DERIVATIONS
An introduction to non associative algebra
(or, Playing havoc with the product rule)

Part 7
Cohomology of nonassociative algebras.
A unified approach

Colloquium
Pacific Summer Unsolved Math Seminar
Fullerton College

Bernard Russo
University of California, Irvine

July 25, 2013
History of these lectures

- PART I  FEBRUARY 8, 2011  ALGEBRAS; DERIVATIONS

- PART II  JULY 21, 2011  TRIPLE SYSTEMS; DERIVATIONS

- PART III  FEBRUARY 28, 2012  MODULES; DERIVATIONS

- PART IV  JULY 26, 2012  COHOMOLOGY (ASSOCIATIVE ALGEBRAS)

- PART V  OCTOBER 25, 2012  THE SECOND COHOMOLOGY GROUP

- PART VI  MARCH 7, 2013  COHOMOLOGY (LIE ALGEBRAS)

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Outline

- Review of Algebras

- Derivations on matrix algebras

- Review of Cohomology

- First Cohomology Group (Derivations)

- Second Cohomology Group (Extensions) (will be done in Part 8, Fall 2013 date TBA)
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ADDITION IS DENOTED BY $a + b$ AND IS REQUIRED TO BE COMMUTATIVE $a + b = b + a$ AND ASSOCIATIVE $(a + b) + c = a + (b + c)$.

MULTIPLICATION IS DENOTED BY $ab$ AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION $(a + b)c = ac + bc$, $a(b + c) = ab + ac$.

AN ALGEBRA IS SAID TO BE ASSOCIATIVE (resp. COMMUTATIVE) IF THE MULTIPLICATION IS ASSOCIATIVE (resp. COMMUTATIVE) (RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE).
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Review of Algebras—Axiomatic approach

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<tr>
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</tr>
<tr>
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MATRIX ADDITION $A + B$

AND

MATRIX MULTIPLICATION $A \times B$

WHICH IS ASSOCIATIVE BUT NOT COMMUTATIVE.

DEFINITION

A DERIVATION ON $M_n(R)$ WITH RESPECT TO MATRIX MULTIPLICATION IS A LINEAR PROCESS $\delta$: $\delta(A + B) = \delta(A) + \delta(B)$

WHICH SATISFIES THE PRODUCT RULE

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The set $M_n(R)$ of $n$ by $n$ matrices is an algebra under
matrix addition $A + B$
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matrix multiplication $A \times B$
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**Definition**

A derivation on $M_n(R)$ with respect to matrix multiplication
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PROPOSITION

FIX A MATRIX $A$ in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$ 

THEN $\delta_A$ IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION (WHICH CAN BE NON-ZERO)

THEOREM (1942 Hochschild)

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS OF THE FORM $\delta_A$ FOR SOME $A$ IN $M_n(\mathbb{R})$. 
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THE BRACKET PRODUCT ON THE SET OF MATRICES

DEFINITION

THE BRACKET PRODUCT ON THE SET $M_n(\mathbb{R})$ OF MATRICES IS DEFINED BY

$$[X, Y] = X \times Y - Y \times X$$

THE SET $M_n(\mathbb{R})$ of $n$ by $n$ MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND BRACKET MULTIPLICATION, WHICH IS NOT ASSOCIATIVE AND NOT COMMUTATIVE.

DEFINITION

A DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS A LINEAR PROCESS $\delta$ WHICH SATISFIES THE PRODUCT RULE

$$\delta([A, B]) = [\delta(A), B] + [A, \delta(B)].$$
THE BRACKET PRODUCT ON THE SET OF MATRICES

**DEFINITION**

The bracket product on the set $M_n(\mathbb{R})$ of matrices is defined by

$$[X, Y] = X \times Y - Y \times X$$

The set $M_n(\mathbb{R})$ of $n$ by $n$ matrices is an algebra under matrix addition and bracket multiplication, which is not associative and not commutative.

