DERIVATIONS
An introduction to non associative algebra
(or, Playing havoc with the product rule)

Series 2—Part 1
A remarkable connection between Jordan algebras and Lie algebras

Colloquium
Fullerton College

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Series 1

- PART I FEBRUARY 8, 2011 **ALGEBRAS; DERIVATIONS**
- PART II JULY 21, 2011 **TRIPLE SYSTEMS; DERIVATIONS**
- PART III FEBRUARY 28, 2012 **MODULES; DERIVATIONS**
- PART IV JULY 26, 2012 **COHOMOLOGY (ASSOCIATIVE ALGEBRAS)**
- PART V OCTOBER 25, 2012 **THE SECOND COHOMOLOGY GROUP**
- PART VI MARCH 7, 2013 **COHOMOLOGY (LIE ALGEBRAS)**
- PART VII JULY 25, 2013 **COHOMOLOGY (JORDAN ALGEBRAS)**
- PART VIII SEPTEMBER 17, 2013 **VANISHING THEOREMS IN DIMENSIONS 1 AND 2 (ASSOCIATIVE ALGEBRAS)**
- PART IX FEBRUARY 18, 2014 **VANISHING THEOREMS IN DIMENSIONS 1 AND 2 (JORDAN ALGEBRAS)**

Series 2

- PART I JULY 24, 2014 **THE REMARKABLE CONNECTION BETWEEN JORDAN ALGEBRAS AND LIE ALGEBRAS**

Outline

- Review of (matrix) Algebras and derivations on them
  (From series 1, part 1)

- Two theorems relating different types of derivations

- Review of (matrix) triple systems and derivations on them
  (From series 1, part 2)

- Two theorems on embedding triple systems into Lie algebras

Only the first two items were covered in the talk. The second two items will be covered in the next lecture (Fall 2014), after a possible revision. However all four items are included in this file.

Introduction

I shall review the definitions of Lie algebra and Jordan algebra (from my talk on February 8, 2011) and show the remarkable connection between them as reflected in the following two (conflicting) quotations:

(Kevin McCrimmon 1978)

"If you open up a Lie algebra and look inside, 9 times out of 10 you will find a Jordan algebra which makes it tick."

(Max Koecher 1967)

"There are no Jordan algebras, there are only Lie algebras."

The relevant definitions and examples from earlier talks in the series will be reviewed beforehand.

Review of Algebras—Axiomatic approach

AN ALGEBRA IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH TWO BINARY OPERATIONS, CALLED ADDITION AND MULTIPLICATION

ADDITION IS DENOTED BY $a+b$ AND IS REQUIRED TO BE COMMUTATIVE $\quad a+b=b+a$
AND ASSOCIATIVE $\quad (a+b)+c=a+(b+c)$

MULTIPLICATION IS DENOTED BY ab AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION
$(a+b)c=ac+bc, \quad a(b+c)=ab+ac$

AN ALGEBRA IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)
(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

Table 1 (FASHIONABLE) ALGEBRAS

commutative algebras $ab = ba$

associative algebras $a(bc) = (ab)c$

Lie algebras $a^2 = 0$, $(ab)c + (bc)a + (ca)b = 0$

Jordan algebras $ab = ba$, $a(a^2b) = a^2(ab)$

DERIVATIONS ON MATRIX ALGEBRAS

THE SET $M_n(\mathbb{R})$ of n by n MATRICES IS AN ALGEBRA UNDER
MATRIX ADDITION $A + B$
AND **MATRIX MULTIPLICATION** $A \times B$
WHICH IS ASSOCIATIVE BUT NOT COMMUTATIVE.

For the Record:

$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$ $\qquad$ $[a_{ij}] \times [b_{ij}] = [\sum_{k=1}^{n} a_{ik} b_{kj}]$

DEFINITION

A DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS A LINEAR PROCESS δ: $\quad \delta(A + B) = \delta(A) + \delta(B)$
WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \times B) = \delta(A) \times B + A \times \delta(B)$$

.

PROPOSITION

FIX A MATRIX A in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION (WHICH ARE CALLED **INNER DERIVATIONS**)

THEOREM (Noether,Wedderburn,Hochschild,Jacobson, Kaplansky,Kadison,Sakai)

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS INNER, THAT IS, OF THE FORM δ_A FOR SOME A IN $M_n(\mathbb{R})$.

We gave a proof of this theorem for $n = 2$ in part 8 of series 1.

THE BRACKET PRODUCT ON THE SET OF MATRICES

DEFINITION

THE **BRACKET PRODUCT** ON THE SET $M_n(\mathbb{R})$ OF MATRICES IS DEFINED BY

$$[X, Y] = X \times Y - Y \times X$$

THE SET $M_n(\mathbb{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND BRACKET MULTIPLICATION, WHICH IS NOT ASSOCIATIVE AND NOT COMMUTATIVE.

DEFINITION

A <u>DERIVATION</u> ON $M_n(\mathbb{R})$ WITH <u>RESPECT TO BRACKET MULTIPLICATION</u> IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta([A, B]) = [\delta(A), B] + [A, \delta(B)].$$

PROPOSITION

FIX A MATRIX A in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = [A, X] = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION (STILL CALLED **INNER DERIVATION**).

