DERIVATIONS
An introduction to non associative algebra
(or, Playing havoc with the product rule)

Series 2—Part 1
A remarkable connection between Jordan algebras and Lie algebras

Colloquium
Fullerton College

Bernard Russo

University of California, Irvine

July 24, 2014
Series 1

- PART I  FEBRUARY 8, 2011  ALGEBRAS; DERIVATIONS
- PART II  JULY 21, 2011  TRIPLE SYSTEMS; DERIVATIONS
- PART III  FEBRUARY 28, 2012  MODULES; DERIVATIONS
- PART IV  JULY 26, 2012  COHOMOLOGY (ASSOCIATIVE ALGEBRAS)
- PART V  OCTOBER 25, 2012  THE SECOND COHOMOLOGY GROUP
- PART VI  MARCH 7, 2013  COHOMOLOGY (LIE ALGEBRAS)
- PART VII  JULY 25, 2013  COHOMOLOGY (JORDAN ALGEBRAS)
- PART VIII  SEPTEMBER 17, 2013  VANISHING THEOREMS IN DIMENSIONS 1 AND 2 (ASSOCIATIVE ALGEBRAS)
- PART IX  FEBRUARY 18, 2014  VANISHING THEOREMS IN DIMENSIONS 1 AND 2 (JORDAN ALGEBRAS)

Series 2

- PART I  JULY 24, 2014  THE REMARKABLE CONNECTION BETWEEN JORDAN ALGEBRAS AND LIE ALGEBRAS
Outline

• Review of (matrix) Algebras and derivations on them  
  (From series 1, part 1)

• Two theorems relating different types of derivations

• Review of (matrix) triple systems and derivations on them  
  (From series 1, part 2)

• Two theorems on embedding triple systems into Lie algebras

Only the first two items were covered in the talk. The second two items will be covered in the next lecture (Fall 2014), after a possible revision. However all four items are included in this file.
Introduction

I shall review the definitions of Lie algebra and Jordan algebra (from my talk on February 8, 2011) and show the remarkable connection between them as reflected in the following two (conflicting) quotations:

*(Kevin McCrimmon 1978)*

”If you open up a Lie algebra and look inside, 9 times out of 10 you will find a Jordan algebra which makes it tick.”

*(Max Koecher 1967)*

”There are no Jordan algebras, there are only Lie algebras.”

The relevant definitions and examples from earlier talks in the series will be reviewed beforehand.
AN ALGEBRA IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH TWO BINARY OPERATIONS, CALLED ADDITION AND MULTIPLICATION.

ADDITION IS DENOTED BY \( a + b \) AND IS REQUIRED TO BE COMMUTATIVE \( a + b = b + a \) AND ASSOCIATIVE \( (a + b) + c = a + (b + c) \).

MULTIPLICATION IS DENOTED BY \( ab \) AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION \( (a + b)c = ac + bc, \quad a(b + c) = ab + ac \).

AN ALGEBRA IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE MULTIPLICATION IS ASSOCIATIVE (RESP. COMMUTATIVE) (RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE).
<table>
<thead>
<tr>
<th>Table 1  (FASHIONABLE) ALGEBRAS</th>
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</thead>
<tbody>
<tr>
<td><strong>commutative algebras</strong>  ( ab = ba )</td>
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<tr>
<td><strong>associative algebras</strong>  ( a(bc) = (ab)c )</td>
</tr>
<tr>
<td><strong>Lie algebras</strong>  ( a^2 = 0 ), ( (ab)c + (bc)a + (ca)b = 0 )</td>
</tr>
<tr>
<td><strong>Jordan algebras</strong>  ( ab = ba ), ( a(a^2b) = a^2(ab) )</td>
</tr>
</tbody>
</table>
DERIVATIONS ON MATRIX ALGEBRAS

THE SET $M_n(\mathbb{R})$ of $n$ by $n$ MATRICES IS AN ALGEBRA UNDER

MATRIX ADDITION $A + B$

AND MATRIX MULTIPLICATION $A \times B$

WHICH IS ASSOCIATIVE BUT NOT COMMUTATIVE.

For the Record:

$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

$[a_{ij}] \times [b_{ij}] = [\sum_{k=1}^{n} a_{ik} b_{kj}]$

DEFINITION

A DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS A LINEAR PROCESS $\delta$:

$\delta(A + B) = \delta(A) + \delta(B)$

WHICH SATISFIES THE PRODUCT RULE

$\delta(A \times B) = \delta(A) \times B + A \times \delta(B)$.
PROPOSITION

FIX A MATRIX $A$ in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$ 

THEN $\delta_A$ IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION (WHICH ARE CALLED INNER DERIVATIONS)

THEOREM (Noether, Wedderburn, Hochschild, Jacobson, Kaplansky, Kadison, Sakai)

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS INNER, THAT IS, OF THE FORM $\delta_A$ FOR SOME $A$ IN $M_n(\mathbb{R})$.

We gave a proof of this theorem for $n = 2$ in part 8 of series 1.
THE BRACKET PRODUCT ON THE SET OF MATRICES

DEFINITION

THE BRACKET PRODUCT ON THE SET $M_n(\mathbb{R})$ OF MATRICES IS DEFINED BY

$$[X, Y] = X \times Y - Y \times X$$

THE SET $M_n(\mathbb{R})$ of $n$ by $n$ MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND BRACKET MULTIPLICATION, WHICH IS NOT ASSOCIATIVE AND NOT COMMUTATIVE.

