $\mathcal{D}$ ARE EXAMPLES OF COMMUTATIVE TRIPLE SYSTEMS.

AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

THE AXIOM WHICH CHARACTERIZES TRIPLE MATRIX MULTIPLICATION IS

$$(abc)de = ab(cde) = a(dcb)e$$

THESE ARE CALLED

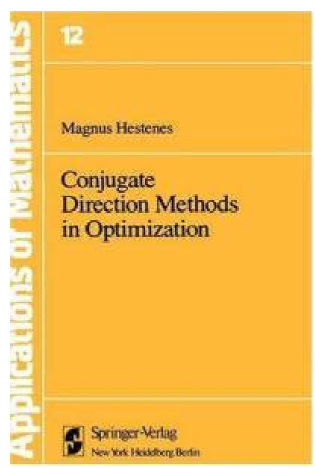
ASSOCIATIVE TRIPLE SYSTEMS

or

HESTENES ALGEBRAS

Magnus Hestenes (1906–1991)

Magnus Rudolph Hestenes was an American mathematician. Together with Cornelius Lanczos and Eduard Stiefel, he invented the conjugate gradient method.



THE AXIOMS WHICH CHARACTERIZE TRIPLE BRACKET MULTIPLICATION ARE

$$aab = 0$$

$$abc + bca + cab = 0$$

$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

THESE ARE CALLED
LIE TRIPLE SYSTEMS

(NATHAN JACOBSON, MAX KOECHER)

Max Koecher (1924–1990)

Max Koecher was a German mathematician. His main research area was the theory of Jordan algebras, where he introduced the KantorKoecherTits construction.

Nathan Jacobson (1910–1999)

THE AXIOMS WHICH CHARACTERIZE TRIPLE CIRCLE MULTIPLICATION ARE

$$abc = cba$$

$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$

THESE ARE CALLED

JORDAN TRIPLE SYSTEMS

Kurt Meyberg (living)

Ottmar Loos + Erhard Neher (both living)

YET ANOTHER SUMMARY

Table 4

TRIPLE SYSTEMS

associative triple systems

$$(abc)de = ab(cde) = a(dcb)e$$

Lie triple systems

$$aab = 0$$

$$abc + bca + cab = 0$$

$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

Jordan triple systems

$$abc = cba$$

$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$

FINAL THOUGHT

THE PHYSICAL UNIVERSE SEEMS TO BE ASSOCIATIVE.

HOW THEN, DO YOU EXPLAIN THE FOLLOWING PHENOMENON?

THEOREM 13[†]

(1985 FRIEDMAN-RUSSO)

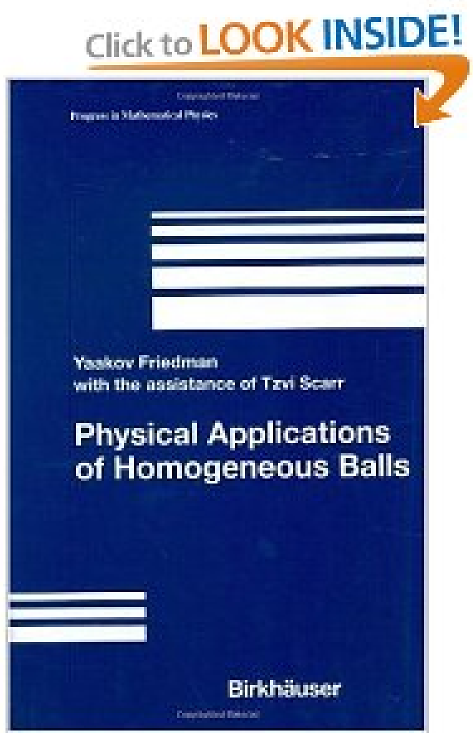
THE RANGE OF A CONTRACTIVE PROJECTION ON $M_n(\mathbf{R})$ (ASSOCIATIVE) IS A JORDAN TRIPLE SYSTEM[‡] (NON-ASSOCIATIVE).

[†]Theorems 11,12 are in part I

[‡]In some triple product

Yaakov Friedman (b. 1948)

Yaakov Friedman is director of research at Jerusalem College of Technology.



BEING MATHEMATICIANS, WE NATURALLY WONDERED ABOUT A CONVERSE:

THEOREM 14
(2008 NEAL-RUSSO)

A LINEAR SUBSPACE OF $M_n(\mathbf{R})$ WHICH IS A JORDAN TRIPLE SYSTEM IN SOME TRIPLE PRODUCT IS THE RANGE OF A CONTRACTIVE PROJECTION ON $M_n(\mathbf{R})$..

Matthew Neal (b. 1972)

Conference on Jordan Algebras
Oberwolfach, Germany 2000

GRADUS AD PARNASSUM
PART II—TRIPLE SYSTEMS

1. Prove Proposition 5
   (Use the notation $\langle abc \rangle$ for ab^tc)
2. Prove Proposition 6
   (Use the notation $[abc]$ for $[[a, b], c]$)
3. Prove Proposition 7
   (Use the notation $\{abc\}$ for $ab^tc + cb^ta$)
4. Show that $M_n(\mathbf{R})$ is a Lie triple system with respect to triple bracket multiplication. In other words, show that the three axioms for Lie triple systems in Table 4 are satisfied if abc denotes $[[a, b], c] = (ab - ba)c - c(ab - ba)$ (a, b and c denote matrices)
   (Use the notation $[abc]$ for $[[a, b], c]$)
5. Show that $M_{m,n}(\mathbf{R})$ is a Jordan triple system with respect to triple circle multiplication. In other words, show that the two axioms for Jordan triple systems in Table 4 are satisfied if abc denotes $ab^tc + cb^ta$ (a, b and c denote matrices)
   (Use the notation $\{abc\}$ for $ab^tc + cb^ta$)

6. Let us write $\delta_{a,b}$ for the linear process

$$\delta_{a,b}(x) = abx$$

in a Lie triple system. Show that $\delta_{a,b}$ is a derivation of the Lie triple system by using the axioms for Lie triple systems in Table 4.

