On the Tits-Kantor-Koecher Lie algebra of a von Neumann algebra

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Outline

1. Introduction

A veritable army of researchers took the theory of derivations of operator algebras to dizzying heights—producing a theory of cohomology of operator algebras as well as much information about automorphisms of operator algebras—Richard Kadison 2000

In addition to associative algebras, cohomology groups are defined for Lie algebras and, to some extent, for Jordan algebras.

Since the structures of Jordan derivations and Lie derivations on von Neumann algebras are well understood, and in view of the above quotation, isn't it time to study the higher dimensional non associative cohomology of a von Neumann algebra?

The paper on which this talk is based is motivated by this rhetorical question.

In the paper we develop a cohomology theory for Jordan triples, including the infinite dimensional ones, by means of the cohomology of TKK Lie algebras.

This enables us to apply Lie cohomological results to the setting of Jordan triples.

2. Jordan triples and TKK Lie algebras

By a *Jordan triple*, we mean a real or complex vector space V, equipped with a Jordan triple product $\{\cdot,\cdot,\cdot\} : V^3 \to V$ which is linear and symmetric in the outer variables, conjugate linear in the middle variable, and satisfies the Jordan triple identity

$$\{x, y, \{a, b, c\}\} = \{\{x, y, a\}, b, c\} - \{a, \{y, x, b\}, c\} + \{a, b, \{x, y, c\}\} \qquad (1)$$

for $a, b, c, x, y \in V$. Given two elements a, b in a Jordan triple V, we define the *box operator* $a \,\square\, b : V \to V$ by $a \,\square\, b(\cdot) = \{a, b, \cdot\}$.

Let V be a Jordan triple. A vector space M over the same scalar field is called a *Jordan triple* V*-module* (cf. [**?**]) if it is equipped with three mappings
$\{\cdot,\cdot,\cdot\}_1 : M \times V \times V \to M, \quad \{\cdot,\cdot,\cdot\}_2 : V \times M \times V \to M, \{\cdot,\cdot,\cdot\}_3 : V \times V \times M \to M$
such that (i) $\{a, b, c\}_1 = \{c, b, a\}_3$;
(ii) $\{\cdot,\cdot,\cdot\}_1$ is linear in the first two variables and conjugate linear in the last variable, $\{\cdot,\cdot,\cdot\}_2$ is conjugate linear in all variables;
(iii) denoting by $\{\cdot,\cdot,\cdot\}$ any of the products $\{\cdot,\cdot,\cdot\}_j$ ($j = 1, 2, 3$), the identity (1) is satisfied whenever one of the above elements is in M and the rest in V.

A Jordan triple is called *nondegenerate* if for each $a \in V$, the condition $\{a, a, a\} = 0$ implies $a = 0$. Given that V is nondegenerate facilitates a simple definition of the TKK Lie algebra $\mathfrak{L}(V) = V \oplus V_0 \oplus V$ of V, with an involution θ (cf. [**?**, p.45]), where $V_0 = \{\sum_j a_j \,\square\, b_j : a_j, b_j \in V\}$, the Lie product is defined by

$$[(x, h, y), (u, k, v)] = (hu - kx,\, [h, k] + x \,\square\, v - u \,\square\, y,\, k^\natural y - h^\natural v), \qquad (2)$$

and for each $h = \sum_i a_i \,\square\, b_i$ in the Lie subalgebra V_0 of $\mathfrak{L}(V)$, the map $h^\natural : V \to V$ is well defined by $h^\natural = \sum_i b_i \,\square\, a_i$.

The involution $\theta : \mathfrak{L}(V) \to \mathfrak{L}(V)$ is given by

$$\theta(x, h, y) = (y, -h^\natural, x) \qquad ((x, h, y) \in \mathfrak{L}(V)).$$

Given $a, b \in V$, the box operator $a \,\square\, b : V \to V$ can also be considered as a mapping from M to M. Similarly, for $u \in V$ and $m \in M$, the "*box operators*"

$$u \,\square\, m,\ m \,\square\, u : V \longrightarrow M$$

are defined in a natural way as $v \mapsto \{u, m, v\}$ and $v \mapsto \{m, u, v\}$ respectively.