DEFINITION

A DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS A LINEAR PROCESS $\delta$ WHICH SATISFIES THE PRODUCT RULE

$$\delta([A, B]) = [\delta(A), B] + [A, \delta(B)].$$
**PROPOSITION**

**FIX** A MATRIX \( A \) in \( M_n(\mathbb{R}) \) **AND DEFINE**

\[
\delta_A(X) = [A, X] = A \times X - X \times A.
\]

**THEN** \( \delta_A \) IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION (STILL CALLED **INNER DERIVATION**).

---

**THEOREM**

EVERY DERIVATION ON \( M_n(\mathbb{R}) \) WITH RESPECT TO BRACKET MULTIPLICATION IS INNER, THAT IS, OF THE FORM \( \delta_A \) FOR SOME \( A \) IN \( M_n(\mathbb{R}) \). \(^a\)

---

\(^a\)Full disclosure: this is actually not true. Check that the map \( X \mapsto \text{trace of } X \cdot I \) is a derivation which is not inner (\( I \) is the identity matrix). The correct statement is that every derivation of a **semisimple** finite dimensional Lie algebra is inner. \( M_n(\mathbb{R}) \) is a semisimple associative algebra under matrix multiplication, a semisimple Jordan algebra under circle multiplication, but not a semisimple Lie algebra under bracket multiplication. Please ignore this footnote until you find out what semisimple means in each context.
THE CIRCLE PRODUCT ON THE SET OF MATRICES

**Definition**

The **circle product** on the set $M_n(\mathbb{R})$ of matrices is defined by

$$X \circ Y = (X \times Y + Y \times X)/2$$

The set $M_n(\mathbb{R})$ of $n$ by $n$ matrices is an algebra under matrix addition and circle multiplication, which is commutative but not associative.

**Definition**

A derivation on $M_n(\mathbb{R})$ with respect to circle multiplication is a linear process $\delta$ which satisfies the product rule

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$
**PROPOSITION**

FIX A MATRIX $A$ in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$  

THEN $\delta_A$ IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION (ALSO CALLED AN INNER DERIVATION IN THIS CONTEXT\(^a\))

\(^a\)However, see the following remark. Also see some of the exercises (Dr. Gradus Ad Parnassum) in part 1 of these lectures

---

**THEOREM (1972-Sinclair)**

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS INNER, THAT IS, OF THE FORM $\delta_A$ FOR SOME $A$ IN $M_n(\mathbb{R})$.

---

**REMARK (1937-Jacobson)**

THE ABOVE PROPOSITION AND THEOREM NEED TO BE MODIFIED FOR THE SUBALGEBRA (WITH RESPECT TO CIRCLE MULTIPLICATION) OF SYMMETRIC MATRICES, FOR EXAMPLE.
### Table 2  $M_n(\mathbb{R})$ (ALGEBRAS)

<table>
<thead>
<tr>
<th></th>
<th>matrix</th>
<th>bracket</th>
<th>circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ab = a \times b$</td>
<td>$[a, b] = ab - ba$</td>
<td>$a \circ b = ab + ba$</td>
<td></td>
</tr>
<tr>
<td>Associative</td>
<td>Lie</td>
<td>Jordan</td>
<td></td>
</tr>
<tr>
<td>$\delta_a(x)$</td>
<td>$\delta_a(x)$</td>
<td>$\delta_a(x)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$ax - xa$</td>
<td>$ax - xa$</td>
<td>$ax - xa$</td>
</tr>
<tr>
<td>or trace($x$)$I$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
$H^1(M_2, M_2) = 0$

**Matrix units**

Let $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

**LEMMA**

- $E_{11} + E_{22} = I$
- $E_{ij}^t = E_{ji}$
- $E_{ij}E_{kl} = \delta_{kl}E_{il}$

**THEOREM 1**

Let $\delta : M_2 \rightarrow M_2$ be a derivation: $\delta$ is linear and $\delta(AB) = A\delta(B) + \delta(A)B$. Then there exists a matrix $K$ such that $\delta(X) = XK - KX$ for $X$ in $M_2$.

**COROLLARY**

$H^1(M_2, M_2) = 0$
\[ H^1(M_2, M_2) = 0 \]

**Matrix units**

Let \( E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \)

**LEMMA**

- \( E_{11} + E_{22} = I \)
- \( E_{ij}^\dagger = E_{ji} \)
- \( E_{ij}E_{kl} = \delta_{kl}E_{il} \)

**THEOREM 1**

Let \( \delta : M_2 \to M_2 \) be a derivation: \( \delta \) is linear and \( \delta(AB) = A\delta(B) + \delta(A)B \). Then there exists a matrix \( K \) such that \( \delta(X) = XK - KX \) for \( X \) in \( M_2 \).

**COROLLARY**

\[ H^1(M_2, M_2) = 0 \]
\[ H^1(M_2, M_2) = 0 \]

**Matrix units**

Let \( E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \)

**Lemma**

\( E_{11} + E_{22} = I \)

\( E_{ij}^t = E_{ji} \)

\( E_{ij}E_{kl} = \delta_{kl}E_{il} \)

**Theorem 1**

Let \( \delta : M_2 \to M_2 \) be a derivation: \( \delta \) is linear and \( \delta(AB) = A\delta(B) + \delta(A)B \). Then there exists a matrix \( K \) such that \( \delta(X) = XK - KX \) for \( X \) in \( M_2 \).

**Corollary**

\[ H^1(M_2, M_2) = 0 \]
$H^1(\mathbb{M}_2, \mathbb{M}_2) = 0$

**Matrix units**
Let $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

**LEMMA**
- $E_{11} + E_{22} = I$
- $E_{ij}^t = E_{ji}$
- $E_{ij}E_{kl} = \delta_{kl}E_{il}$

**THEOREM 1**
Let $\delta : \mathbb{M}_2 \to \mathbb{M}_2$ be a derivation: $\delta$ is linear and $\delta(AB) = A\delta(B) + \delta(A)B$. Then there exists a matrix $K$ such that $\delta(X) = XK - KX$ for $X$ in $\mathbb{M}_2$.