**DEFINITION**

A derivation on $M_n(\mathbb{R})$ with respect to bracket multiplication is a linear process $\delta$ which satisfies the product rule

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FIX A MATRIX $A$ in $M_n(\mathbb{R})$ AND DEFINE

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THEN $\delta_A$ IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION

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EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS OF THE FORM $\delta_A$ FOR SOME $A$ IN $M_n(\mathbb{R})$. 
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### PROPOSITION

Fix a matrix $A$ in $M_n(\mathbb{R})$ and define

$$\delta_A(X) = [A, X] = A \times X - X \times A.$$  

Then $\delta_A$ is a derivation with respect to bracket multiplication.

### THEOREM

Every derivation on $M_n(\mathbb{R})$ with respect to bracket multiplication is of the form $\delta_A$ for some $A$ in $M_n(\mathbb{R})$. 
**THE CIRCLE PRODUCT ON THE SET OF MATRICES**

**DEFINITION**

**The Circle Product** on the set $M_n(\mathbb{R})$ of matrices is defined by

$$X \circ Y = \frac{(X \times Y + Y \times X)}{2}$$

The set $M_n(\mathbb{R})$ of $n$ by $n$ matrices is an algebra under matrix addition and circle multiplication, which is commutative but not associative.

**DEFINITION**

A derivation on $M_n(\mathbb{R})$ with respect to circle multiplication is a linear process $\delta$ which satisfies the product rule

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$
THE CIRCLE PRODUCT ON THE SET OF MATRICES

**DEFINITION**

THE **CIRCLE PRODUCT** ON THE SET $M_n(\mathbb{R})$ OF MATRICES IS DEFINED BY

$$X \circ Y = (X \times Y + Y \times X)/2$$

THE SET $M_n(\mathbb{R})$ of $n$ by $n$ MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND CIRCLE MULTIPLICATION, WHICH IS COMMUTATIVE BUT NOT ASSOCIATIVE.

**DEFINITION**

A DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS A LINEAR PROCESS $\delta$ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$
THE CIRCLE PRODUCT ON THE SET OF MATRICES

**DEFINITION**

THE CIRCLE PRODUCT ON THE SET $M_n(\mathbb{R})$ OF MATRICES IS DEFINED BY

$$X \circ Y = \frac{(X \times Y + Y \times X)}{2}$$

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THE CIRCLE PRODUCT ON THE SET OF MATRICES

**DEFINITION**

The **Circle Product** on the set $M_n(R)$ of matrices is defined by

$$X \circ Y = (X \times Y + Y \times X)/2$$

The set $M_n(R)$ of $n$ by $n$ matrices is an algebra under matrix addition and circle multiplication, which is commutative but not associative.

**DEFINITION**

A **derivation** on $M_n(R)$ with respect to circle multiplication is a linear process $\delta$ which satisfies the product rule

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$
PROPOSITION

FIX A MATRIX $A$ in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$  

THEN $\delta_A$ IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION.

THEOREM (1972-Sinclair)

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS OF THE FORM $\delta_A$ FOR SOME $A$ IN $M_n(\mathbb{R})$.

REMARK (1937-Jacobson)

THE ABOVE PROPOSITION AND THEOREM NEED TO BE MODIFIED FOR THE SUBALGEBRA (WITH RESPECT TO CIRCLE MULTIPLICATION) OF SYMMETRIC MATRICES.
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Table 2 \( M_n(R) \) (ALGEBRAS)

<table>
<thead>
<tr>
<th>matrix</th>
<th>bracket</th>
<th>circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ab = a \times b )</td>
<td>([a, b] = ab - ba)</td>
<td>( a \circ b = ab + ba)</td>
</tr>
<tr>
<td>Th. 2</td>
<td>Th.3</td>
<td>Th.4</td>
</tr>
<tr>
<td>( \delta_a(x) ) = ( ax - xa )</td>
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</table>
Review of Cohomology

**NOTATION**
n is a positive integer, \( n = 1, 2, \ldots \)
f is a function of \( n \) variables
\( F \) is a function of \( n + 1 \) variables (\( n + 2 \) variables?)
x_1, x_2, \ldots, x_{n+1} \) belong to an algebra \( A \)
f(\( y_1, \ldots, y_n \)) and \( F(\( y_1, \ldots, y_{n+1} \)) \) also belong to \( A \)