Given a Lie algebra $\mathfrak{L}$ and a module X over $\mathfrak{L}$, we denote the action of $\mathfrak{L}$ on X by

$$(\ell, x) \in \mathfrak{L} \times X \mapsto \ell.x \in X \text{ so that } [\ell, \ell'].x = \ell'.(\ell.x) - \ell.(\ell'.x).$$

Let M_0 be the linear span of $\{u \square m, n \square v : u, v \in V, m, n \in M\}$ in the vector space $L(V, M)$ of linear maps from V to M.

M_0 is the space of *inner structural transformations* $\mathrm{Instrl}\,(V, M)$ (see [**?**, Section 7]) . (More about that later)

Extending the above product by linearity, we can define an action of V_0 on M_0 by

$$(h, \varphi) \in V_0 \times M_0 \mapsto [h, \varphi] \in M_0,$$

where $h = \sum a_i \square b_i$ and $\varphi = \sum_j (m_j \square u_j + v_j \square n_j)$.

Lemma

M_0 is a V_0-module of the Lie algebra V_0.

Let V be a Jordan triple and $\mathfrak{L}(V)$ its TKK Lie algebra. Given a triple V-module M, let $\mathcal{L}(M) = M \oplus M_0 \oplus M$ and define the action

$$((a,h,b),(m,\varphi,n)) \in \mathfrak{L}(V) \times \mathcal{L}(M) \mapsto (a,h,b).(m,\varphi,n) \in \mathcal{L}(M)$$

by $$(a,h,b).(m,\varphi,n) = (hm - \varphi a,\ [h,\varphi] + a \square n - m \square b,\ \varphi^\natural b - h^\natural(n)), \quad (3)$$

where, for $h = \sum_i a_i \square b_i$ and $\varphi = \sum_i u_i \square m_i + \sum_j n_j \square v_j$, we have the following natural definitions

$$hm = \sum_i \{a_i, b_i, m\}, \quad \varphi a = \sum_i \{u_i, m_i, a\} + \sum_j \{n_j, v_j, a\},$$

and

$$\varphi^\natural = \sum_i m_i \square u_i + \sum_j v_j \square n_j.$$

Theorem

Let V be a Jordan triple and let $\mathfrak{L}(V)$ be its TKK Lie algebra. Let M be a triple V-module. Then $\mathcal{L}(M)$ is a Lie $\mathfrak{L}(V)$-module.

3. Cohomology of Lie algebras with involution

Given an involutive Lie algebra $(\mathfrak{L}, \theta)$, an $(\mathfrak{L}, \theta)$-*module* is a (left) $\mathfrak{L}$-module $\mathfrak{M}$, equipped with an involution $\widetilde{\theta} : \mathfrak{M} \to \mathfrak{M}$ satisfying

$$\widetilde{\theta}(\ell.\mu) = \theta(\ell).\widetilde{\theta}(\mu) \qquad (\ell \in \mathfrak{L}, \mu \in \mathfrak{M}).$$

We also call $\mathfrak{M}$ an *involutive* $\mathfrak{L}$*-module* if θ is understood. For $\ell \in \mathfrak{L}$ and $\mu \in \mathfrak{M}$, we define

$$[\ell, \mu] := \ell.\mu \quad \text{and} \quad [\mu, \ell] := -\ell.\mu.$$

A k-linear map $\psi : \mathfrak{L}^k \to \mathfrak{M}$ is called θ*-invariant* if

$$\psi(\theta x_1, \cdots, \theta x_k) = \widetilde{\theta}\psi(x_1, \cdots, x_k) \quad \text{for} \quad (x_1, \cdots, x_k) \in \mathfrak{L} \times \cdots \times \mathfrak{L}.$$