THEOREM

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS INNER, THAT IS, OF THE FORM δ_A FOR SOME A IN $M_n(\mathbb{R})$. [a]

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[a] Full disclosure: this is actually not true. Check that the map $X \mapsto$ (trace of X)I is a derivation which is not inner (I is the identity matrix). The correct statement is that every derivation of a semisimple finite dimensional Lie algebra is inner. $M_n(\mathbb{R})$ is a semisimple associative algebra under matrix multiplication, a semisimple Jordan algebra under circle multiplication, but not a semisimple Lie algebra under bracket multiplication. Please ignore this footnote until you find out what semisimple means in each context

THE CIRCLE PRODUCT ON THE SET OF MATRICES

DEFINITION

THE **CIRCLE PRODUCT** ON THE SET $M_n(\mathbb{R})$ OF MATRICES IS DEFINED BY

$$X \circ Y = (X \times Y + Y \times X)/2$$

THE SET $M_n(\mathbb{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND CIRCLE MULTIPLICATION, WHICH IS COMMUTATIVE BUT NOT ASSOCIATIVE.

DEFINITION

A DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$

PROPOSITION

FIX A MATRIX A in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION (ALSO CALLED AN INNER DERIVATION IN THIS CONTEXT[a])

[a] However, see the following remark. Also see some of the exercises (Dr. Gradus Ad Parnassum) in part 1 of these lectures

THEOREM (1972-Sinclair)

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS INNER, THAT IS, OF THE FORM δ_A FOR SOME A IN $M_n(\mathbb{R})$.

REMARK (1937-Jacobson)
THE ABOVE PROPOSITION AND THEOREM NEED TO BE MODIFIED FOR THE SUBALGEBRA (WITH RESPECT TO CIRCLE MULTIPLICATION) OF SYMMETRIC MATRICES, FOR EXAMPLE.

Table 2 $M_n(\mathbb{R})$ (ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a, b] = ab - ba$	$a \circ b = ab + ba$
Associative	Lie	Jordan
$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$
	or trace$(x)I$	

$H^1(M_2, M_2) = 0$

Matrix units

Let $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

LEMMA

- $E_{11} + E_{22} = I$
- $E_{ij}^t = E_{ji}$
- $E_{ij}E_{kl} = \delta_{kl}E_{il}$

THEOREM 1

Let $\delta : M_2 \to M_2$ be a derivation: δ is linear and $\delta(AB) = A\delta(B) + \delta(A)B$. Then there exists a matrix K such that $\delta(X) = XK - KX$ for X in M_2.

COROLLARY

$H^1(M_2, M_2) = 0$

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LEMMA

- $E_{11} + E_{22} = I$
- $E_{ij}^t = E_{ji}$
- $E_{ij}E_{kl} = \delta_{kl}E_{il}$

THEOREM 1

Let $\delta : M_2 \to M_2$ be a derivation: δ is linear and $\delta(AB) = A\delta(B) + \delta(A)B$. Then there exists a matrix K such that $\delta(X) = XK - KX$ for X in M_2.

COROLLARY

$H^1(M_2, M_2) = 0$

PROOF OF THEOREM 1

$$
\begin{aligned}
0 &= \delta(1) = \delta(E_{11} + E_{22}) = \delta(E_{11}) + \delta(E_{22}) \\
&= \delta(E_{11}E_{11}) + \delta(E_{21}E_{12}) \\
&= E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} + E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} \\
&= E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) + \delta(E_{11})E_{11} + \delta(E_{21})E_{12}.
\end{aligned}
$$

Let $K = E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) = -\delta(E_{11})E_{11} - \delta(E_{21})E_{12}$. Then

- $KE_{11} = -\delta(E_{11})E_{11} \quad , \quad E_{11}K = E_{11}\delta(E_{11})$
- $KE_{12} = -\delta(E_{11})E_{12} \quad , \quad E_{12}K = E_{11}\delta(E_{12})$
- $KE_{21} = -\delta(E_{21})E_{11} \quad , \quad E_{21}K = E_{21}\delta(E_{11})$
- $KE_{22} = -\delta(E_{21})E_{12} \quad , \quad E_{22}K = E_{21}\delta(E_{12})$

- $E_{11}K - KE_{11} = E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} = \delta(E_{11}E_{11}) = \delta(E_{11})$
- $E_{12}K - KE_{12} = E_{11}\delta(E_{12}) + \delta(E_{11})E_{12} = \delta(E_{11}E_{12}) = \delta(E_{12})$
- $E_{21}K - KE_{21} = E_{21}\delta(E_{11}) + \delta(E_{21})E_{11} = \delta(E_{21}E_{11}) = \delta(E_{21})$
- $E_{22}K - KE_{22} = E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} = \delta(E_{21}E_{12}) = \delta(E_{22})$ Q.E.D.

PROOF OF THEOREM 1

$$\begin{aligned} 0 &= \delta(1) = \delta(E_{11}+E_{22}) = \delta(E_{11}) + \delta(E_{22}) \\ &= \delta(E_{11}E_{11}) + \delta(E_{21}E_{12}) \\ &= E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} + E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} \\ &= E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) + \delta(E_{11})E_{11} + \delta(E_{21})E_{12}. \end{aligned}$$

Let $K = E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) = -\delta(E_{11})E_{11} - \delta(E_{21})E_{12}$. Then

- $KE_{11} = -\delta(E_{11})E_{11}$, $E_{11}K = E_{11}\delta(E_{11})$
- $KE_{12} = -\delta(E_{11})E_{12}$, $E_{12}K = E_{11}\delta(E_{12})$
- $KE_{21} = -\delta(E_{21})E_{11}$, $E_{21}K = E_{21}\delta(E_{11})$
- $KE_{22} = -\delta(E_{21})E_{12}$, $E_{22}K = E_{21}\delta(E_{12})$

- $E_{11}K - KE_{11} = E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} = \delta(E_{11}E_{11}) = \delta(E_{11})$
- $E_{12}K - KE_{12} = E_{11}\delta(E_{12}) + \delta(E_{11})E_{12} = \delta(E_{11}E_{12}) = \delta(E_{12})$
- $E_{21}K - KE_{21} = E_{21}\delta(E_{11}) + \delta(E_{21})E_{11} = \delta(E_{21}E_{11}) = \delta(E_{21})$
- $E_{22}K - KE_{22} = E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} = \delta(E_{21}E_{12}) = \delta(E_{22})$ Q.E.D.

