Recent Advances in the Theory of Derivations on Jordan structures

Jordan Geometric Analysis and Applications
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Outline

1. Jordan Derivations

2. Jordan Weak*-Amenability

3. Jordan 2-cocycles

4. Algebras of unbounded operators

5. Local and 2-local derivations

NOTE: Jordan will mean Jordan algebra or Jordan triple
(and usually both)

1. Jordan Derivations

Building on earlier work of Kadison, Sakai proved that very derivation $\delta : M \to M$ of a von Neumann algebra into itself is inner (1966).

$$\delta(ab) = a\delta(b) + \delta(a)b \quad , \quad \delta(x) = \text{ ad}a(x) = ax - xa$$

Thus the first Hochschild cohomology group $H^1(M, M)$ vanishes for any von Neumann algebra M.

Building on earlier work of Bunce and Paschke, Haagerup showed in 1983 that every derivation $\delta : M \to M_*$ of a von Neumann algebra into its predual is inner, and as a consquence that every C*-algebra is weakly amenable. .

$$\delta(ab) = a.\delta(b) + \delta(a).b \quad , \quad \delta(x) = \text{ ad}\varphi(x) = \varphi.x - x.\varphi$$
$$\varphi.x(y) = \varphi(xy) \quad , \quad x.\varphi(y) = \varphi(yx)$$

Thus the first Hochschild cohomology group $H^1(M, M_*)$ vanishes for any von Neumann algebra M. (So does $H^1(A, A^*)$ for every C*-algebra)

PROPOSITION 1 (special case of Upmeier 1980)

Let M be any von Neumann algebra. Then every Jordan derivation of M is an inner Jordan derivation. Thus the first "Jordan cohomology group" $H^1_J(M,M)$ vanishes for any von Neumann algebra M.

Earlier History

FD SS char 0: **Jacobson** 1949, 1951; char $\neq 2$: **Harris** 1959

Definition

When x_0 is an element in a Jordan Banach A-module, X, over a Jordan Banach algebra [a], A, for each $b \in A$, the mapping $\delta_{x_0,b} = [L(b), L(x_0)] : A \to X$,

$$\delta_{x_0,b}(a) := (x_0 \circ a) \circ b - (b \circ a) \circ x_0, \ (a \in A),$$

is a Jordan derivation. Finite sums of derivations of this form are called **inner Jordan derivations**.

[a] For purposes of this talk, Jordan algebra means an associative algebra with the product $a \circ b = (ab + ba)/2$, so for a Jordan derivation $D(a^2) = 2a \circ D(a)$ is enough.

Commutators in von Neumann algebras

Pearcy-Topping '69; Fack-delaHarpe '80

If M is a finite von Neumann algebra, then every element of M of central trace zero is a finite sum of commutators

Halmos '52,'54; Brown-Pearcy-Topping '68; Halpern '69

If M is properly infinite (no finite central projections), then every element of M is a finite sum of commutators

Thus for any von Neumann algebra, we have $M = Z(M) + [M, M]$, where $Z(M)$ is the center of M and $[M, M]$ is the set of finite sums of commutators in M.

PROOF of PROPOSITION 1

Suppose δ is a Jordan derivation of M. Then δ is an associative derivation (Sinclair) and by Kadison-Sakai, $\delta(x) = ax - xa$ where $a = z + \sum [x_i, y_i]$ with $z \in Z(M)$ and $x_i, y_i \in M$. Since $\mathrm{ad}\,[x, y] = 4[L(x), L(y)]$, δ is an inner Jordan derivation. Q.E.D.

PROPOSITION 2
(special case of Ho-Martinez-Peralta-Russo 2002)

Every Jordan triple derivation of M is an inner triple derivation. Thus $H_t^1(M, M) = 0$

Earlier History

FD SS char 0: **Meyberg** 1972: Jordan Pair: **Loos** 1977, **Kühn-Rosendahl** 1978

Definition

Let E be a Jordan triple[a] and let X be a triple E-module. For each $b \in E$ and each $x_0 \in X$, the mapping $\delta = L(b, x_0) - L(x_0, b) : E \to X$, defined by

$$\delta(a) = \{b, x_0, a\} - \{x_0, b, a\} \quad (a \in E), \tag{1}$$

is a triple derivation from E into X. Finite sums of derivations of the form $\delta(b, x_0)$ are called **inner triple derivations**.

[a] For purposes of this talk, a Jordan triple is an associative *-algebra with the triple product $\{a, b, c\} = (ab^*c + cb^*a)/2$ and a triple derivation satisfies $\delta\{a, b, c\} = \{\delta a, b, c\} + \{a, \delta b, c\} + \{a, b, \delta c\}$

LEMMA

Let A be a unital Banach *-algebra equipped with the ternary product given by $\{a, b, c\} = \frac{1}{2}\,(ab^*c + cb^*a)$ and the Jordan product $a \circ b = (ab + ba)/2$.

