DERIVATIONS
An introduction to non associative algebra (or, Playing havoc with the product rule)

Part 7
Cohomology of non associative algebras A unified approach

Colloquium
Pacific Summer Unsolved Math Seminar
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History of these lectures

- PART I FEBRUARY 8, 2011 **ALGEBRAS; DERIVATIONS**

- PART II JULY 21, 2011 **TRIPLE SYSTEMS; DERIVATIONS**

- PART III FEBRUARY 28, 2012 **MODULES; DERIVATIONS**

- PART IV JULY 26, 2012 **COHOMOLOGY (ASSOCIATIVE ALGEBRAS)**

- PART V OCTOBER 25, 2012 **THE SECOND COHOMOLOGY GROUP**

- PART VI MARCH 7, 2013 **COHOMOLOGY (LIE ALGEBRAS)**

(today)

- PART VII JULY 25, 2013 **COHOMOLOGY (JORDAN ALGEBRAS)**

Outline

- Review of Algebras
- Derivations on matrix algebras
- Review of Cohomology
- First Cohomology Group (Derivations)
- Second Cohomology Group (Extensions)
(will be done in Part 8, Fall 2013 date TBA)

Introduction

I will present a unified approach to 1 and 2 dimensional cohomology of the three main types of algebras we have been studying, namely, associative algebras, Lie algebras and Jordan algebras.

The relevant definitions and examples from earlier talks in the series will be reviewed beforehand.

Contact will be made, through this new perspective, with some of the results on derivations and on extensions which were presented earlier in the series.

Review of Algebras—Axiomatic approach

AN ALGEBRA IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH TWO BINARY OPERATIONS, CALLED ADDITION AND MULTIPLICATION

ADDITION IS DENOTED BY $a + b$ AND IS REQUIRED TO BE
COMMUTATIVE $\quad a + b = b + a$
AND ASSOCIATIVE $\quad (a + b) + c = a + (b + c)$

MULTIPLICATION IS DENOTED BY ab AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION
$(a + b)c = ac + bc, \quad a(b + c) = ab + ac$

AN ALGEBRA IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)
(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

Table 1 (FASHIONABLE) ALGEBRAS

commutative algebras $ab = ba$

associative algebras $a(bc) = (ab)c$

Lie algebras $a^2 = 0$, $(ab)c + (bc)a + (ca)b = 0$

Jordan algebras $ab = ba$, $a(a^2b) = a^2(ab)$

DERIVATIONS ON MATRIX ALGEBRAS

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER
MATRIX ADDITION $A + B$
AND
MATRIX MULTIPLICATION $A \times B$
WHICH IS ASSOCIATIVE BUT NOT COMMUTATIVE.

DEFINITION

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS A LINEAR PROCESS δ: $\delta(A+B) = \delta(A) + \delta(B)$
WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \times B) = \delta(A) \times B + A \times \delta(B).$$

PROPOSITION

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION (WHICH CAN BE NON-ZERO)

THEOREM (1942 Hochschild)

EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

THE BRACKET PRODUCT ON THE SET OF MATRICES

DEFINITION

THE **BRACKET PRODUCT** ON THE SET $M_n(\mathbf{R})$ OF MATRICES IS DEFINED BY

$$[X, Y] = X \times Y - Y \times X$$

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND BRACKET MULTIPLICATION, WHICH IS NOT ASSOCIATIVE AND NOT COMMUTATIVE.

DEFINITION

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta([A, B]) = [\delta(A), B] + [A, \delta(B)].$$

PROPOSITION

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = [A, X] = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION

THEOREM

EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

THE CIRCLE PRODUCT ON THE SET OF MATRICES

DEFINITION

THE **CIRCLE PRODUCT** ON THE SET $M_n(\mathbf{R})$ OF MATRICES IS DEFINED BY

$$X \circ Y = (X \times Y + Y \times X)/2$$

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND CIRCLE MULTIPLICATION, WHICH IS COMMUTATIVE BUT NOT ASSOCIATIVE.

DEFINITION

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$

PROPOSITION

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION

THEOREM (1972-Sinclair)

EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

REMARK (1937-Jacobson)
THE ABOVE PROPOSITION AND THEOREM NEED TO BE MODIFIED FOR THE SUBALGEBRA (WITH RESPECT TO CIRCLE MULTIPLICATION) OF SYMMETRIC MATRICES.