**The basic formula of homological algebra**

\[
F(x_1, \ldots, x_n, x_{n+1}) = \\
x_1 f(x_2, \ldots, x_{n+1}) \\
- f(x_1x_2, x_3, \ldots, x_{n+1}) \\
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F(x_1, \ldots, x_n, x_{n+1}) = x_1 f(x_2, \ldots, x_{n+1}) - f(x_1 x_2, x_3, \ldots, x_{n+1}) + f(x_1, x_2 x_3, x_4, \ldots, x_{n+1}) - \cdots \\
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**HIERARCHY**

\(x_1, x_2, \ldots, x_n\) are points (or vectors)

\(f\) and \(F\) are functions—they take points to points

\(T\), defined by \(T(f) = F\) is a transformation—takes functions to functions

points \(x_1, \ldots, x_{n+1}\) and \(f(y_1, \ldots, y_n)\) will belong to an algebra \(A\)

functions \(f\) will be either constant, linear or multilinear (hence so will \(F\))

transformation \(T\) is linear

**SHORT FORM OF THE FORMULA**

\[
(Tf)(x_1, \ldots, x_n, x_{n+1})
\]

\[
= x_1 f(x_2, \ldots, x_{n+1})
\]

\[
+ \sum_{j=1}^{n} (-1)^{j} f(x_1, \ldots, x_j x_{j+1}, \ldots, x_{n+1})
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\[
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**SHORT FORM OF THE FORMULA**

$$(Tf)(x_1, \ldots, x_n, x_{n+1})$$

$$= x_1 f(x_2, \ldots, x_{n+1})$$

$$+ \sum_{j=1}^{n} (-1)^j f(x_1, \ldots, x_j x_{j+1}, \ldots, x_{n+1})$$

$$+ (-1)^{n+1} f(x_1, \ldots, x_n) x_{n+1}$$
HIERARCHY

$x_1, x_2, \ldots, x_n$ are points (or vectors)

$f$ and $F$ are functions—they take points to points

$T$, defined by $T(f) = F$ is a transformation—takes functions to functions

points $x_1, \ldots, x_{n+1}$ and $f(y_1, \ldots, y_n)$ will belong to an algebra $A$

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FIRST CASES

\( n = 0 \)

If \( f \) is any constant function from \( A \) to \( A \), say, \( f(x) = b \) for all \( x \) in \( A \), where \( b \) is a fixed element of \( A \), we have, consistent with the basic formula, a linear function \( T_0(f) \):

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T_0(f)(x_1) = x_1 b - bx_1
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If \( f \) is a bilinear function from \( A \times A \) to \( A \), then \( T_2(f) \) is a trilinear function

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Kernel and Image of a linear transformation

$G : X \rightarrow Y$

Since $X$ and $Y$ are vector spaces, they are in particular, commutative groups.

**Kernel** of $G$ (also called nullspace of $G$) is

$\ker G = \{x \in X : G(x) = 0\}$

This is a subgroup of $X$

**Image** of $G$ is

$\text{im } G = \{G(x) : x \in X\}$

This is a subgroup of $Y$

$G = T_0$

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FIRST COHOMOLOGY GROUP

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**FIRST COHOMOLOGY GROUP**

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### Kernel and Image of a linear transformation

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### Example

Let $G = T_0$,  

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ker \( T_0 = \{ b \in A : xb - bx = 0 \text{ for all } x \in A \} \) (center of \( A \))

im \( T_0 \) = the set of all linear maps of \( A \) of the form \( x \mapsto xb - bx \),

in other words, the set of all inner derivations of \( A \)

ker \( T_0 \) is a subgroup of \( A \)

im \( T_0 \) is a subgroup of \( L(A) \)
Kernel and Image of a linear transformation

$G : X \to Y$

Since $X$ and $Y$ are vector spaces, they are in particular, commutative groups.