PROOF OF THEOREM 1

$$\begin{aligned} 0 &= \delta(1) = \delta(E_{11}+E_{22}) = \delta(E_{11}) + \delta(E_{22}) \\ &= \delta(E_{11}E_{11}) + \delta(E_{21}E_{12}) \\ &= E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} + E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} \\ &= E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) + \delta(E_{11})E_{11} + \delta(E_{21})E_{12}. \end{aligned}$$

Let $K = E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) = -\delta(E_{11})E_{11} - \delta(E_{21})E_{12}$. Then

- $KE_{11} = -\delta(E_{11})E_{11}$, $E_{11}K = E_{11}\delta(E_{11})$
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- $KE_{21} = -\delta(E_{21})E_{11}$, $E_{21}K = E_{21}\delta(E_{11})$
- $KE_{22} = -\delta(E_{21})E_{12}$, $E_{22}K = E_{21}\delta(E_{12})$

- $E_{11}K - KE_{11} = E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} = \delta(E_{11}E_{11}) = \delta(E_{11})$
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- $E_{21}K - KE_{21} = E_{21}\delta(E_{11}) + \delta(E_{21})E_{11} = \delta(E_{21}E_{11}) = \delta(E_{21})$
- $E_{22}K - KE_{22} = E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} = \delta(E_{21}E_{12}) = \delta(E_{22})$ Q.E.D.

Return to Axiomatic approach

If A is an associative algebra, we can make it into a Lie algebra, denoted A^- by defining $[a, b] = ab - ba$ and into a Jordan algebra, denoted A^+ by defining $a \circ b = (ab + ba)/2$.

Examples: $A = \mathcal{C} \Rightarrow A^- = A$ with all products $[a, b] = 0$, $A^+ = A$ with $a \circ b = ab$

NOT VERY INTERESTING

$A = M_n(\mathbb{R})$ is more interesting!

Types of derivations

Derivation

$\delta(ab) = a\delta(b) + \delta(a)b$

Lie derivation

$\delta([a, b]) = [a, \delta(b)] + [\delta(a), b]$

Jordan derivation

$\delta(a \circ b) = a \circ \delta(b) + \delta(a) \circ b$

Trivial Exercise

A derivation is also a Lie derivation and a Jordan derivation.

Converses

These are the two theorems relating different types of derivations

Theorem 1

A Jordan derivation is a derivation ($A = M_n(\mathbb{R})$)

Example

There is a Lie derivation which is not a derivation ($A = M_n(\mathbb{R})$, namely $\delta(x) = trace(x)I$

Theorem 2

Every Lie derivation is the sum of a derivation and a linear operator of the above form ($A = M_n(\mathbb{R})$)

Diagonals

Let $d = \frac{1}{n}\sum_{i,j} e_{ij} \otimes e_{ji}$. Then d is a **diagonal** for $M_n(\mathbb{C})$, that is,
$\pi(d) = 1$ and $a \cdot d = d \cdot a$ for all $a \in M_n(\mathbb{C})$, where
$\pi(x \otimes y) = xy$, $a \cdot x \otimes y = (ax) \otimes y$ and $x \otimes y \cdot a = x \otimes (ya)$.
Explicitly, $\pi(d) = \frac{1}{n}\sum_{i,j} e_{ij}e_{ji} = 1$, $\frac{1}{n}\sum_{i,j}(ae_{ij}) \otimes e_{ji} = \frac{1}{n}\sum_{i,j} e_{ij} \otimes (e_{ji}a)$

The symmetric nature of d implies $\frac{1}{n}\sum_{i,j}(e_{ij}a) \otimes e_{ji} = \frac{1}{n}\sum_{i,j} e_{ij} \otimes (ae_{ji})$
For any linear transformation D, apply $1 \otimes D$ and then π, to get
$\frac{1}{n}\sum_{i,j}(ae_{ij})D(e_{ji}) = \frac{1}{n}\sum_{i,j} e_{ij}D(e_{ji}a)$ and $\frac{1}{n}\sum_{i,j}(e_{ij}a)D(e_{ji}) = \frac{1}{n}\sum_{i,j} e_{ij}D(ae_{ji})$

Proof of Theorem 1 (Barry Johnson 1996)

Let $D : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a Jordan derivation

Define $x = \frac{1}{n}\sum_{i,j} e_{ij}De_{ji}$. Then

$ax = \frac{1}{n}\sum_{i,j} ae_{ij}De_{ji} = \frac{1}{n}\sum_{i,j} e_{ij}D(e_{ji}a)$

$D(e_{ji}a) + D(ae_{ji}) = e_{ji}Da + (Da)e_{ji} + (De_{ji})a + aDe_{ji}$

$ax = \frac{1}{n}\sum_{i,j} e_{ij}[e_{ji}Da + D(e_{ji})a + (Da)e_{ji} + aDe_{ji} - D(ae_{ji})]$