- Let D be an inner derivation, that is, $D = \operatorname{ad} a$ for some a in A. Then D is a *-derivation whenever $a^* = -a$. Conversely, if D is a *-derivation, then $a^* = -a + z$ for some z in the center of A.
- Every triple derivation is the sum of a Jordan *-derivation and an inner triple derivation.

PROOF of PROPOSITION 2

It suffices to show that a self-adjoint Jordan derivation δ of M in an inner triple derivation. Such a δ is an associative derivation (Sinclair) and by Kadison-Sakai and the Lemma, $\delta(x) = ax - xa$ where $a^* + a = z$ is a self adjoint element of the center of M. Since $M = Z(M) + [M, M]$, where $Z(M)$ denotes the center of M, we can therefore write

$$a = z' + \sum_j [b_j + ic_j, b_j' + ic_j'],$$

where b_j, b_j', c_j, c_j' are self adjoint elements of M and $z' \in Z(M)$.

It follows that

$$0 = a^* + a - z = (z')^* + z' - z + 2i \sum_j ([c_j, b_j'] + [b_j, c_j'])$$

so that $\sum_j ([c_j, b_j'] + [b_j, c_j'])$ belongs to the center of M. We now have

$$\delta = \text{ad}\, a = \text{ad} \sum_j ([b_j, b_j'] - [c_j, c_j']) \qquad (2)$$

We have just seen that a self adjoint Jordan derivation δ of M has the form (2). A direct calculation shows that δ is equal to the inner triple derivation

$$\sum_j \left(L(b_j, 2b_j') - L(2b_j', b_j) - L(c_j, 2c_j') + L(2c_j', c_j) \right).$$

Thus, every triple derivation is inner.

2. Jordan weak*-amenability

This section represents joint work with Robert Pluta

Theorem 1 Let M be a von Neumann algebra.

(a) If every Jordan derivation of M into M_* is approximated in norm by inner Jordan derivations, then M is finite.

(b) Conversely, if M is a finite von Neumann algebra acting on a separable Hilbert space or if M is a finite factor, then every Jordan derivation of M into M_* is approximated in norm by inner Jordan derivations.

Corollary (Cohomological characterization of finiteness)

If M acts on a separable Hilbert space, or if M is a factor, then M is finite if and only if every Jordan derivation of M into M_* is approximated in norm by inner Jordan derivations.

Theorem 1 and its corollary hold with Jordan derivation replaced by Jordan triple derivation. An important role in the proofs is played by commutators in the predual of a von Neumann algebra. (More about that later)

Theorem 2 Let M be an infinite factor

The complex vector space of Jordan derivations of M into M_*, modulo the norm closure of the inner Jordan derivations, has dimension 1.

Corollary (Zero-One Law)

If M is a factor, the linear space of Jordan derivations into the predual, modulo the norm closure of the inner Jordan derivations, has dimension 0 or 1: It is zero if the factor is finite; and it is 1 if the factor is infinite.

Theorem 2 and its corollary hold with Jordan derivation replaced by Jordan triple derivation

Summary: If M is a factor,

$$M \text{ is infinite } \Leftrightarrow \frac{\text{Jordan Derivations into } M_*}{\textbf{Norm closure} \text{ of inner } \textbf{Jordan} \text{ derivations into } M_*} \sim \mathbb{C}$$

$$M \text{ is finite } \Leftrightarrow \frac{\text{Jordan Derivations into } M_*}{\textbf{Norm closure} \text{ of inner } \textbf{Jordan} \text{ derivations into } M_*} = 0$$

$$M \text{ is infinite } \Leftrightarrow \frac{\text{Jordan triple Derivations into } M_*}{\textbf{Norm closure} \text{ of inner } \textbf{triple} \text{ derivations into } M_*} \sim \mathbb{R}$$

$$M \text{ is finite } \Leftrightarrow \frac{\text{Jordan triple Derivations into } M_*}{\textbf{Norm closure} \text{ of inner } \textbf{triple} \text{ derivations into } M_*} = 0$$

Which von Neumann algebras are Jordan weak*-amenable? That is, every Jordan derivation into the predual is inner

Short answer: commutative, finite dimensional. Any others?

Proposition 3 Let M be a finite von Neumann algebra.

(a) If M acts on a separable Hilbert space or is a factor (hence admits a faithful normal finite trace tr), and if $\mathrm{tr}^{-1}(0) = [M_*, M]$, then M is Jordan weak*-amenable. (Extended trace)

(b) If M is a factor and M is Jordan weak*-amenable, then $\mathrm{tr}^{-1}(0) = [M_*, M]$.