**COROLLARY**
$H^1(\mathbb{M}_2, \mathbb{M}_2) = 0$
PROOF OF THEOREM 1

\[
0 = \delta(1) = \delta(E_{11} + E_{22}) = \delta(E_{11}) + \delta(E_{22})
\]

\[
= \delta(E_{11}E_{11}) + \delta(E_{21}E_{12})
\]

\[
= E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} + E_{21}\delta(E_{12}) + \delta(E_{21})E_{12}
\]

\[
= E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) + \delta(E_{11})E_{11} + \delta(E_{21})E_{12}.
\]

Let \( K = E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) = -\delta(E_{11})E_{11} - \delta(E_{21})E_{12} \). Then

\[
\begin{align*}
KE_{11} &= -\delta(E_{11})E_{11} \quad , \quad E_{11}K = E_{11}\delta(E_{11}) \\
KE_{12} &= -\delta(E_{11})E_{12} \quad , \quad E_{12}K = E_{11}\delta(E_{12}) \\
KE_{21} &= -\delta(E_{21})E_{11} \quad , \quad E_{21}K = E_{21}\delta(E_{11}) \\
KE_{22} &= -\delta(E_{21})E_{12} \quad , \quad E_{22}K = E_{21}\delta(E_{12})
\end{align*}
\]

\[
\begin{align*}
E_{11}K - KE_{11} &= E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} = \delta(E_{11}E_{11}) = \delta(E_{11}) \\
E_{12}K - KE_{12} &= E_{11}\delta(E_{12}) + \delta(E_{11})E_{12} = \delta(E_{11}E_{12}) = \delta(E_{12}) \\
E_{21}K - KE_{21} &= E_{21}\delta(E_{11}) + \delta(E_{21})E_{11} = \delta(E_{21}E_{11}) = \delta(E_{21}) \\
E_{22}K - KE_{22} &= E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} = \delta(E_{21}E_{12}) = \delta(E_{22}) \text{ Q.E.D.}
\end{align*}
\]
PROOF OF THEOREM 1

0  =  \delta(1) = \delta(E_{11} + E_{22}) = \delta(E_{11}) + \delta(E_{22})
    =  \delta(E_{11}E_{11}) + \delta(E_{21}E_{12})
    =  E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} + E_{21}\delta(E_{12}) + \delta(E_{21})E_{12}
    =  E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) + \delta(E_{11})E_{11} + \delta(E_{21})E_{12}.

Let \( K = E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) = -\delta(E_{11})E_{11} - \delta(E_{21})E_{12}. \) Then

\begin{align*}
\blacktriangleright KE_{11} &= -\delta(E_{11})E_{11} , \quad E_{11}K = E_{11}\delta(E_{11}) \\
\blacktriangleright KE_{12} &= -\delta(E_{11})E_{12} , \quad E_{12}K = E_{11}\delta(E_{12}) \\
\blacktriangleright KE_{21} &= -\delta(E_{21})E_{11} , \quad E_{21}K = E_{21}\delta(E_{11}) \\
\blacktriangleright KE_{22} &= -\delta(E_{21})E_{12} , \quad E_{22}K = E_{21}\delta(E_{12})
\end{align*}

\begin{align*}
\blacktriangleright E_{11}K - KE_{11} &= E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} = \delta(E_{11}E_{11}) = \delta(E_{11}) \\
\blacktriangleright E_{12}K - KE_{12} &= E_{11}\delta(E_{12}) + \delta(E_{11})E_{12} = \delta(E_{11}E_{12}) = \delta(E_{12}) \\
\blacktriangleright E_{21}K - KE_{21} &= E_{21}\delta(E_{11}) + \delta(E_{21})E_{11} = \delta(E_{21}E_{11}) = \delta(E_{21}) \\
\blacktriangleright E_{22}K - KE_{22} &= E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} = \delta(E_{21}E_{12}) = \delta(E_{22}) Q.E.D.
\end{align*}
0 = \delta(1) = \delta(E_{11} + E_{22}) = \delta(E_{11}) + \delta(E_{22})
= \delta(E_{11} E_{11}) + \delta(E_{21} E_{12})
= E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} + E_{21}\delta(E_{12}) + \delta(E_{21})E_{12}
= E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) + \delta(E_{11})E_{11} + \delta(E_{21})E_{12}.

Let \( K = E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) = -\delta(E_{11})E_{11} - \delta(E_{21})E_{12} \). Then

- \( KE_{11} = -\delta(E_{11})E_{11} \), \( E_{11}K = E_{11}\delta(E_{11}) \)
- \( KE_{12} = -\delta(E_{11})E_{12} \), \( E_{12}K = E_{11}\delta(E_{12}) \)
- \( KE_{21} = -\delta(E_{21})E_{11} \), \( E_{21}K = E_{21}\delta(E_{11}) \)
- \( KE_{22} = -\delta(E_{21})E_{12} \), \( E_{22}K = E_{21}\delta(E_{12}) \)

- \( E_{11}K - KE_{11} = E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} = \delta(E_{11} E_{11}) = \delta(E_{11}) \)
- \( E_{12}K - KE_{12} = E_{11}\delta(E_{12}) + \delta(E_{11})E_{12} = \delta(E_{11} E_{12}) = \delta(E_{12}) \)
- \( E_{21}K - KE_{21} = E_{21}\delta(E_{11}) + \delta(E_{21})E_{11} = \delta(E_{21} E_{11}) = \delta(E_{21}) \)
- \( E_{22}K - KE_{22} = E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} = \delta(E_{21} E_{12}) = \delta(E_{22}) \) Q.E.D.
If $A$ is an associative algebra, we can make it into a Lie algebra, denoted $A^-$ by defining $[a, b] = ab - ba$ and into a Jordan algebra, denoted $A^+$ by defining $a \circ b = (ab + ba)/2$.

Examples: $A = C \Rightarrow A^- = A$ with all products $[a, b] = 0$, $A^+ = A$ with $a \circ b = ab$

NOT VERY INTERESTING

$A = M_n(\mathbb{R})$ is more interesting!
Types of derivations

**Derivation**

\[ \delta(ab) = a\delta(b) + \delta(a)b \]

**Lie derivation**

\[ \delta([a, b]) = [a, \delta(b)] + [\delta(a), b] \]

**Jordan derivation**

\[ \delta(a \circ b) = a \circ \delta(b) + \delta(a) \circ b \]

**Trivial Exercise**

A derivation is also a Lie derivation and a Jordan derivation.
Converses

These are the two theorems relating different types of derivations

**Theorem 1**
A Jordan derivation is a derivation \((A = M_n(\mathbb{R}))\)

**Example**
There is a Lie derivation which is not a derivation \((A = M_n(\mathbb{R}))\), namely \(\delta(x) = \text{trace}(x)I\)