$ax = Da + xa + \Delta(a) + 0$, where

$\Delta(a) = \frac{1}{n}\sum_{i,j} e_{ij}(Da)e_{ji}$ (recall that $\frac{1}{n}\sum_{i,j}(e_{ij}a)D(e_{ji}) = \frac{1}{n}\sum_{i,j} e_{ij}D(ae_{ji})$)

Δ is a Jordan derivation with $a\Delta(b) = \Delta(b)a$, that is,
$\frac{1}{n}\sum_{i,j} ae_{ij}(Db)e_{ji} = \frac{1}{n}\sum_{i,j} e_{ij}(Db)e_{ji}a$
Proof: Apply $R_{Db} \otimes 1$, then π to $\frac{1}{n}\sum_{i,j}(ae_{ij}) \otimes e_{ji} = \frac{1}{n}\sum_{i,j} e_{ij} \otimes (e_{ji}a)$

Start over with D replaced by Δ
$x_0 = \frac{1}{n}\sum_{i,j} e_{ij}\Delta(e_{ji})$
$ax_0 = \Delta a + x_0 a + \frac{1}{n}\sum_{i,j} e_{ij}\Delta(a)(e_{ji}) = 2\Delta a + x_0 a$

$\Delta a = \frac{1}{2}(ax_0 - x_0 a)$
$Da = ax - xa - \Delta a = a(x - \frac{1}{2}x_0) - (x - \frac{1}{2}x_0)a$
is an inner associative derivation. Q.E.D.

The proof of the Lie derivation result goes along the same lines

Proof of Theorem 2 (Barry Johnson 1996)

Let $D : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a Lie derivation

Define $x = \frac{1}{n}\sum_{i,j} e_{ij} De_{ji}$. Then

$ax = \frac{1}{n}\sum_{i,j} ae_{ij} De_{ji} = \frac{1}{n}\sum_{i,j} e_{ij} D(e_{ji}a)$

$D(e_{ji}a) - D(ae_{ji}) = e_{ji}Da - (Da)e_{ji} + (De_{ji})a - aDe_{ji}$

$ax = \frac{1}{n}\sum_{i,j} e_{ij}[e_{ji}Da + D(e_{ji})a - (Da)e_{ji} - aDe_{ji} + D(ae_{ji})]$

$ax = Da + xa - \Delta(a) + 0$, where

$\Delta(a) = \frac{1}{n}\sum_{i,j} e_{ij}(Da)e_{ji}$ (recall that $\frac{1}{n}\sum_{i,j}(e_{ij}a)D(e_{ji}) = \frac{1}{n}\sum_{i,j} e_{ij}D(ae_{ji})$)

Δ is a Lie derivation with $a\Delta(b) = \Delta(b)a$, that is, so it vanishes on commutators and so $Da = ax - xa + \Delta(a)$ as required.

DERIVATIONS ON RECTANGULAR MATRICES

MULTIPLICATION DOES NOT MAKE SENSE ON $M_{m,n}(\mathbb{R})$ if $m \neq n$.
NOT TO WORRY!
WE CAN FORM A TRIPLE PRODUCT $X \times Y^t \times Z$
(TRIPLE MATRIX MULTIPLICATION)

COMMUTATIVE AND ASSOCIATIVE DON'T MAKE SENSE HERE. RIGHT?
WRONG!!

$$(X \times Y^t \times Z) \times A^t \times B = X \times Y^t \times (Z \times A^t \times B)$$

DEFINITION

A DERIVATION ON $M_{m,n}(\mathbb{R})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION
IS A LINEAR PROCESS δ WHICH SATISFIES THE (TRIPLE) PRODUCT RULE
$\delta(A \times B^t \times C) =$
$\delta(A) \times B^t \times C + A \times \delta(B)^t \times C + A \times B^t \times \delta(C)$

PROPOSITION

FOR TWO MATRICES A, B in $M_{m,n}(\mathbb{R})$,
DEFINE $\delta_{A,B}(X) =$

$$A \times B^t \times X + X \times B^t \times A - B \times A^t \times X - X \times A^t \times B$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION

THEOREM

EVERY DERIVATION ON $M_{m,n}(\mathbb{R})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

REMARK

THESE RESULTS HOLD TRUE AND ARE OF INTEREST FOR THE CASE $m = n$.

TRIPLE BRACKET MULTIPLICATION

LET'S GO BACK FOR A MOMENT TO SQUARE MATRICES AND THE BRACKET MULTIPLICATION.
MOTIVATED BY THE LAST REMARK, WE DEFINE THE TRIPLE BRACKET MULTIPLICATION TO BE $[[X, Y], Z]$

DEFINITION

A <u>DERIVATION</u> ON $M_n(\mathbb{R})$ WITH <u>RESPECT TO TRIPLE BRACKET MULTIPLICATION</u>
IS A LINEAR PROCESS δ WHICH SATISFIES THE TRIPLE PRODUCT RULE
$\delta([[A, B], C]) = [[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$

PROPOSITION

FIX TWO MATRICES A, B IN $M_n(\mathbb{R})$ AND DEFINE $\delta_{A,B}(X) = [[A, B], X]$ THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION.