**Kernel** of $G$ (also called **nullspace** of $G$) is

$\ker G = \{ x \in X : G(x) = 0 \}$

This is a subgroup of $X$

**Image** of $G$ is

$\text{im } G = \{ G(x) : x \in X \}$

This is a subgroup of $Y$

\[
G = T_0 \\
X = A \text{ (the algebra)} \\
Y = L(A) \text{ (all linear transformations on } A) \\
T_0(f)(x_1) = x_1 b - b x_1 \\
\ker T_0 = \{ b \in A : xb - bx = 0 \text{ for all } x \in A \} \text{ (center of } A) \\
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$\text{ker } T_0$ is a subgroup of $A$

$\text{im } T_0$ is a subgroup of $L(A)$
\[ G = T_1 \]

\[ X = L(A) \] (linear transformations on \(A\))

\[ Y = L^2(A) \] (bilinear transformations on \(A \times A\))

\[ T_1(f)(x_1, x_2) = x_1 f(x_2) - f(x_1x_2) + f(x_1)x_2 \]

\( \ker T_1 = \{ f \in L(A) : T_1 f(x_1, x_2) = 0 \text{ for all } x_1, x_2 \in A \} \) = the set of all derivations of \(A\)

\( \text{im } T_1 \) = the set of all bilinear maps of \(A \times A\) of the form

\[ (x_1, x_2) \mapsto x_1 f(x_2) - f(x_1x_2) + f(x_1)x_2, \]

for some linear function \(f \in L(A)\).

\( \ker T_1 \) is a subgroup of \(L(A)\)

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\[ \begin{align*}
G & = T_1 \\
X & = L(A) \text{ (linear transformations on } A) \\
Y & = L^2(A) \text{ (bilinear transformations on } A \times A) \\
T_1(f)(x_1, x_2) & = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2 \\
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im T_1 & = \text{the set of all bilinear maps of } A \times A \text{ of the form} \\
& \quad (x_1, x_2) \mapsto x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2, \\
& \quad \text{for some linear function } f \in L(A). \\
\ker T_1 & \text{ is a subgroup of } L(A) \\
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\[
L^0(A) \xrightarrow{T_0} L(A) \xrightarrow{T_1} L^2(A) \xrightarrow{T_2} L^3(A) \cdot \cdot \cdot 
\]

**FACTS:**
\[
T_1 \circ T_0 = 0 \\
T_2 \circ T_1 = 0 \\
\cdot \cdot \cdot \\
T_{n+1} \circ T_n = 0 \\
\cdot \cdot \cdot 
\]

Therefore

\[
\text{im } T_n \subset \ker T_{n+1} \subset L^n(A) 
\]

and therefore

\[
\text{im } T_n \text{ is a subgroup of } \ker T_{n+1} 
\]
\[ L^0(A) \xrightarrow{T_0} L(A) \xrightarrow{T_1} L^2(A) \xrightarrow{T_2} L^3(A) \cdots \]

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\[ \vdots \]
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\( \ldots \)

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\( \ldots \)

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and therefore

\( \text{im } T_n \) is a subgroup of \( \ker T_{n+1} \)
\[ L^0(A) \xrightarrow{T_0} L(A) \xrightarrow{T_1} L^2(A) \xrightarrow{T_2} L^3(A) \cdots \]

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\( T_2 \circ T_1 = 0 \)
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\[ \vdots \]

Therefore

\[ \text{im } T_n \subseteq \text{ker } T_{n+1} \subseteq L^n(A) \]
and therefore

\[ \text{im } T_n \text{ is a subgroup of ker } T_{n+1} \]
\[ L^0(A) \xrightarrow{T_0} L(A) \xrightarrow{T_1} L^2(A) \xrightarrow{T_2} L^3(A) \cdots \]

**FACTS:**

\[ T_1 \circ T_0 = 0 \]
\[ T_2 \circ T_1 = 0 \]

\[ \cdots \]
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\[ \cdots \]

*Therefore*

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and therefore

\[ \text{im } T_n \text{ is a subgroup of } \ker T_{n+1} \]
$L^0(A) \xrightarrow{T_0} L(A) \xrightarrow{T_1} L^2(A) \xrightarrow{T_2} L^3(A) \cdots$