(Use the notation $[abc]$ for the triple product in any Lie triple system, so that, for example, $\delta_{a,b}(x)$ is denoted by $[abx]$)

7. Let us write $\delta_{a,b}$ for the linear process

$$\delta_{a,b}(x) = abx - bax$$

in a Jordan triple system. Show that $\delta_{a,b}$ is a derivation of the Jordan triple system by using the axioms for Jordan triple systems in Table 4.

(Use the notation $\{abc\}$ for the triple product in any Jordan triple system, so that, for example, $\delta_{a,b}(x) = \{abx\} - \{bax\}$)

8. On the Jordan algebra $M_n(\mathbf{R})$ with the circle product $a \circ b = ab + ba$, define a triple product

$$\{abc\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b.$$

Show that $M_n(\mathbf{R})$ is a Jordan triple system with this triple product.
Hint: show that $\{abc\} = 2a \times b \times c + 2c \times b \times a$
9. On the vector space $M_n(\mathbf{R})$, define a triple product $\langle abc \rangle = abc$ (matrix multiplication without the transpose in the middle). Formulate the definition of a derivation of the resulting triple system, and state and prove a result corresponding to Proposition 5. Is this triple system associative?
10. In an associative algebra, define a triple product $\langle abc \rangle$ to be $(ab)c$. Show that the algebra becomes an associative triple system with this triple product.
11. In an associative triple system with triple product denoted $\langle abc \rangle$, define a binary product ab to be $\langle aub \rangle$, where u is a fixed element. Show that the triple system becomes an associative algebra with this product.

12. In a Lie algebra with product denoted by $[a, b]$, define a triple product $[abc]$ to be $[[a, b], c]$. Show that the Lie algebra becomes a Lie triple system with this triple product.
13. Let A be an algebra (associative, Lie, or Jordan; it doesn't matter). Show that the set $\mathcal{D} := \text{Der}\,(A)$ of all derivations of A is a Lie subalgebra of $\text{End}\,(A)$. That is $\mathcal{D}$ is a linear subspace of the vector space of linear transformations on A, and if $D_1, D_2 \in \mathcal{D}$, then $D_1 D_2 - D_2 D_1 \in \mathcal{D}$.
14. Let A be a triple system (associative, Lie, or Jordan; it doesn't matter). Show that the set $\mathcal{D} := \text{Der}\,(A)$ of derivations of A is a Lie subalgebra of $\text{End}\,(A)$. That is $\mathcal{D}$ is a linear subspace of the vector space of linear transformations on A, and if $D_1, D_2 \in \mathcal{D}$, then $D_1 D_2 - D_2 D_1 \in \mathcal{D}$.

END OF PART II

GRADUS AD PARNASSUM
PART III
ALGEBRAS AND TRIPLE SYSTEMS
(SNEAK PREVIEW)

1. In an arbitrary Jordan triple system, with triple product denoted by $\{abc\}$, define a triple product by

$$[abc] = \{abc\} - \{bac\}.$$

   Show that the Jordan triple system becomes a Lie triple system with this new triple product.

2. In an arbitrary associative triple system, with triple product denoted by $\langle abc\rangle$, define a triple product by

$$[xyz] = \langle xyz\rangle - \langle yxz\rangle - \langle zxy\rangle + \langle zyx\rangle.$$

   Show that the associative triple system becomes a Lie triple system with this new triple product.

3. In an arbitrary Jordan algebra, with product denoted by xy, define a triple product by $[xyz] = x(yz) - y(xz)$. Show that the Jordan algebra becomes a Lie triple system with this new triple product.
4. In an arbitrary Jordan triple system, with triple product denoted by $\{abc\}$, fix an element y and define a binary product by

$$ab = \{ayb\}.$$

Show that the Jordan triple system becomes a Jordan algebra with this (binary) product.
5. In an arbitrary Jordan algebra with multiplication denoted by ab, define a triple product

$$\{abc\} = (ab)c + (cb)a - (ac)b.$$

Show that the Jordan algebra becomes a Jordan triple system with this triple product. (cf. Problem 8)

6. Show that every Lie triple system, with triple product denoted $[abc]$ is a subspace of some Lie algebra, with product denoted $[a, b]$, such that $[abc] = [[a, b], c]$.
7. Find out what a semisimple associative algebra is and prove that every derivation of a finite dimensional semisimple associative algebra is inner, that is, of the form $x \mapsto ax - xa$ for some fixed a in the algebra.
8. Find out what a semisimple Lie algebra is and prove that every derivation of a finite dimensional semisimple Lie algebra is inner, that is, of the form $x \mapsto [a, x]$ for some fixed a in the algebra.
9. Find out what a semisimple Jordan algebra is and prove that every derivation of a finite dimensional semisimple Jordan algebra is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}(a_i(b_ix) - b_i(a_ix))$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the algebra.

10. Find out what a semisimple associative triple system is and prove that every derivation of a finite dimensional semisimple associative triple system is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}(\langle a_i b_i x\rangle - \langle b_i a_i x\rangle)$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the associative triple system.
11. Find out what a semisimple Lie triple system is and prove that every derivation of a finite dimensional semisimple Lie triple system is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}[a_i b_i x]$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the Lie triple system.
12. Find out what a semisimple Jordan triple system is and prove that every derivation of a finite dimensional semisimple Jordan triple system is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}(\{a_i b_i x\} - \{b_i a_i x\})$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the Jordan triple system.