**Theorem 2**
Every Lie derivation is the sum of a derivation and a linear operator of the above form \((A = M_n(\mathbb{R}))\)
Let $d = \frac{1}{n} \sum_{i,j} e_{ij} \otimes e_{ji}$. Then $d$ is a **diagonal** for $M_n(\mathbb{C})$, that is, 
\[ \pi(d) = 1 \text{ and } a \cdot d = d \cdot a \text{ for all } a \in M_n(\mathbb{C}), \]
where
\[ \pi(x \otimes y) = xy, \ a \cdot x \otimes y = (ax) \otimes y \text{ and } x \otimes y \cdot a = x \otimes (ya). \]
Explicitly, 
\[ \pi(d) = \frac{1}{n} \sum_{i,j} e_{ij} e_{ji} = 1, \ \frac{1}{n} \sum_{i,j} (ae_{ij}) \otimes e_{ji} = \frac{1}{n} \sum_{i,j} e_{ij} \otimes (e_{ji}a) \]

The symmetric nature of $d$ implies 
\[ \frac{1}{n} \sum_{i,j} (e_{ij}a) \otimes e_{ji} = \frac{1}{n} \sum_{i,j} e_{ij} \otimes (ae_{ji}) \]
For any linear transformation $D$, apply $1 \otimes D$ and then $\pi$, to get
\begin{align*}
\frac{1}{n} \sum_{i,j} (ae_{ij}) D(e_{ji}) &= \frac{1}{n} \sum_{i,j} e_{ij} D(e_{ji}a) \text{ and } \frac{1}{n} \sum_{i,j} (e_{ij}a) D(e_{ji}) = \frac{1}{n} \sum_{i,j} e_{ij} D(ae_{ji})
\end{align*}
Proof of Theorem 1 (Barry Johnson 1996)

Let \( D : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) be a Jordan derivation

Define \( x = \frac{1}{n} \sum_{i,j} e_{ij}De_{ji} \). Then

\[
ax = \frac{1}{n} \sum_{i,j} ae_{ij}De_{ji} = \frac{1}{n} \sum_{i,j} e_{ij}D(e_{ji}a)
\]

\[
D(e_{ji}a) + D(ae_{ji}) = e_{ji}Da + (Da)e_{ji} + (De_{ji})a + aDe_{ji}
\]

\[
ax = \frac{1}{n} \sum_{i,j} e_{ij}[e_{ji}Da + D(e_{ji})a + (Da)e_{ji} + aDe_{ji} - D(ae_{ji})]
\]

\[
ax = Da + xa + \Delta(a) + 0, \text{ where}
\]

\[
\Delta(a) = \frac{1}{n} \sum_{i,j} e_{ij}(Da)e_{ji} \quad \text{(recall that} \quad \frac{1}{n} \sum_{i,j}(e_{ij}a)D(e_{ji}) = \frac{1}{n} \sum_{i,j} e_{ij}D(ae_{ji}))
\]
\( \Delta \) is a Jordan derivation with \( a \Delta (b) = \Delta (b) a \), that is,
\[
\frac{1}{n} \sum_{i,j} a e_{ij} (Db) e_{ji} = \frac{1}{n} \sum_{i,j} e_{ij} (Db) e_{ji} a
\]
Proof: Apply \( R_{Db} \otimes 1 \), then \( \pi \) to \( \frac{1}{n} \sum_{i,j} (ae_{ij}) \otimes e_{ji} = \frac{1}{n} \sum_{i,j} e_{ij} \otimes (e_{ji} a) \)

Start over with \( D \) replaced by \( \Delta \)
\[
x_0 = \frac{1}{n} \sum_{i,j} e_{ij} \Delta (e_{ji})
\]
\[
ax_0 = \Delta a + x_0 a + \frac{1}{n} \sum_{i,j} e_{ij} \Delta (a)(e_{ji}) = 2\Delta a + x_0 a
\]

\[
\Delta a = \frac{1}{2} (ax_0 - x_0 a)
\]
\[
Da = ax - xa - \Delta a = a(x - \frac{1}{2} x_0) - (x - \frac{1}{2} x_0) a
\]
is an inner associative derivation. Q.E.D.

The proof of the Lie derivation result goes along the same lines
Proof of Theorem 2 (Barry Johnson 1996)

Let \( D : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) be a Lie derivation

Define \( x = \frac{1}{n} \sum_{i,j} e_{ij} D e_{ji} \). Then

\[
ax = \frac{1}{n} \sum_{i,j} ae_{ij} D e_{ji} = \frac{1}{n} \sum_{i,j} e_{ij} D(e_{ji} a)
\]

\[
D(e_{ji} a) - D(ae_{ji}) = e_{ji} Da - (Da)e_{ji} + (De_{ji})a - aDe_{ji}
\]

\[
ax = \frac{1}{n} \sum_{i,j} e_{ij} [e_{ji} Da + D(e_{ji}) a - (Da)e_{ji} - aDe_{ji} + D(ae_{ji})]
\]

\[
ax = Da + xa - \Delta(a) + 0,
\]

where

\[
\Delta(a) = \frac{1}{n} \sum_{i,j} e_{ij} (Da)e_{ji} \quad \text{(recall that} \quad \frac{1}{n} \sum_{i,j} (e_{ij} a) D(e_{ji}) = \frac{1}{n} \sum_{i,j} e_{ij} D(ae_{ji}) \text{)}
\]

\( \Delta \) is a Lie derivation with \( a\Delta(b) = \Delta(b)a \), that is, so it vanishes on commutators and so \( Da = ax - xa + \Delta(a) \) as required.
MULTIPLICATION DOES NOT MAKE SENSE ON $M_{m,n}(\mathbb{R})$ if $m \neq n$.
NOT TO WORRY!
WE CAN FORM A TRIPLE PRODUCT $X \times Y^t \times Z$
(TRIPLE MATRIX MULTIPLICATION)

COMMUTATIVE AND ASSOCIATIVE DON’T MAKE SENSE HERE. RIGHT?
WRONG!!