Let $(\mathfrak{L},\theta)$ be an involutive Lie algebra and $\mathfrak{M}$ an $(\mathfrak{L},\theta)$-module. We define

$$A^0(\mathfrak{L},\mathfrak{M})=\mathfrak{M} \text{ and } A^0_\theta(\mathfrak{L},\mathfrak{M})=\{\mu\in\mathfrak{M}:\widetilde{\theta}\mu=\mu\}.$$

For $k=1,2,\ldots$, we let

$$A^k(\mathfrak{L},\mathfrak{M})=\{\psi:\mathfrak{L}^k\to\mathfrak{M}\mid \psi \text{ is k-linear and alternating}\} \quad \text{and}$$

$$A^k_\theta(\mathfrak{L},\mathfrak{M})=\{\psi\in A^k(\mathfrak{L},\mathfrak{M}):\mid \psi \text{ is } \theta\text{-invariant}\}.$$

For $k=0,1,2,\ldots$, we define the *coboundary operator* $d_k: A^k(\mathfrak{L},\mathfrak{M})\to A^{k+1}(\mathfrak{L},\mathfrak{M})$ by $d_0m(x)=x.m$ and for $k\geq 1$,

$$\begin{aligned}(d_k\psi)(x_1,\ldots,x_{k+1}) &= \sum_{\ell=1}^{k+1}(-1)^{\ell+1}x_\ell.\psi(x_1,\ldots,\widehat{x}_\ell,\ldots,x_{k+1}) \\ &+\sum_{1\leq i<j\leq k+1}(-1)^{i+j}\psi([x_i,x_j],\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_{k+1})\end{aligned} \tag{4}$$

where the symbol $\widehat{z}$ indicates the omission of z.

The restriction of d_k to the subspace $A^k_\theta(\mathfrak{L},\mathfrak{M})$, still denoted by d_k, has range $A^k_\theta(\mathfrak{L},\mathfrak{M})$ Also, we have $d_k d_{k-1} = 0$ for $k = 1, 2, \ldots$ (cf. [?, p. 167]) and the two cochain complexes

$$A^0(\mathfrak{L},\mathfrak{M}) \xrightarrow{d_0} A^1(\mathfrak{L},\mathfrak{M}) \xrightarrow{d_1} A^2(\mathfrak{L},\mathfrak{M}) \xrightarrow{d_2} \cdots$$

$$A^0_\theta(\mathfrak{L},\mathfrak{M}) \xrightarrow{d_0} A^1_\theta(\mathfrak{L},\mathfrak{M}) \xrightarrow{d_1} A^2_\theta(\mathfrak{L},\mathfrak{M}) \xrightarrow{d_2} \cdots .$$

As usual, we define the *k-th cohomology group of $\mathfrak{L}$ with coefficients in* $\mathfrak{M}$ to be

$$H^k(\mathfrak{L},\mathfrak{M}) = \ker d_k / d_{k-1}(A^{k-1}(\mathfrak{L},\mathfrak{M})) = \ker d_k / \mathrm{im}\, d_{k-1}$$

for $k = 1, 2, \ldots$ and define $H^0(\mathfrak{L},\mathfrak{M}) = \ker d_0$.

We define the *k-th involutive cohomology group of $(\mathfrak{L},\theta)$ with coefficients in an $(\mathfrak{L},\theta)$-module* $\mathfrak{M}$ to be the quotient

$$H^k_\theta(\mathfrak{L},\mathfrak{M}) = \ker d_k / d_{k-1}(A^{k-1}_\theta(\mathfrak{L},\mathfrak{M})) = \ker d_k / \mathrm{im}\, d_{k-1}$$

for $k = 1, 2, \ldots$ and define $H^0_\theta(\mathfrak{L},\mathfrak{M}) = \ker d_0 \subset H^0(\mathfrak{L},\mathfrak{M})$.