Table 2 $M_n(\mathbf{R})$ (ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a, b] = ab - ba$	$a \circ b = ab + ba$
Th. 2	Th.3	Th.4
$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$

Review of Cohomology

NOTATION

n is a positive integer, $n = 1, 2, \cdots$
f is a function of n variables
F is a function of $n+1$ variables ($n+2$ variables?)
$x_1, x_2, \cdots, x_{n+1}$ belong to an algebra A
$f(y_1, \ldots, y_n)$ and $F(y_1, \cdots, y_{n+1})$ also belong to A

The basic formula of homological algebra

$F(x_1, \ldots, x_n, x_{n+1}) =$
$x_1 f(x_2, \ldots, x_{n+1})$
$-f(x_1 x_2, x_3, \ldots, x_{n+1})$
$+f(x_1, x_2 x_3, x_4, \ldots, x_{n+1})$
$- \cdots$
$\pm f(x_1, x_2, \ldots, x_n x_{n+1})$
$\mp f(x_1, \ldots, x_n) x_{n+1}$

HIERARCHY

$x_1, x_2, \ldots, x_n$ are points (or vectors)
f and F are functions—they take points to points
T, defined by $T(f) = F$ is a transformation—takes functions to functions
points $x_1, \ldots, x_{n+1}$ and $f(y_1, \ldots, y_n)$ will belong to an algebra A
functions f will be either constant, linear or multilinear (hence so will F)
transformation T is linear

SHORT FORM OF THE FORMULA

$$(Tf)(x_1, \ldots, x_n, x_{n+1})$$

$$= x_1 f(x_2, \ldots, x_{n+1})$$

$$+ \sum_{j=1}^{n} (-1)^j f(x_1, \ldots, x_j x_{j+1}, \ldots, x_{n+1})$$

$$+(-1)^{n+1} f(x_1, \ldots, x_n) x_{n+1}$$

FIRST CASES

$\underline{n=0}$
If f is any constant function from A to A, say, $f(x)=b$ for all x in A, where b is a fixed element of A, we have, consistent with the basic formula, a linear function $T_0(f)$:

$$T_0(f)(x_1) = x_1 b - b x_1$$

$\underline{n=1}$
If f is a linear function from A to A, then $T_1(f)$ is a bilinear function

$$T_1(f)(x_1,x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$$

$\underline{n=2}$
If f is a bilinear function from $A \times A$ to A, then $T_2(f)$ is a trilinear function

$$T_2(f)(x_1,x_2,x_3) =$$

$$x_1 f(x_2,x_3) - f(x_1 x_2, x_3) + f(x_1, x_2 x_3) - f(x_1,x_2) x_3$$

FIRST COHOMOLOGY GROUP

Kernel and Image of a linear transformation

$G : X \to Y$

Since X and Y are vector spaces, they are in particular, commutative groups.

Kernel of G (also called **nullspace** of G) is

$\ker G = \{x \in X : G(x) = 0\}$

This is a subgroup of X

Image of G is

$\text{im}\, G = \{G(x) : x \in X\}$

This is a subgroup of Y

$\underline{G = T_0}$

$X = A$ (the algebra)

$Y = L(A)$ (all linear transformations on A)

$T_0(f)(x_1) = x_1 b - b x_1$

$\ker T_0 = \{b \in A : xb - bx = 0 \text{ for all } x \in A\}$ (center of A)

$\text{im}\, T_0 =$ the set of all linear maps of A of the form $x \mapsto xb - bx$,

in other words, the set of all inner derivations of A

$\ker T_0$ is a subgroup of A

$\text{im}\, T_0$ is a subgroup of $L(A)$

$\underline{G = T_1}$
$X = L(A)$ (linear transformations on A)
$Y = L^2(A)$ (bilinear transformations on $A \times A$)
$T_1(f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$
$\ker T_1 = \{f \in L(A) : T_1 f(x_1, x_2) = 0 \text{ for all } x_1, x_2 \in A\} =$ the set of all derivations of A
$\text{im } T_1 =$ the set of all bilinear maps of $A \times A$ of the form

$$(x_1, x_2) \mapsto x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2,$$

for some linear function $f \in L(A)$.
$\ker T_1$ is a subgroup of $L(A)$
$\text{im } T_1$ is a subgroup of $L^2(A)$

$L^0(A) \xrightarrow{T_0} L(A) \xrightarrow{T_1} L^2(A) \xrightarrow{T_2} L^3(A) \cdots$
FACTS: $T_1 \circ T_0 = 0$
$T_2 \circ T_1 = 0$
$\ldots$
$T_{n+1} \circ T_n = 0$
$\ldots$

Therefore

im $T_n \subset$ ker $T_{n+1} \subset L^n(A)$
and therefore
im T_n is a subgroup of ker T_{n+1}

im $T_0 \subset$ ker T_1
says
Every inner derivation is a derivation
im $T_1 \subset$ ker T_2
says
for every linear map f, the bilinear map F defined by

$$F(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$$

satisfies the equation

$$x_1 F(x_2, x_3) - F(x_1 x_2, x_3) +$$
$$F(x_1, x_2 x_3) - F(x_1, x_2) x_3 = 0$$

for every $x_1, x_2, x_3 \in A$.