THEOREM

EVERY DERIVATION OF $M_n(\mathbb{R})$ WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION IS A SUM OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

TRIPLE CIRCLE MULTIPLICATION

LET'S RETURN TO RECTANGULAR MATRICES AND FORM THE TRIPLE CIRCLE MULTIPLICATION
$(A \times B^t \times C + C \times B^t \times A)/2$
For sanity's sake, let us write this as

$$\{A, B, C\} = (A \times B^t \times C + C \times B^t \times A)/2$$

DEFINITION

A DERIVATION ON $M_{m,n}(\mathbb{R})$ WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION
IS A LINEAR PROCESS δ WHICH SATISFIES THE TRIPLE PRODUCT RULE
$\delta(\{A, B, C\}) = \{\delta(A), B, C\} + \{A, \delta(B), C\} + \{A, B, \delta(C)\}$

PROPOSITION

FIX TWO MATRICES A, B IN $M_{m,n}(\mathbb{R})$ AND DEFINE

$$\delta_{A,B}(X) = \{A, B, X\} - \{B, A, X\}$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION.

THEOREM

EVERY DERIVATION OF $M_{m,n}(\mathbb{R})$ WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

IT IS TIME FOR SUMMARY OF THE PRECEDING

Table 3 $M_{m,n}(\mathbb{R})$ (TRIPLE SYSTEMS)

triple matrix	triple bracket	triple circle
ab^tc	$[[a,b],c]$	$ab^tc + cb^ta$
Th. 8	Th.9	Th.10
$\delta_{a,b}(x)$ $=$ ab^tx $+xb^ta$ $-ba^tx$ $-xa^tb$	$\delta_{a,b}(x)$ $=$ abx $+xba$ $-bax$ $-xab$	$\delta_{a,b}(x)$ $=$ ab^tx $+xb^ta$ $-ba^tx$ $-xa^tb$
(sums)	(sums) $(m = n)$	(sums)

Table 2 $M_n(\mathbb{R})$ (ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a, b] = ab - ba$	$a \circ b = ab + ba$
$\delta_a(x)$ $=$ $ax - xa$	$\delta_a(x)$ $=$ $ax - xa$	$\delta_a(x)$ $=$ $ax - xa$

Table 3 $M_{m,n}(\mathbb{R})$ (TRIPLE SYSTEMS)

triple matrix	triple bracket	triple circle
$< abc >= ab^t c$	$[abc] = [[a, b], c]$	$\{abc\} = ab^t c + cb^t a$
$\delta_{a,b}(x)$ $=$ $ab^t x$ $+xb^t a$ $-ba^t x$ $-xa^t b$	$\delta_{a,b}(x)$ $=$ abx $+xba$ $-bax$ $-xab$	$\delta_{a,b}(x)$ $=$ $ab^t x$ $+xb^t a$ $-ba^t x$ $-xa^t b$
(sums)	(sums) $(m = n)$	(sums)

AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

AN TRIPLE SYSTEM IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH ONE BINARY OPERATION, CALLED ADDITION AND ONE TERNARY OPERATION CALLED TRIPLE MULTIPLICATION
ADDITION IS DENOTED BY
$a + b$
AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE
$a + b = b + a, \quad (a + b) + c = a + (b + c)$
TRIPLE MULTIPLICATION IS DENOTED
abc
AND IS REQUIRED TO BE LINEAR IN EACH VARIABLE

$(a + b)cd = acd + bcd$
$a(b + c)d = abd + acd$
$ab(c + d) = abc + abd$

SIMPLE BUT IMPORTANT EXAMPLES OF TRIPLE SYSTEMS CAN BE FORMED FROM ANY ALGEBRA

IF ab DENOTES THE ALGEBRA PRODUCT, JUST DEFINE A TRIPLE MULTIPLICATION TO BE $(ab)c$

LET'S SEE HOW THIS WORKS IN THE ALGEBRAS WE INTRODUCED IN PART I

$\mathcal{C}, \mathcal{D}$; $fgh = (fg)h$

$(M_n(\mathbb{R}), \times)$; $abc = a \times b \times c$ or $a \times b^t \times c$

$(M_n(\mathbb{R}), [,])$; $abc = [[a, b], c]$

$(M_n(\mathbb{R}), \circ)$; $abc = (a \circ b) \circ c$ (**NO GO!**)

A TRIPLE SYSTEM IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE) (RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE) IN THE TRIPLE CONTEXT THIS MEANS THE FOLLOWING

ASSOCIATIVE

$ab(cde) = (abc)de = a(bcd)e$

OR $ab(cde) = (abc)de = a(dcb)e$

COMMUTATIVE: $abc = cba$

THE TRIPLE SYSTEMS $\mathcal{C}$, $\mathcal{D}$ AND $(M_n(\mathbb{R}), \times)$ ARE EXAMPLES OF ASSOCIATIVE TRIPLE SYSTEMS.

$\mathcal{C}$ AND $\mathcal{D}$ ARE EXAMPLES OF COMMUTATIVE TRIPLE SYSTEMS.

THE AXIOM WHICH CHARACTERIZES TRIPLE MATRIX MULTIPLICATION IS

$(abc)de = ab(cde) = a(dcb)e$

THESE ARE CALLED
ASSOCIATIVE TRIPLE SYSTEMS
or
HESTENES ALGEBRAS

THE AXIOMS WHICH CHARACTERIZE TRIPLE BRACKET MULTIPLICATION ARE
$aab = 0$
$abc + bca + cab = 0$
$de(abc) = (dea)bc + a(deb)c + ab(dec)$
THESE ARE CALLED
LIE TRIPLE SYSTEMS
(NATHAN JACOBSON, MAX KOECHER)

THE AXIOMS WHICH CHARACTERIZE TRIPLE CIRCLE MULTIPLICATION ARE

$abc = cba$

$de(abc) = (dea)bc - a(edb)c + ab(dec)$

THESE ARE CALLED
JORDAN TRIPLE SYSTEMS

YET ANOTHER SUMMARY

Table 4 TRIPLE SYSTEMS

associative triple systems
$(abc)de = ab(cde) = a(dcb)e$
Lie triple systems
$aab = 0$
$abc + bca + cab = 0$
$de(abc) = (dea)bc + a(deb)c + ab(dec)$
Jordan triple systems
$abc = cba$
$de(abc) = (dea)bc - a(edb)c + ab(dec)$