4. Cohomology of Jordan triples

Let V be a Jordan triple and let $\mathfrak{L}(V) = V \oplus V_0 \oplus V$ be its TKK Lie algebra with the involution $\theta(a, h, b) = (b, -h^\natural, a)$. Given a V-module M, we have shown that $\mathfrak{L}(M) = M \oplus M_0 \oplus M$ is an $\mathfrak{L}(V)$-module. We define an induced involution $\widetilde{\theta} : \mathfrak{L}(M) \to \mathfrak{L}(M)$ by

$$\widetilde{\theta}(m, \varphi, n) = (n, -\varphi^\natural, m)$$

for $(m, \varphi, n) \in M \oplus M_0 \oplus M$.

Lemma

$\mathfrak{L}(M)$ is an $(\mathfrak{L}(V), \theta)$-module, that is, we have $\widetilde{\theta}(\ell.\mu) = \theta(\ell).\widetilde{\theta}(\mu)$ for $\ell \in \mathfrak{L}(V)$ and $\mu \in \mathfrak{L}(M)$.

Let $\mathfrak{k}(V) = \{(v, h, v) \in \mathfrak{L}(V) : h^\natural = -h\}$ be the 1-eigenspace of the involution θ (see [**?**, p.48]), which is a real Lie subalgebra of $\mathfrak{L}(V)$, and let $\mathfrak{k}(M) = \{(m, \varphi, m) \in \mathfrak{L}(M) : \varphi = -\varphi^\natural\}$ be the 1-eigenspace of $\widetilde{\theta}$. Then $\mathfrak{k}(M)$ is a Lie module over the Lie algebra $\mathfrak{k}(V)$.

We will construct cohomology groups of a Jordan triple V with coefficients in a V-module M using the cohomology groups of $\mathfrak{L}(V)$ with coefficients $\mathfrak{L}(M)$.

For a real Jordan triple V, one can also make use of the cohomology groups of the real Lie algebra $\mathfrak{k}(V)$ with coefficients $\mathfrak{k}(M)$.

Let V be a Jordan triple. V is identified as the subspace $\{(v, 0, 0) : v \in V\}$ of the TKK Lie algebra $\mathfrak{L}(V)$.

For a triple V-module M, there is a natural embedding of M into $\mathfrak{L}(M) = M \oplus M_0 \oplus M$ given by $\iota : m \in M \mapsto (m, 0, 0) \in \mathfrak{L}(M)$ and we will identify M with $\iota(M)$.

We denote by $\iota_p : \mathfrak{L}(M) \to \iota(M)$ the natural projection $\iota_p(m, \varphi, n) = (m, 0, 0)$.

We define $A^0(V, M) = M$ and for $k = 1, 2, \ldots$, we denote by $A^k(V, M)$ the vector space of all alternating k-linear maps $\omega : V^k = \overbrace{V \times \cdots \times V}^{k-\text{times}} \to M$.

To motivate the definition of an extension $\mathfrak{L}_k(\omega) \in A^k(\mathfrak{L}(V), \mathfrak{L}(M))$ of an element $\omega \in A^k(V, M)$, for $k \geq 1$, we first consider the case $k = 1$ and note that $\omega \in A^1(V, M)$ is a Jordan triple derivation if and only if

$$\omega \circ (a \square b) - (a \square b) \circ \omega = \omega(a) \square b + a \square \omega(b).$$

Let us call a linear transformation $\omega : V \to M$ *extendable* if the following condition holds:

$$\sum_i a_i \square b_i = 0 \Rightarrow \sum_i (\omega(a_i) \square b_i + a_i \square \omega(b_i)) = 0.$$

Thus a Jordan triple derivation is extendable, and if ω is any extendable transformation in $A^1(V, M)$, then the map

$$\mathfrak{L}_1(\omega)(x_1 \oplus a_1 \square b_1 \oplus y_1) := (\, \omega(x_1), \, \omega(a_1) \square b_1 + a_1 \square \omega(b_1), \; \omega(y_1)\,)$$

is well defined and extends linearly to an element $\mathfrak{L}_1(\omega) \in A^1(\mathfrak{L}(V), \mathfrak{L}(M))$, in which case we call $\mathfrak{L}_1(\omega)$ the *Lie extension* of ω on the Lie algebra $\mathfrak{L}(V)$.