$$(X \times Y^t \times Z) \times A^t \times B = X \times Y^t \times (Z \times A^t \times B)$$
DEFINITION
A DERIVATION ON $M_{m,n}(\mathbb{R})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION IS A LINEAR PROCESS $\delta$ WHICH SATISFIES THE (TRIPLE) PRODUCT RULE

$$\delta(A \times B^t \times C) =$$

$$\delta(A) \times B^t \times C + A \times \delta(B)^t \times C + A \times B^t \times \delta(C)$$

PROPOSITION
FOR TWO MATRICES $A, B$ in $M_{m,n}(\mathbb{R})$, DEFINE $\delta_{A,B}(X) =$

$$A \times B^t \times X + X \times B^t \times A - B \times A^t \times X - X \times A^t \times B$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION
THEOREM

EVERY DERIVATION ON $M_{m,n}(\mathbb{R})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION IS A SUM OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

REMARK

THESE RESULTS HOLD TRUE AND ARE OF INTEREST FOR THE CASE $m = n$. 
TRIPLE BRACKET MULTIPLICATION

LET’S GO BACK FOR A MOMENT TO SQUARE MATRICES AND THE BRACKET MULTIPLICATION.
MOTIVATED BY THE LAST REMARK, WE DEFINE THE TRIPLE BRACKET MULTIPLICATION TO BE $[[X, Y], Z]$.

DEFINITION

A DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION IS A LINEAR PROCESS $\delta$ WHICH SATISFIES THE TRIPLE PRODUCT RULE

$\delta([[A, B], C]) = [[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$.
PROPOSITION

FIX TWO MATRICES $A, B$ IN $M_n(\mathbb{R})$ AND DEFINE $\delta_{A,B}(X) = [[A, B], X]$ THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION.

THEOREM

EVERY DERIVATION OF $M_n(\mathbb{R})$ WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION IS A SUM OF DERIVATIONS OF THE FORM $\delta_{A,B}$.
TRIPLE CIRCLE MULTIPLICATION

Let’s return to rectangular matrices and form the triple circle multiplication
\((A \times B^t \times C + C \times B^t \times A)/2\)

For sanity’s sake, let us write this as

\[\{A, B, C\} = (A \times B^t \times C + C \times B^t \times A)/2\]

DEFINITION

A derivation on \(M_{m,n}(\mathbb{R})\) with respect to triple circle multiplication
is a linear process \(\delta\) which satisfies the triple product rule
\(\delta(\{A, B, C\}) = \{\delta(A), B, C\} + \{A, \delta(B), C\} + \{A, B, \delta(C)\}\)
**PROPOSITION**

Fix two matrices $A, B$ in $M_{m,n}(\mathbb{R})$ and define

$$
\delta_{A,B}(X) = \{A, B, X\} - \{B, A, X\}
$$

Then $\delta_{A,B}$ is a derivation with respect to triple circle multiplication.

**THEOREM**

Every derivation of $M_{m,n}(\mathbb{R})$ with respect to triple circle multiplication is a **sum** of derivations of the form $\delta_{A,B}$. 
It is time for summary of the preceding

Table 3 $M_{m,n}(\mathbb{R})$ (TRIPLE SYSTEMS)

<table>
<thead>
<tr>
<th>triple matrix</th>
<th>triple bracket</th>
<th>triple circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ab^tc$</td>
<td>$[a, b], c$</td>
<td>$ab^tc + cb^ta$</td>
</tr>
<tr>
<td>Th. 8</td>
<td>Th.9</td>
<td>Th.10</td>
</tr>
<tr>
<td>$\delta_{a,b}(x)$</td>
<td>$\delta_{a,b}(x)$</td>
<td>$\delta_{a,b}(x)$</td>
</tr>
<tr>
<td>$= ab^tx$</td>
<td>$= abx$</td>
<td>$= ab^tx$</td>
</tr>
<tr>
<td>$+ xb^ta$</td>
<td>$+ xba$</td>
<td>$+ xb^ta$</td>
</tr>
<tr>
<td>$- ba^tx$</td>
<td>$- bax$</td>
<td>$- ba^tx$</td>
</tr>
<tr>
<td>$- xa^tb$</td>
<td>$- xab$</td>
<td>$- xa^tb$</td>
</tr>
<tr>
<td>(sums)</td>
<td>(sums)</td>
<td>(sums)</td>
</tr>
<tr>
<td>($m = n$)</td>
<td>($m = n$)</td>
<td>($m = n$)</td>
</tr>
</tbody>
</table>
### Table 2 $M_n(\mathbb{R})$ (ALGEBRAS)

<table>
<thead>
<tr>
<th>matrix</th>
<th>bracket</th>
<th>circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ab = a \times b$</td>
<td>$[a, b] = ab - ba$</td>
<td>$a \circ b = ab + ba$</td>
</tr>
<tr>
<td>$\delta_a(x)$</td>
<td>$\delta_a(x)$</td>
<td>$\delta_a(x)$</td>
</tr>
<tr>
<td>$ax - xa$</td>
<td>$ax - xa$</td>
<td>$ax - xa$</td>
</tr>
</tbody>
</table>

### Table 3 $M_{m,n}(\mathbb{R})$ (TRIPLE SYSTEMS)

<table>
<thead>
<tr>
<th>triple matrix</th>
<th>triple bracket</th>
<th>triple circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; abc \geq ab^t c$</td>
<td>$[abc] = [[a, b], c]$</td>
<td>${abc} = ab^t c + cb^t a$</td>
</tr>
<tr>
<td>$\delta_{a,b}(x)$</td>
<td>$\delta_{a,b}(x)$</td>
<td>$\delta_{a,b}(x)$</td>
</tr>
<tr>
<td>$ab^t x$</td>
<td>$abx$</td>
<td>$ab^t x$</td>
</tr>
<tr>
<td>$+xb^t a$</td>
<td>$+xba$</td>
<td>$+xb^t a$</td>
</tr>
<tr>
<td>$-ba^t x$</td>
<td>$-bax$</td>
<td>$-ba^t x$</td>
</tr>
<tr>
<td>$-xa^t b$</td>
<td>$-xab$</td>
<td>$-xa^t b$</td>
</tr>
<tr>
<td>(sums)</td>
<td>(sums)</td>
<td>(sums)</td>
</tr>
</tbody>
</table>