The cohomology groups of A are defined as the quotient groups

$$H^n(A) = \frac{\ker T_n}{\text{im } T_{n-1}}$$

$(n = 1, 2, \ldots)$

Thus

$$H^1(A) = \frac{\ker T_1}{\text{im } T_0} = \frac{\text{derivations}}{\text{inner derivations}}$$

$$H^2(A) = \frac{\ker T_2}{\text{im } T_1} = \quad ?$$

The theorem that every derivation of $M_n(\mathbf{R})$ is inner (that is, of the form δ_a for some $a \in M_n(\mathbf{R})$) can now be restated as:
"the cohomology group $H^1(M_n(\mathbf{R}))$ is the trivial one element group"

COHOMOLOGY OF LIE ALGEBRAS

GO TO PART 6, PAGES 57-74
http://staffwww.fullcoll.edu/dclahane/colloquium/russotalk030713.pdf

COHOMOLOGY OF JORDAN ALGEBRAS

$$n = 0$$

ASSOCIATIVE

$f : A \to A$ is a constant function, say $f(x) = b$ for all x
$T_0(f) : A \to A$ is a linear function
$T_0(f)(x_1) = x_1 b - b x_1$

LIE

$f : A \to A$ is a constant function, say $f(x) = b$ for all x
$T_0(f) : A \to A$ is a linear function
$T_0(f)(x_1) = [b, x_1]$

JORDAN

$f \in A \times A$ is an ordered pair, say $f = (a, b)$
$T_0(f) : A \to A$ is a linear function
$T_0(f)(x_1) = a \circ (b \circ x_1) - b \circ (a \circ x_1)$

$n = 1$

ASSOCIATIVE

$f : A \to A$ is a linear function

$T_1(f) : A \times A \to A$ is a bilinear function

$T_1(f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$

LIE

$f : A \to A$ is a linear function

$T_1(f) : A \times A \to A$ is a skew-symmetric bilinear function

$T_1(f)(x_1, x_2) = -[f(x_2), x_1] + [f(x_1), x_2] - f([x_1, x_2])$

JORDAN

$f : A \to A$ is a linear function

$T_1(f) : A \times A \to A$ is a symmetric bilinear function

$T_1(f)(x_1, x_2) = x_1 \circ f(x_2) - f(x_1 \circ x_2) + f(x_1) \circ x_2$

$n = 2$

ASSOCIATIVE

$f : A \times A \to A$ is a bilinear function
$T_2(f) : A \times A \times A \to A$ is a trilinear function
$T_2(f)(x_1, x_2, x_3) = x_1 f(x_2, x_3) - f(x_1x_2, x_3) - f(x_1, x_2x_3) + f(x_1, x_2)x_3$

LIE

$f : A \times A \to A$ is a skew-symmetric bilinear function
$T_2(f) : A \times A \times A \to A$ is a skew-symmetric trilinear function

$$\begin{aligned} T_2(f)(x_1, x_2, x_3) &= [f(x_2, x_3), x_1] - [f(x_1, x_3), x_2] + [f(x_1, x_2), x_3] \\ &- f(x_3, [x_1, x_2]) + f(x_2, [x_1, x_3]) - f(x_1, [x_2, x_3]) \end{aligned}$$

JORDAN

FALL 2013

CLOSING

INTERPRETATION OF COHOMOLOGY GROUPS

FIRST COHOMOLOGY GROUP
DERIVATIONS (AND INNER DERIVATIONS)

SECOND COHOMOLOGY GROUP
EXTENSIONS (AND SPLIT EXTENSIONS)

VANISHING THEOREMS

FOR EACH CLASS OF ALGEBRAS (ASSOCIATIVE, LIE, JORDAN), UNDER WHAT CONDITIONS IS $H^n(A) = 0$, ESPECIALLY FOR $n = 1, 2$