For $k > 1$, given a k-linear mapping $\omega : V^k \to M$, we say that ω is *extendable* if it satisfies the following condition under the assumption $\sum_i u_i \square v_i = 0$:

$$\sum_i (\omega(u_i, a_2, \ldots, a_k) \square (v_i + b_2 + \cdots + b_k))$$

$$+ \sum_i ((u_i + a_2 + \cdots + a_k) \square \omega(v_i, b_2, \ldots, b_k)) = 0,$$

for all $a_2, \ldots, a_k, b_2, \ldots, b_k \in V$.

For an extendable ω, we can unambiguously define a k-linear map $\mathfrak{L}_k(\omega) : \mathfrak{L}(V)^k \to \mathfrak{L}(M)$ as the linear extension (in each variable) of

$$\mathfrak{L}_k(\omega)(x_1 \oplus a_1 \square b_1 \oplus y_1, \, x_2 \oplus a_2 \square b_2 \oplus y_2, \, \cdots, \, x_k \oplus a_k \square b_k \oplus y_k) = \tag{5}$$
$$(\omega(x_1, \ldots, x_k), \sum_{j=1}^{k} \omega(a_1, \ldots, a_k) \square b_j + \sum_{j=1}^{k} a_j \square \omega(b_1, \ldots, b_k), \omega(y_1, \ldots, y_k)).$$

We call $\mathfrak{L}_k(\omega)$ the *Lie extension* of ω and $\mathfrak{L}_k(\omega) \in A^k(\mathfrak{L}(V), \mathfrak{L}(M))$. Moreover, $\mathfrak{L}_k(\omega) \in A^k_\theta(\mathfrak{L}(V), \mathfrak{L}(M))$ if and only if k is odd.

This lemma enables us to define the following extension map on the subspace $A^k(V,M)'$ of extendable maps in $A^k(V,M)$:

$$\mathfrak{L}_k : \omega \in A^k(V,M)' \mapsto \mathfrak{L}_k(\omega) \in A^k(\mathfrak{L}(V), \mathfrak{L}(M)).$$

Conversely, given $\psi \in A^k(\mathfrak{L}(V), \mathfrak{L}(M))$ for $k = 1, 2, \ldots$, one can define an alternating map

$$J^k(\psi) : V^k \to M$$

by

$$J^k(\psi)(x_1, \ldots, x_k) = \iota_p \psi(\,(x_1, 0, 0), \,\ldots\,, (x_k, 0, 0)\,)$$

for $(x_1, \ldots, x_k) \in V^k$.

We define $J^0 : \mathfrak{L}(M) \to \iota(M) \approx M = A^0(V,M)$ by

$$J^0(m, \varphi, n) = (m, 0, 0) \qquad ((m, \varphi, n) \in \mathfrak{L}(M)).$$

We call $J^k(\psi)$ the *Jordan restriction* of ψ in $A^k(V,M)$ and sometimes write J for J^k if the index k is understood.