(possibly) $m = n$
AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

AN **TRIPLE SYSTEM** IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH ONE BINARY OPERATION, CALLED **ADDITION** AND ONE TERNARY OPERATION CALLED **TRIPLE MULTIPLICATION**

**ADDITION** IS DENOTED BY
\[ a + b \]
AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE
\[ a + b = b + a, \quad (a + b) + c = a + (b + c) \]

**TRIPLE MULTIPLICATION** IS DENOTED
\[ abc \]
AND IS REQUIRED TO BE LINEAR IN EACH VARIABLE

\[ (a + b)cd = acd + bcd \]
\[ a(b + c)d = abd + acd \]
\[ ab(c + d) = abc + abd \]
Simple but important examples of triple systems can be formed from any algebra. If \( ab \) denotes the algebra product, just define a triple multiplication to be \((ab)c\).

Let’s see how this works in the algebras we introduced in Part I:

\[ \mathcal{C}, \mathcal{D}; \ fgh = (fg)h \]

\[ (M_n(\mathbb{R}), \times); \ abc = a \times b \times c \text{ or } a \times b^t \times c \]

\[ (M_n(\mathbb{R}), [ , ]); \ abc = [[a, b], c] \]

\[ (M_n(\mathbb{R}), \circ); \ abc = (a \circ b) \circ c \text{ (NO GO!)} \]
A triple system is said to be **associative** (resp. **commutative**) if the multiplication is associative (resp. commutative) (recall that addition is always commutative and associative) in the triple context this means the following:

**Associative**
\[ ab(cde) = (abc)de = a(bcd)e \]

or
\[ ab(cde) = (abc)de = a(dcb)e \]

**Commutative:** \[ abc = cba \]

The triple systems \( C \), \( D \) and \((M_n(\mathbb{R}), \times)\) are examples of associative triple systems. \( C \) and \( D \) are examples of commutative triple systems.
THE AXIOM WHICH CHARACTERIZES TRIPLE MATRIX MULTIPLICATION IS

\[(abc)de = ab(cde) = a(dcb)e\]

THESE ARE CALLED
ASSOCIATIVE TRIPLE SYSTEMS
or
HESTENES ALGEBRAS

THE AXIOMS WHICH CHARACTERIZE TRIPLE BRACKET MULTIPLICATION ARE

\[aab = 0\]
\[abc + bca + cab = 0\]
\[de(abc) = (dea)bc + a(deb)c + ab(dec)\]

THESE ARE CALLED
LIE TRIPLE SYSTEMS
(NATHAN JACOBSON, MAX KOECHER)
THE AXIOMS WHICH CHARACTERIZE TRIPLE CIRCLE MULTIPLICATION ARE

\[
abc = cba
\]

\[
de(abc) = (dea)bc - a(edb)c + ab(dec)
\]

THESE ARE CALLED JORDAN TRIPLE SYSTEMS
Table 4  TRIPLE SYSTEMS

<table>
<thead>
<tr>
<th>Type of Triple System</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Associative triple systems</td>
<td>$(abc)de = ab(cde) = a(dcb)e$</td>
</tr>
<tr>
<td>Lie triple systems</td>
<td>$aab = 0$</td>
</tr>
<tr>
<td></td>
<td>$abc + bca + cab = 0$</td>
</tr>
<tr>
<td></td>
<td>$de(abc) = (dea)bc + a(deb)c + ab(dec)$</td>
</tr>
<tr>
<td>Jordan triple systems</td>
<td>$abc = cba$</td>
</tr>
<tr>
<td></td>
<td>$de(abc) = (dea)bc - a(edb)c + ab(dec)$</td>
</tr>
</tbody>
</table>

Table 1  (FASHIONABLE) ALGEBRAS

<table>
<thead>
<tr>
<th>Type of Algebra</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commutative algebras</td>
<td>$ab = ba$</td>
</tr>
<tr>
<td>Associative algebras</td>
<td>$a(bc) = (ab)c$</td>
</tr>
<tr>
<td>Lie algebras</td>
<td>$a^2 = 0$, $(ab)c + (bc)a + (ca)b = 0$</td>
</tr>
<tr>
<td>Jordan algebras</td>
<td>$ab = ba$, $a(a^2b) = a^2(ab)$</td>
</tr>
</tbody>
</table>
Theorem

Every derivation of a finite dimensional semisimple Lie triple system $F$ is a sum of derivations of the form $\delta_{A,B}$, for some A’s and B’s in the triple system. These derivations are called inner derivations and their set is denoted $\text{Inder } F$.

Proof

Let $F$ be a finite dimensional semisimple Lie triple system (over a field of characteristic 0) and suppose that $D$ is a derivation of $F$. Let $L$ be the Lie algebra $(\text{Inder } F) \oplus F$ with product

$$[(H_1, x_1), (H_2, x_2)] = ([H_1, H_2] + L(x_1, x_2), H_1 x_2 - H_2 x_1).$$

A derivation of $L$ is defined by $\delta(H \oplus a) = [D, H] \oplus Da$. Together with the definition of semisimple Lie triple system, it is proved in the lecture notes of Meyberg (Lectures on algebras and triple systems 1972) that $F$ semisimple implies $L$ semisimple. Thus there exists $U = H_1 \oplus a_1 \in L$ such that $\delta(X) = [U, X]$ for all $X \in L$. Then $0 \oplus Da = \delta(0 \oplus a) = [H_1 + a_1, 0 \oplus a] = L(a_1, a) \oplus H_1 a$ so $L(a_1, a) = 0$ and $D = H_1 \in \text{Inder } F$. 
Theorem
Every derivation of a finite dimensional semisimple Jordan triple system is inner.