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$T_2 \circ T_1 = 0$

$\cdots$

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\( \text{im } T_0 \subset \ker T_1 \)
says
Every inner derivation is a derivation
\( \text{im } T_1 \subset \ker T_2 \)
says
for every linear map \( f \), the bilinear map \( F \) defined by
\[
F(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1)x_2
\]
satisfies the equation
\[
x_1 F(x_2, x_3) - F(x_1 x_2, x_3) + F(x_1, x_2 x_3) - F(x_1, x_2)x_3 = 0
\]
for every \( x_1, x_2, x_3 \in A \).
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im \, T_0 \subset \ker \, T_1 \\
\text{says} \\
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\]
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\text{im } T_0 \subset \ker T_1

does

Every inner derivation is a derivation

\text{im } T_1 \subset \ker T_2

does

for every linear map \( f \), the bilinear map \( F \) defined by

\[ F(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2 \]

does not satisfy the equation

\[ x_1 F(x_2, x_3) - F(x_1 x_2, x_3) + F(x_1, x_2 x_3) - F(x_1, x_2) x_3 = 0 \]

for every \( x_1, x_2, x_3 \in A \).
The cohomology groups of $A$ are defined as the quotient groups

$$H^n(A) = \frac{\ker T_n}{\text{im } T_{n-1}}$$

($n = 1, 2, \ldots$)

Thus

$$H^1(A) = \frac{\ker T_1}{\text{im } T_0} = \frac{\text{derivations}}{\text{inner derivations}}$$

$$H^2(A) = \frac{\ker T_2}{\text{im } T_1} = ?$$

The theorem that every derivation of $M_n(\mathbb{R})$ is inner (that is, of the form $\delta_a$ for some $a \in M_n(\mathbb{R})$) can now be restated as:

"the cohomology group $H^1(M_n(\mathbb{R}))$ is the trivial one element group"
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COHOMOLOGY OF JORDAN ALGEBRAS

\[ n = 0 \]

ASSOCIATIVE

\[ f : A \to A \text{ is a constant function, say } f(x) = b \text{ for all } x \]
\[ T_0(f) : A \to A \text{ is a linear function} \]
\[ T_0(f)(x_1) = x_1 b - bx_1 \]

LIE

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\[ T_0(f)(x_1) = [b, x_1] \]

JORDAN

\[ f \in A \times A \text{ is an ordered pair, say } f = (a, b) \]
\[ T_0(f) : A \to A \text{ is a linear function} \]
\[ T_0(f)(x_1) = a \circ (b \circ x_1) - b \circ (a \circ x_1) \]
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ASSOCIATIVE

\( f : A \rightarrow A \) is a constant function, say \( f(x) = b \) for all \( x \)

\( T_0(f) : A \rightarrow A \) is a linear function

\( T_0(f)(x_1) = x_1 b - bx_1 \)

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COHOMOLOGY OF JORDAN ALGEBRAS

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JORDAN

FALL 2013
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**FALL 2013**
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JORDAN

FALL 2013
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\[ T_2(f) : A \times A \times A \to A \text{ is a skew-symmetric trilinear function} \]

\[ T_2(f)(x_1, x_2, x_3) = [f(x_2, x_3), x_1] - [f(x_1, x_3), x_2] + [f(x_1, x_2), x_3] \]
\[ - f(x_3, [x_1, x_2]) + f(x_2, [x_1, x_3]) - f(x_1, [x_2, x_3]) \]

**JORDAN**

**FALL 2013**
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INTERPRETATION OF COHOMOLOGY GROUPS

FIRST COHOMOLOGY GROUP
DERIVATIONS (AND INNER DERIVATIONS)

SECOND COHOMOLOGY GROUP
EXTENSIONS (AND SPLIT EXTENSIONS)

VANISHING THEOREMS
FOR EACH CLASS OF ALGEBRAS (ASSOCIATIVE, LIE, JORDAN), UNDER WHAT CONDITIONS IS $H^n(A) = 0$, ESPECIALLY FOR $n = 1, 2$
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