We now define the two classes of cohomology groups for a Jordan triple V with coefficients M. First, for $H^k(V,M)$, and then for $H^k_\theta(V,M)$:

$$\begin{array}{ccccccc}
\mathfrak{L}(M) = A^0(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_0} & A^1(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_1} & A^2(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_2} & \cdot \\
\downarrow J^0 & & \downarrow J^1 & & \downarrow J^2 & & \cdot \\
M = A^0(V,M) & & A^1(V,M) & & A^2(V,M) & & \cdot
\end{array}$$

For $k = 0,1,2,\ldots$, the *k-th cohomology groups* $H^k(V,M)$ are defined by

$$\begin{aligned}
H^0(V,M) &= J^0(\ker d_0) = J^0\{(m,\varphi,n) : (u,h,v).(m,\varphi,n) = 0,\ \forall (u,h,v) \in \mathfrak{L}(V) \\
&= \{m \in M : m \,\square\, v = 0,\ \forall v \in V\} = \{0\}
\end{aligned}$$

$$H^k(V,M) = Z^k(V,M)/B^k(V,M) \qquad (k = 1,2,\ldots)$$

$$Z^k(V,M) = J^k(Z^k(\mathfrak{L}(V),\mathfrak{L}(M))), \quad Z^k(\mathfrak{L}(V),\mathfrak{L}(M)) = \ker d_k$$

$$B^k(V,M) = J^k(B^k(\mathfrak{L}(V),\mathfrak{L}(M))), \quad B^k(\mathfrak{L}(V),\mathfrak{L}(M)) = \mathrm{im}\, d_{k-1}.$$

$$
\begin{array}{ccccccc}
A^0_\theta(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_0} & A^1_\theta(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_1} & A^2_\theta(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_2} & \cdots \\
\downarrow J^0 & & \downarrow J^1 & & \downarrow J^2 & & \cdots \\
M = A^0(V,M) & & A^1(V,M) & & A^2(V,M) & & \cdots
\end{array}
$$

For $k = 0, 1, 2\ldots$, the *k-th involutive cohomology groups* $H^k_\theta(V,M)$ are

$$H^0(V,M) = J^0(\ker d_0) = \{0\}$$

$$H^k_\theta(V,M) = Z^k_\theta(V,M)/B^k_\theta(V,M) \qquad (k = 1, 2, \ldots)$$

$$Z^k_\theta(V,M) = J^k(Z^k_\theta(\mathfrak{L}(V),\mathfrak{L}(M))), \quad Z^k_\theta(\mathfrak{L}(V),\mathfrak{L}(M))) = \ker d_k|_{A^k_\theta(\mathfrak{L}(V),\mathfrak{L}(M))}$$

$$B^k_\theta(V,M) = J^k(B^k_\theta(\mathfrak{L}(V),\mathfrak{L}(M))), B^k_\theta(\mathfrak{L}(V),\mathfrak{L}(M)) = d_{k-1}(A^{k-1}_\theta(\mathfrak{L}(V),\mathfrak{L}(M))).$$

We call elements in $H^k(V,M)$ the *Jordan triple k-cocycles*, and the ones in $H^k_\theta(V,M)$ the *involutive Jordan triple k-cocycles*. Customarily, elements in $B^k(V,M)$ and $B^k_\theta(V,M)$are called the *coboundaries*.

Some immediate consequences of the construction

Let V be a Jordan triple with TKK Lie algebra $(\mathfrak{L}(V), \theta)$. If the k-th Lie cohomology group $H^k(\mathfrak{L}(V), \mathfrak{L}(M))$ vanishes, then $H^k(V, M) = \{0\}$ and $H^k_\theta(V, M) = \{0\}$.

From Whitehead's lemmas for semisimple Lie algebras,

$$H^1(\mathfrak{L}(V), \mathfrak{L}(M)) = H^2(\mathfrak{L}(V), \mathfrak{L}(M)) = \{0\}, \qquad \text{we have}$$

Let V be a finite dimensional Jordan triple with semisimple TKK Lie algebra $\mathfrak{L}(V)$. Then for any finite dimensional V-module M, we have $H^1(V, M) = H^2(V, M) = \{0\}$. In particular, every triple derivation from V to M is inner.

In fact, we have $H^k(\mathfrak{L}(V), \mathfrak{L}(M)) = \{0\}$ for all $k \geq 3$ if $\mathfrak{L}(M)$ is a nontrivial irreducible module over $\mathfrak{L}(V)$.