Table 1 (FASHIONABLE) ALGEBRAS

commutative algebras $ab = ba$
associative algebras $a(bc) = (ab)c$
Lie algebras $a^2 = 0$, $(ab)c + (bc)a + (ca)b = 0$
Jordan algebras $ab = ba,\ a(a^2b) = a^2(ab)$

Theorem

Every derivation of a finite dimensional semisimple Lie triple system F is a sum of derivations of the form $\delta_{A,B}$, for some A's and B's in the triple system. These derivations are called inner derivations and their set is denoted Inder F.

Proof

Let F be a finite dimensional semisimple Lie triple system (over a field of characteristic 0) and suppose that D is a derivation of F. Let L be the Lie algebra $(\text{Inder}\, F) \oplus F$ with product

$$[(H_1, x_1), (H_2, x_2)] = ([H_1, H_2] + L(x_1, x_2), H_1x_2 - H_2x_1).$$

A derivation of L is defined by $\delta(H \oplus a) = [D, H] \oplus Da$. Together with the definition of semisimple Lie triple system, it is proved in the lecture notes of Meyberg (Lectures on algebras and triple systems 1972) that F semisimple implies L semisimple. Thus there exists $U = H_1 \oplus a_1 \in L$ such that $\delta(X) = [U, X]$ for all $X \in L$. Then $0 \oplus Da = \delta(0 \oplus a) = [H_1 + a_1, 0 \oplus a] = L(a_1, a) \oplus H_1 a$ so $L(a_1, a) = 0$ and $D = H_1 \in \text{Inder}\, F$.

Theorem

Every derivation of a finite dimensional semisimple Jordan triple system is inner.

The TKK construction (Tits-Kantor-Koecher)

Let V be a Jordan triple and let $\mathcal{L}(V)$ be its TKK Lie algebra .
$\mathcal{L}(V) = V \oplus V_0 \oplus V$ and the Lie product is given by

$$[(x,h,y),(u,k,v)] = (hu - kx,\, [h,k] + x\,\square\, v - u\,\square\, y,\, k^\natural y - h^\natural v).$$

Here, $a\,\square\, b$ is the left multiplication operator $x \mapsto \{abx\}$ (also called the box operator), $V_0 = \mathrm{span}\{V\,\square\, V\}$ is a Lie subalgebra of $\mathcal{L}(V)$ and for $h = \sum_i a_i\,\square\, b_i \in V_0$, the map $h^\natural : V \to V$ is defined by

$$h^\natural = \sum_i b_i\,\square\, a_i.$$

We can show the correspondence of derivations $\delta : V \to V$ and $D : \mathcal{L}(V) \to \mathcal{L}(V)$ for Jordan triple V and its TKK Lie algebra $\mathcal{L}(V)$.

Let $\theta : \mathcal{L}(V) \to \mathcal{L}(V)$ be the main involution $\theta(x \oplus h \oplus y) = y \oplus -h^{\natural} \oplus x$

Lemma

Let $\delta : V \to V$ be a derivation of a Jordan triple V, with TKK Lie algebra $(\mathcal{L}(V), \theta)$. Then there is a derivation $D : \mathcal{L}(V) \to \mathcal{L}(V)$ satisfying

$$D(V) \subset V \quad \text{and} \quad D\theta = \theta D.$$

Proof

Given $a, b \in V$, we define

$$\begin{aligned} D(a,0,0) &= (\delta a,0,0) \\ D(0,0,b) &= (0,0,\delta b) \\ D(0,a\Box\, b,0) &= (0,\, \delta a\Box\, b + a\Box\, \delta b,\, 0) \end{aligned}$$

and extend D linearly on $\mathcal{L}(V)$. Then D is a derivation of $\mathcal{L}(V)$ and evidently, $D(V) \subset V$.

It is readily seen that $D\theta = \theta D$, since

$$\begin{aligned} D\theta(0,a\Box\, b,0) &= D(0,-b\Box\, a,0) \\ &= (0,\, -\delta b\Box\, a - b\Box\, \delta a,\, 0) \\ &= \theta(0,\, \delta a\Box\, b + a\Box\, \delta b,\, 0) \\ &= \theta D(0,a\Box\, b,0). \end{aligned}$$

Lemma

Let V be a Jordan triple with TKK Lie algebra $(\mathcal{L}(V), \theta)$. Given a derivation $D : \mathcal{L}(V) \to \mathcal{L}(V)$ satisfying $D(V) \subset V$ and $D\theta = \theta D$, the restriction $D|_V : V \to V$ is a triple derivation.

Theorem

Let V be a Jordan triple with TKK Lie algebra $(\mathcal{L}(V), \theta)$. There is a one-one correspondence between the triple derivations of V and the Lie derivations $D : \mathcal{L}(V) \to \mathcal{L}(V)$ satisfying $D(V) \subset V$ and $D\theta = \theta D$.

Lemma

Let V be a Jordan triple with TKK Lie algebra $(\mathcal{L}(V), \theta)$. Let $D : \mathcal{L}(V) \to \mathcal{L}(V)$ be a Lie inner derivation such that $D(V) \subset V$. Then the restriction $D|_V$ is a triple inner derivation of V.

Corollary

Let δ be a derivation of a finite dimensional semisimple Jordan triple V. Then δ is a triple inner derivation of V.