The TKK construction (Tits-Kantor-Koecher)
Let $V$ be a Jordan triple and let $\mathcal{L}(V)$ be its TKK Lie algebra. $\mathcal{L}(V) = V \oplus V_0 \oplus V$ and the Lie product is given by

$$[(x, h, y), (u, k, v)] = (hu - kx, [h, k] + x \boxtimes v - u \boxtimes y, k^\flat y - h^\flat v).$$

Here, $a \boxtimes b$ is the left multiplication operator $x \mapsto \{abx\}$ (also called the box operator), $V_0 = \text{span}\{V \boxtimes V\}$ is a Lie subalgebra of $\mathcal{L}(V)$ and for $h = \sum_i a_i \boxtimes b_i \in V_0$, the map $h^\flat : V \rightarrow V$ is defined by

$$h^\flat = \sum_i b_i \boxtimes a_i.$$
We can show the correspondence of derivations $\delta : V \to V$ and $D : \mathcal{L}(V) \to \mathcal{L}(V)$ for Jordan triple $V$ and its TKK Lie algebra $\mathcal{L}(V)$.

Let $\theta : \mathcal{L}(V) \to \mathcal{L}(V)$ be the main involution $\theta(x \oplus h \oplus y) = y \oplus -h^\circ \oplus x$.

**Lemma**

Let $\delta : V \to V$ be a derivation of a Jordan triple $V$, with TKK Lie algebra $(\mathcal{L}(V), \theta)$. Then there is a derivation $D : \mathcal{L}(V) \to \mathcal{L}(V)$ satisfying

$$D(V) \subset V \quad \text{and} \quad D\theta = \theta D.$$
Proof

Given \( a, b \in V \), we define

\[
D(a, 0, 0) = (\delta a, 0, 0) \\
D(0, 0, b) = (0, 0, \delta b) \\
D(0, a \square b, 0) = (0, \delta a \square b + a \square \delta b, 0)
\]

and extend \( D \) linearly on \( \mathcal{L}(V) \). Then \( D \) is a derivation of \( \mathcal{L}(V) \) and evidently, \( D(V) \subset V \).

It is readily seen that \( D \theta = \theta D \), since

\[
D\theta(0, a \square b, 0) = D(0, -b \square a, 0) \\
= (0, -\delta b \square a - b \square \delta a, 0) \\
= \theta(0, \delta a \square b + a \square \delta b, 0) \\
= \theta D(0, a \square b, 0).
\]
**Lemma**

Let $V$ be a Jordan triple with TKK Lie algebra $(\mathcal{L}(V), \theta)$. Given a derivation $D : \mathcal{L}(V) \to \mathcal{L}(V)$ satisfying $D(V) \subset V$ and $D\theta = \theta D$, the restriction $D|_V : V \to V$ is a triple derivation.

**Theorem**

Let $V$ be a Jordan triple with TKK Lie algebra $(\mathcal{L}(V), \theta)$. There is a one-one correspondence between the triple derivations of $V$ and the Lie derivations $D : \mathcal{L}(V) \to \mathcal{L}(V)$ satisfying $D(V) \subset V$ and $D\theta = \theta D$. 
**Lemma**

Let $V$ be a Jordan triple with TKK Lie algebra $(\mathcal{L}(V), \theta)$. Let $D : \mathcal{L}(V) \to \mathcal{L}(V)$ be a Lie inner derivation such that $D(V) \subset V$. Then the restriction $D|_V$ is a triple inner derivation of $V$.

**Corollary**

Let $\delta$ be a derivation of a finite dimensional semisimple Jordan triple $V$. Then $\delta$ is a triple inner derivation of $V$.

**Proof**

The TKK Lie algebra $\mathcal{L}(V)$ is semisimple. Hence the result follows from the Lie result and the Lemma.
The proof of the last Lemma is instructive. The steps are as follows.

1. \( D(x, k, y) = [(x, k, y), (a, h, b)] \) for some \((a, h, b) \in \mathcal{L}(V)\)

2. \( D(x, 0, 0) = [(x, 0, 0), (a, h, b)] = (-h(x), x \boxplus b, 0) \)

3. \( \delta(x) = -h(x) = -\sum_i \alpha_i \boxminus \beta_i(x) \)

4. \( D(0, 0, y) = [(0, 0, y), (a, h, b)] = (0, -a \boxplus y, h^b(y)) \)

5. \( \delta(x) = -h^b(x) = \sum_i \beta_i \boxminus \alpha_i(x) \)

6. \( \delta(x) = \frac{1}{2} \sum_i (\beta_i \boxminus \alpha_i - \alpha_i \boxminus \beta_i)(x) \)
Let $h$ be a Hochschild 1-cocycle, that is, a linear map

$$h : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

satisfying

$$h(ab) - ah(b) - h(a)b = 0. \tag{1}$$

We are going to show that there is an element $x \in M_n(\mathbb{C})$ such that

$$h(a) = xa - ax. \tag{2}$$

It is enough to prove that (2) holds with $a \in \{e_{ij}\}$, that is

$$h(e_{ij}) = xe_{ij} - e_{ij}x. \tag{3}$$

The element $x$ can be defined by determining its coordinates $x_{ij}$ from other information, that is

$$x = \sum_{p,q} x_{pq} e_{pq}. \tag{4}$$

Define $\gamma_{ijpq}$ by

$$h(e_{ij}) = \sum_{p,q} \gamma_{ijpq} e_{pq}. \tag{5}$$
Then (3), (4) and (5) lead to the system of linear vector equations

\[ \sum_{p,q} \gamma_{ijpq} e_{pq} = \sum_{p,q} \delta_{qi} x_{pq} e_{pj} - \sum_{p,q} \delta_{jp} x_{pq} e_{iq}. \]  

(6)

with \( n^2 \) unknowns \( x_{ij} \). Then any solution of (6) proves (3) and hence (2).

It is a special case of a theorem of Hochschild (1945) that such an \( x \) exists. We shall find \( x \) by solving the equations. We shall now restrict to \( n = 2 \).