5.1 Examples of Jordan triple cocycles

If $\omega \in A^2(V, M)$ is extendable with $\mathfrak{L}_2(\omega) \in Z^2(\mathfrak{L}(V), \mathfrak{L}(M))$, then $\omega = 0$.

Insert Example 5.4 here

We have seen in the first example that there are no non-zero extendable elements $\omega \in Z^2(V,M)$ with $\mathfrak{L}_2(\omega) \in Z^2(\mathfrak{L}(V),\mathfrak{L}(M))$. The next example examines this phenomenon for extendable $\omega \in A^3(V,M)$ with $\mathfrak{L}_3(\omega) \in Z^3_\theta(\mathfrak{L}(V),\mathfrak{L}(M))$.

For $a,b \in V$ and $m \in M$, $[a,b] := a\Box b - b\Box a$ and $[m,a] := m\Box a - a\Box m$.

Let ω be an extendable element of $A^3(V,M)$. Then its Lie extension $\mathfrak{L}_3(\omega)$ is a Lie 3-cocycle in $A^3_\theta(\mathfrak{k}(V),\mathfrak{k}(M))$ if and only if ω satisfies the following three conditions:

$$[a,b]\omega(x,y,z) = \omega([a,b]x,y,z) + \omega(x,[a,b]y,z) + \omega(x,y,[a,b]z) \qquad (6)$$

for all $a,b,x,y,z \in V$;

$$[\omega(a,b,c),d] = [\omega(d,b,c),a] = [\omega(a,b,d),c] = [\omega(a,d,c),b] \qquad (7)$$

for all $a,b,c,d \in V$; and

$$[\omega(x,y,[a,b]z),c] = 0. \qquad (8)$$

for all $x,y,z,a,b,c \in V$.

5.2 Examples of TKK Lie algebras

Let A be a unital associative algebra with Lie product the commutator $[x,y]=xy-yx$, Jordan product the anti-commutator $x\circ y=(xy+yx)/2$ and Jordan triple product $\{xyz\}=(xyz+zyx)/2$ (or $\{xyz\}=(xy^*z+zy^*x)/2$ if A has an involution). Denote by $Z(A)$ the center of A and by $[A,A]$ the set of finite sums of commutators.

Proposition ([?, Chapter 12], [?, pp. 809–810])

Let A be a unital associative algebra with or without an involution considered as a Jordan triple system. If $Z(A)\cap[A,A]=\{0\}$, then the mapping $(x,a\,\square\,b,y)\mapsto\begin{bmatrix} ab & x \\ y & -ba\end{bmatrix}$ is an isomorphism of the TKK Lie algebra $\mathfrak{L}(A)$ onto the Lie subalgebra

$$\left\{\begin{bmatrix} u+\sum[v_i,w_i] & x \\ y & -u+\sum[v_i,w_i]\end{bmatrix} : u,x,y,v_i,w_i\in A\right\} \qquad (9)$$

of the Lie algebra $M_2(A)$ with the commutator product.

Proposition

Let V be a finite von Neumann algebra. Then $\mathfrak{L}(V)$ is isomorphic to the Lie algebra $[M_2(V), M_2(V)]$.

In a properly infinite von Neumann algebra, the assumption $Z(A) \cap [A, A] = \{0\}$ fails since $A = [A, A]$.

This assumption also fails in the Murray-von Neumann algebra of measurable operators affiliated with a factor of type II_1 ([**?**]).

For a finite factor of type I_n, the proposition states that the classical Lie algebras $sl(2n, \mathbb{C})$ of type A are TKK Lie algebras.

Similarly, the TKK Lie algebra of a Cartan factor of type 3 on an n-dimensional Hilbert space is the classical Lie algebra $sp(2n, \mathbb{C})$ of type C ([**?**, Theorem 3,p. 131]).