Proof

The TKK Lie algebra $\mathcal{L}(V)$ is semisimple. Hence the result follows from the Lie result and the Lemma.

The proof of the last Lemma is instructive. The steps are as follows.

1. $D(x,k,y) = [(x,k,y),(a,h,b)]$ for some $(a,h,b) \in \mathcal{L}(V)$
2. $D(x,0,0) = [(x,0,0),(a,h,b)] = (-h(x), x \,\square\, b, 0)$
3. $\delta(x) = -h(x) = -\sum_i \alpha_i \,\square\, \beta_i(x)$
4. $D(0,0,y) = [(0,0,y),(a,h,b)] = (0, -a \,\square\, y, h^\natural(y))$
5. $\delta(x) = -h^\natural(x) = \sum_i \beta_i \,\square\, \alpha_i(x)$
6. $\delta(x) = \frac{1}{2}\sum_i(\beta_i \,\square\, \alpha_i - \alpha_i \,\square\, \beta_i)(x)$

APPENDIX: SOLVING LINEAR EQUATIONS

Let h be a Hochschild 1-cocycle, that is, a linear map

$$h : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

satisfying

$$h(ab) - ah(b) - h(a)b = 0. \tag{1}$$

We are going to show that there is an element $x \in M_n(\mathbb{C})$ such that

$$h(a) = xa - ax. \tag{2}$$

It is enough to prove that (2) holds with $a \in \{e_{ij}\}$, that is

$$h(e_{ij}) = xe_{ij} - e_{ij}x. \tag{3}$$

The element x can be defined by determining its coordinates x_{ij} from other information, that is

$$x = \sum_{p,q} x_{pq} e_{pq}. \tag{4}$$

Define γ_{ijpq} by

$$h(e_{ij}) = \sum_{p,q} \gamma_{ijpq} e_{pq}. \tag{5}$$

Then (3), (4) and (5) lead to the system of linear vector equations

$$\sum_{p,q} \gamma_{ijpq} e_{pq} = \sum_{p,q} \delta_{qi} x_{pq} e_{pj} - \sum_{p,q} \delta_{jp} x_{pq} e_{iq}. \tag{6}$$

with n^2 unknowns x_{ij}. Then any solution of (6) proves (3) and hence (2).

It is a special case of a theorem of Hochschild (1945) that such an x exists. We shall find x by solving the equations. We shall now restrict to $n = 2$.
From (6) we have for fixed i, j,

$$\begin{aligned} &\gamma_{ij11} e_{11} + \gamma_{ij12} e_{12} + \gamma_{ij21} e_{21} + \gamma_{ij22} e_{22} \\ &= \delta_{1i} x_{11} e_{1j} + \delta_{2i} x_{12} e_{1j} + \delta_{1i} x_{21} e_{2j} + \delta_{2i} x_{22} e_{2j} \\ &-\delta_{j1} x_{11} e_{i1} - \delta_{j1} x_{12} e_{i2} - \delta_{j2} x_{21} e_{i1} - \delta_{j2} x_{22} e_{i2}. \end{aligned} \tag{7}$$

With (i, j) successively equal to $(1, 1), (1, 2), (2, 1), (2, 2)$ we obtain the following 16 equations.

Coefficient of e_{11}

$$1 \quad (i,j) = (1,1) \; \gamma_{1111} = x_{11} - x_{11} = 0$$
$$2 \quad (i,j) = (1,2) \; \gamma_{1211} = -x_{21}$$
$$3 \quad (i,j) = (2,1) \; \gamma_{2111} = x_{12}$$
$$4 \quad (i,j) = (2,2) \; \gamma_{2211} = 0$$

Coefficient of e_{12}

$$5 \quad (i,j) = (1,1) \; \gamma_{1112} = -x_{12}$$
$$6 \quad (i,j) = (1,2) \; \gamma_{1212} = x_{11} - x_{22}$$
$$7 \quad (i,j) = (2,1) \; \gamma_{2112} = 0$$
$$8 \quad (i,j) = (2,2) \; \gamma_{2212} = x_{12}$$

Coefficient of e_{21}

$$9 \quad (i,j) = (1,1) \; \gamma_{1121} = x_{21}$$
$$10 \quad (i,j) = (1,2) \; \gamma_{1221} = 0$$
$$11 \quad (i,j) = (2,1) \; \gamma_{2121} = x_{22} - x_{11}$$
$$12 \quad (i,j) = (2,2) \; \gamma_{2221} = -x_{21}$$

Coefficient of e_{22}

$$13 \ (i,j) = (1,1) \ \gamma_{1122} = 0$$
$$14 \ (i,j) = (1,2) \ \gamma_{1222} = x_{21}$$
$$15 \ (i,j) = (2,1) \ \gamma_{2122} = -x_{12}$$
$$16 \ (i,j) = (2,2) \ \gamma_{2222} = x_{22} - x_{22} = 0$$

These 16 equations have the following formal solution, which is subject to the validity of the relations between the γ_{ijpq}, which is verified below.

- $x_{21} = -\gamma_{1211} = \gamma_{1121} = \gamma_{1222} = -\gamma_{2221}$
- $x_{12} = \gamma_{2111} = -\gamma_{1112} = \gamma_{2212} = -\gamma_{2122}$
- $x_{22} = -\gamma_{1221} = -\gamma_{2221}$
- $x_{11} = x_{22} + \gamma_{1212} = x_{22} - \gamma_{2121}$
- $0 = \gamma_{1111} = \gamma_{2211} = \gamma_{1122} = \gamma_{2222} = \gamma_{2112} = \gamma_{1221}$

Thus all solutions are given by

$$x = \begin{bmatrix} \gamma_{1212} + x_{22} & 0 \\ \gamma_{1121} & x_{22} \end{bmatrix} \quad (x_{22} \in \mathbb{C})$$

proving the theorem.