From (6) we have for fixed \( i, j \),

\[ \gamma_{ij11} e_{11} + \gamma_{ij12} e_{12} + \gamma_{ij21} e_{21} + \gamma_{ij22} e_{22} \]

\[ = \delta_{1i} x_{11} e_{1j} + \delta_{2i} x_{12} e_{1j} + \delta_{1i} x_{21} e_{2j} + \delta_{2i} x_{22} e_{2j} \]

\[ - \delta_{j1} x_{11} e_{i1} - \delta_{j1} x_{12} e_{i2} - \delta_{j2} x_{21} e_{i1} - \delta_{j2} x_{22} e_{i2}. \]  

(7)

With \((i, j)\) successively equal to \((1, 1), (1, 2), (2, 1), (2, 2)\) we obtain the following 16 equations.
Coefficient of $e_{11}$

1. $(i, j) = (1, 1)$ $\gamma_{1111} = x_{11} - x_{11} = 0$
2. $(i, j) = (1, 2)$ $\gamma_{1211} = -x_{21}$
3. $(i, j) = (2, 1)$ $\gamma_{2111} = x_{12}$
4. $(i, j) = (2, 2)$ $\gamma_{2211} = 0$

Coefficient of $e_{12}$

5. $(i, j) = (1, 1)$ $\gamma_{1112} = -x_{12}$
6. $(i, j) = (1, 2)$ $\gamma_{1212} = x_{11} - x_{22}$
7. $(i, j) = (2, 1)$ $\gamma_{2112} = 0$
8. $(i, j) = (2, 2)$ $\gamma_{2212} = x_{12}$

Coefficient of $e_{21}$

9. $(i, j) = (1, 1)$ $\gamma_{1121} = x_{21}$
10. $(i, j) = (1, 2)$ $\gamma_{1221} = 0$
11. $(i, j) = (2, 1)$ $\gamma_{2121} = x_{22} - x_{11}$
12. $(i, j) = (2, 2)$ $\gamma_{2221} = -x_{21}$
Coefficient of $e_{22}$

13 \((i, j) = (1, 1)\) $\gamma_{1122} = 0$
14 \((i, j) = (1, 2)\) $\gamma_{1222} = x_{21}$
15 \((i, j) = (2, 1)\) $\gamma_{2122} = -x_{12}$
16 \((i, j) = (2, 2)\) $\gamma_{2222} = x_{22} - x_{22} = 0$

These 16 equations have the following formal solution, which is subject to the validity of the relations between the $\gamma_{ijpq}$, which is verified below.

- $x_{21} = -\gamma_{1211} = \gamma_{1121} = \gamma_{1222} = -\gamma_{2221}$
- $x_{12} = \gamma_{2111} = -\gamma_{1112} = \gamma_{2212} = -\gamma_{2122}$
- $x_{22} = -\gamma_{1221} = -\gamma_{2221}$
- $x_{11} = x_{22} + \gamma_{1212} = x_{22} - \gamma_{2121}$
- $0 = \gamma_{1111} = \gamma_{2211} = \gamma_{1122} = \gamma_{2222} = \gamma_{2112} = \gamma_{1221}$

Thus all solutions are given by

$$x = \begin{bmatrix} \gamma_{1212} + x_{22} & 0 \\ \gamma_{1121} & x_{22} \end{bmatrix} \quad (x_{22} \in \mathbb{C})$$

proving the theorem.
We now find the relations between the $\gamma$s. From (1) and (5), we have for fixed $i, j, k, l$,

$$
0 = \delta_{jk}(\gamma_{i11}e_{11} + \gamma_{i12}e_{12} + \gamma_{i21}e_{21} + \gamma_{i22}e_{22})
- \delta_{j1}\gamma_{k11}e_{i1} - \delta_{j1}\gamma_{k12}e_{i2} - \delta_{j2}\gamma_{k21}e_{i1} - \delta_{j2}\gamma_{k22}e_{i2}
- \delta_{1k}\gamma_{ij11}e_{1l} - \delta_{2k}\gamma_{ij12}e_{1l} - \delta_{1k}\gamma_{ij21}e_{2l} - \delta_{2k}\gamma_{ij22}e_{2l}.
$$

(8)

With $(i, j)$ successively equal to $(1, 1), (1, 2), (2, 1), (2, 2)$ and $(k, l)$ successively equal to $(1, 1), (1, 2), (2, 1), (2, 2)$ we obtain the following 64 equations giving the necessary conditions on the quantities $\gamma_{ijkl}$.
## Coefficient of $e_{11}$

<table>
<thead>
<tr>
<th>$eq$</th>
<th>$ij$</th>
<th>$kl$</th>
<th>$= 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11</td>
<td>11</td>
<td>$-\gamma_{1111}$</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>12</td>
<td>$\gamma_{1211} - \gamma_{2111}$</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>21</td>
<td>$\gamma_{2111} - \gamma_{1112}$</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
<td>22</td>
<td>$-\gamma_{2211}$</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>11</td>
<td>$-\gamma_{1111} - \gamma_{1121} - \gamma_{1211}$</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>12</td>
<td>$-\gamma_{1211} - \gamma_{1221}$</td>
</tr>
<tr>
<td>7</td>
<td>12</td>
<td>21</td>
<td>$\gamma_{1111} - \gamma_{2121} - \gamma_{1212}$</td>
</tr>
<tr>
<td>8</td>
<td>12</td>
<td>22</td>
<td>$\gamma_{1211} - \gamma_{2211} - \gamma_{2221}$</td>
</tr>
<tr>
<td>9</td>
<td>21</td>
<td>11</td>
<td>$\gamma_{1111} - \gamma_{2111}$</td>
</tr>
<tr>
<td>10</td>
<td>21</td>
<td>12</td>
<td>$\gamma_{2211}$</td>
</tr>
<tr>
<td>11</td>
<td>21</td>
<td>21</td>
<td>$-\gamma_{2112}$</td>
</tr>
<tr>
<td>12</td>
<td>21</td>
<td>22</td>
<td>$0$</td>
</tr>
<tr>
<td>13</td>
<td>22</td>
<td>11</td>
<td>$-\gamma_{2211}$</td>
</tr>
<tr>
<td>14</td>
<td>22</td>
<td>12</td>
<td>$0$</td>
</tr>
<tr>
<td>15</td>
<td>22</td>
<td>21</td>
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## Coefficient of $e_{12}$

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**Coefficient of $e_{22}$**

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