More examples of TKK Lie algebras can be found in [**?**, 1.4] and [**?**, Chapter III].

6. Structural transformations

Let V be a Jordan triple and M a triple V-module. A mapping $\omega : V \to M$ is called an *inner triple derivation* if it is of the form

$$\omega = \sum_{i=1}^{k} (m_i \,\Box\, v_i - v_i \,\Box\, m_i) \in M_0$$

for some $m_1, \ldots, m_k \in M$ and $v_1, \ldots, v_k \in V$. Note that $\omega^\natural = -\omega$ and $(0, \omega, 0) \in \mathfrak{k}(M)$.

$B^1_\theta(V, M)$ coincides with the space of inner triple derivations from V to M.

$Z^1_\theta(V, M)$ coincides with the set of triple derivations of V

The first involutive cohomology group $H^1_\theta(V, M) = Z^1_\theta(V, M)/B^1_\theta(V, M)$ is the space of triple derivations modulo the inner triple derivations of V into M. This will be generalized shortly.

A (conjugate-) linear transformation $S : V \to M$ is said to be a *structural transformation* if there exists a (conjugate-) linear transformation $S^* : V \to M$ such that

$$S\{xyx\} + \{x(S^*y)x\} = \{xySx\}$$

$$S^*\{xyx\} + \{x(Sy)x\} = \{xyS^*x\}.$$

By polarization, this property is equivalent to

$$S\{xyz\} + \{x(S^*y)z\} = \{zySx\} + \{xySz\}$$

$$S^*\{xyz\} + \{x(Sy)z\} = \{zyS^*x\} + \{xyS^*z\}.$$

- A triple derivation D is a a structural transformation S with $S^* = -S$.
- The space of *inner structural transformations* coincides with the space M_0.
- Triple derivations which are inner structural transformations are inner
- If ω is a structural transformation, then $\omega - \omega^*$ is a triple derivation
- If ω is a triple derivation, $i\omega$ is a structural transformation, inner if ω is inner.

Proposition

Let ψ be a Lie derivation of $\mathfrak{L}(V)$ into $\mathfrak{L}(M)$. Then

(i) $J(\psi) : V \to M$ is a structural transformation with $(J\psi)^* = -J\psi'$ where $\psi' = \widetilde{\theta}\psi\theta$.

(ii) If ψ is θ-invariant, then $\psi' = \psi$ and $J\psi$ is a triple derivation.

(iii) If ψ is an inner derivation then $J\psi$ is an inner structural transformation. In particular, if ψ is a θ-invariant inner derivation then $J\psi$ is an inner triple derivation.

Conversely, let ω be a structural transformation.

(iv) The mapping $D = \frac{1}{2}\mathfrak{L}_1(\omega - \omega^*) : \mathfrak{L}(V) \to \mathfrak{L}(M)$ defined by

$$D(x, a\,\square\, b, y) = \frac{1}{2}(\omega(x)-\omega^*(x), \omega(a)\,\square\, b - a\,\square\,\omega^*(b) - \omega^*(a)\,\square\, b + a\,\square\,\omega(b), \omega(y)$$

is a derivation of the Lie algebra $\mathfrak{L}(V)$ into $\mathfrak{L}(M)$.

(v) D is θ-invariant if and only if ω is a triple derivation, that is, $\omega^* = -\omega$.

(vi) If ω is an inner structural transformation then D is an inner derivation. In particular, if ω is an inner triple derivation then D is a θ-invariant inner derivation.

The following theorem provides some significant infinite dimensional examples of Lie algebras in which every derivation is inner. Its proof is in the spirit of [**?**].

Theorem

Let V be a von Neumann algebra considered as a Jordan triple system with the triple product $\{xyz\} = (xy^*z + zy^*x)/2$. Then every structural transformation on V is an inner structural transformation. Hence, every derivation of the TKK Lie algebra $\mathfrak{L}(V)$ is inner.