We now find the relations between the γs. From (1) and (5), we have for fixed $i, j, k.l$,

$$\begin{aligned} 0 = \delta_{jk}(\gamma_{il11}e_{11} + \gamma_{il12}e_{12} + \gamma_{il21}e_{21} + \gamma_{il22}e_{22}) \\ - \delta_{j1}\gamma_{kl11}e_{i1} - \delta_{j1}\gamma_{kl12}e_{i2} - \delta_{j2}\gamma_{kl21}e_{i1} - \delta_{j2}\gamma_{kl22}e_{i2} \\ -\delta_{1k}\gamma_{ij11}e_{1l} - \delta_{2k}\gamma_{ij12}e_{1l} - \delta_{1k}\gamma_{ij21}e_{2l} - \delta_{2k}\gamma_{ij22}e_{2l}. \end{aligned} \tag{8}$$

With (i, j) successively equal to $(1, 1), (1, 2), (2, 1), (2, 2)$ and (k, l) successively equal to $(1, 1), (1, 2), (2, 1), (2, 2)$ we obtain the following 64 equations giving the necessary conditions on the quantities γ_{ijkl}

Coefficient of e_{11}

eq	ij	kl	$= 0$
1	11	11	$-\gamma_{1111}$
2	11	12	$\gamma_{1211} - \gamma_{2111}$
3	11	21	$\gamma_{2111} - \gamma_{1112}$
4	11	22	$-\gamma_{2211}$
5	12	11	$-\gamma_{1111} - \gamma_{1121} - \gamma_{1211}$
6	12	12	$-\gamma_{1211} - \gamma_{1221}$
7	12	21	$\gamma_{1111} - \gamma_{2121} - \gamma_{1212}$
8	12	22	$\gamma_{1211} - \gamma_{2211} - \gamma_{2221}$
9	21	11	$\gamma_{1111} - \gamma_{2111}$
10	21	12	γ_{2211}
11	21	21	$-\gamma_{2112}$
12	21	22	0
13	22	11	$-\gamma_{2211}$
14	22	12	0
15	22	21	$\gamma_{2111} - \gamma_{2212}$
16	22	22	$\gamma_{2211} - \gamma_{2222}$

Coefficient of e_{12}

eq	ij	kl	$=0$
17	11	11	$\gamma_{1112}-\gamma_{1112}=0$
18	11	12	$\gamma_{1212}-\gamma_{1212}-\gamma_{1111}$
19	11	21	$-\gamma_{2112}$
20	11	22	$-\gamma_{2212}-\gamma_{1112}$
21	12	11	$-\gamma_{1122}$
22	12	12	$-\gamma_{1221}-\gamma_{1222}-\gamma_{1211}$
23	12	21	$\gamma_{1112}-\gamma_{2122}$
24	12	22	$\gamma_{1212}-\gamma_{2222}-\gamma_{1212}$
25	21	11	γ_{2112}
26	21	12	$\gamma_{2212}-\gamma_{2111}$
27	21	21	0
28	21	22	$-\gamma_{2112}$
29	22	11	0
30	22	12	$-\gamma_{2211}$
31	22	21	$\gamma_{2112}-\gamma_{2212}$
32	22	22	$\gamma_{2212}-\gamma_{2212}=0$

Coefficient of e_{21}

eq	ij	kl	$=0$
33	11	11	$\gamma_{1121}-\gamma_{1121}$
34	11	12	γ_{1221}
35	11	21	$-\gamma_{1122}$
36	11	22	0
37	12	11	$-\gamma_{1221}$
38	12	12	0
39	12	21	$\gamma_{1121}-\gamma_{1222}$
40	12	22	γ_{1221}
41	21	11	$\gamma_{2121}-\gamma_{1111}-\gamma_{2121}=-\gamma_{1111}$
42	21	12	$\gamma_{2221}-\gamma_{1211}$
43	21	21	$-\gamma_{2111}-\gamma_{2122}$
44	21	22	γ_{2211}
45	22	11	$-\gamma_{1121}-\gamma_{2221}$
46	22	12	$-\gamma_{1221}$
47	22	21	$\gamma_{2121}-\gamma_{2121}-\gamma_{2222}=-\gamma_{2222}$
48	22	22	$\gamma_{2221}-\gamma_{2221}=0$

Coefficient of e_{22}

eq	ij	kl	$= 0$
491	11	11	$-\gamma_{1122}$
50	11	12	$\gamma_{1222} - \gamma_{1121}$
51	11	21	0
52	11	22	0
53	12	11	$-\gamma_{1221}$
54	12	12	0
55	12	21	γ_{1122}
56	12	22	γ_{1222}
57	21	11	$\gamma_{2122} - \gamma_{2112}$
58	21	12	$\gamma_{2222} - \gamma_{2212} - \gamma_{2121}$
59	21	21	$-\gamma_{2112}$
60	21	22	$-\gamma_{2212} - \gamma_{2122}$
61	22	11	$-\gamma_{1122}$
62	22	12	$-\gamma_{1222} - \gamma_{2221}$
63	22	21	$\gamma_{2122} - \gamma_{2122} = 0$
64	22	22	$\gamma_{2222} - \gamma_{2222} - \gamma_{2222} = -3\gamma_{2222}$