Corollary

No factor of type II_1 is Jordan weak*-amenable

Proposition 3 and its corollary hold with Jordan derivation replaced by Jordan triple derivation

If M is a finite von Neumann algebra of type I_n, with $n < \infty$, we can assume

$$M = L^\infty(\Omega, \mu, M_n(\mathbb{C})) = M_n(L^\infty(\Omega, \mu)),$$

$$M_* = L^1(\Omega, \mu, M_n(\mathbb{C})_*)) = M_n(L^1(\Omega, \mu))$$

and

$$Z(M) = L^\infty(\Omega, \mu)1.$$

It is known that the center valued trace on M is given by

$$\mathrm{TR}\,(x) = \frac{1}{n}(\sum_1^n x_{ii})1 \quad , \quad \text{for } x = [x_{ij}] \in M$$

We thus define, for a finite von Neumann algebra of type I_n which has a faithful normal finite trace tr ,

$$\mathrm{TR}\,(\psi) = \frac{1}{n}(\sum_1^n \psi_{ii})\mathrm{tr} \quad , \quad \text{for } \psi = [\psi_{ij}] \in M_*.$$

(a) If $\mathrm{TR}(\psi) = 0$, then ψ vanishes on the center $Z(M)$ of M.

(b) $\psi^* = -\psi$ on $Z(M)$ if and only if $\mathrm{tr}\,(\psi(\omega))$ is purely imaginary for almost every ω.

$$\begin{aligned}
\psi(x) &= \int_\Omega \langle \psi(\omega), x(\omega)\rangle \, d\mu(\omega) = \int_\Omega \mathrm{tr}\,(\psi(\omega)x(\omega))\, d\mu(\omega) \\
&= \int_\Omega \mathrm{tr}\,([\sum_k \psi_{ik}(\omega)x_{kj}(\omega)])\, d\mu(\omega) \\
&= \int_\Omega (\sum_i \sum_k \psi_{ik}(\omega)x_{ki}(\omega))\, d\mu(\omega) \\
&= \int_\Omega (\sum_k \psi_{kk}(\omega)x_{kk}(\omega))\, d\mu(\omega) \\
&= \int_\Omega (\sum_k \psi_{kk}(\omega))f(\omega)\, d\mu(\omega) = 0
\end{aligned}$$

proving (a). As for (b), use $\psi(f \cdot 1) = \int_\Omega f(\omega)\mathrm{tr}\,(\psi(\omega))\, d\mu(\omega)$.

Proposition 4

Let M be a finite von Neumann algebra of type I_n with $n < \infty$, which admits a faithful normal finite trace tr (equivalently, M is countably decomposable $= \sigma$-finite). Then M is Jordan weak*-amenable if and only if

$$\mathrm{TR}^{-1}(0) = [M_*, M].$$

Corollary

Let M be a finite von Neumann algebra of type I_n admitting a faithful normal finite trace tr. If $\mathrm{tr}^{-1}(0) = [M_*, M]$, then M is Jordan weak*-amenable.

Proposition 4 and its corollary hold with Jordan derivation replaced by Jordan triple derivation

Problem

Is a finite von Neumann algebra of type I Jordan weak*-amenable? or triple weak*-amenable? If M admits a faithful normal finite trace, is

$$\mathrm{TR}^{-1}(0) = [M_*, M]?$$

"A veritable army of researchers took the theory of derivations of operator algebras to dizzying heights—producing a theory of cohomology of operator algebras as well as much information about automorphisms of operator algebras."

—Dick Kadison (Which Singer is that? 2000)

It is conjectured that all of the Hochschild cohomology groups $H^n(A, A)$ of a von Neumann algebra A vanish and that this is known to be true for most of them. In addition to associative algebras, cohomology groups are defined for Lie algebras and to some extent, for Jordan algebras. Since the structures of Jordan derivations and Lie derivations on von Neumann algebras are well understood, isn't it time to study the higher dimensional non associative cohomology of a von Neumann algebra? This section will be an introduction to the first and second Jordan cohomology groups of a von Neumann algebra. (Spoiler alert: Very little is known about the second Jordan cohomology group.)

3. Jordan 2-cocycles

Let M be a von Neumann algebra. A **Hochschild 2-cocycle** is a bilinear map $f : M \times M \to M$ satisfying

$$af(b,c) - f(ab,c) + f(a,bc) - f(a,b)c = 0 \tag{3}$$

EXAMPLE: **Hochschild 2-coboundary**

$$f(a,b) = a\mu(b) - \mu(ab) + \mu(a)b \quad , \qquad \mu : M \to M \text{ linear}$$

A **Jordan 2-cocycle** is a bilinear map $f : M \times M \to M$ satisfying

$$f(a,b) = f(b,a) \text{ (symmetric)}$$

$$\begin{aligned} &f(a^2, a\circ b) + f(a,b)\circ a^2 + f(a,a)\circ(a\circ b) \\ &-f(a^2\circ b, a) - f(a^2,b)\circ a - (f(a,a)\circ b)\circ a = 0 \end{aligned} \tag{4}$$

EXAMPLE: **Jordan 2-coboundary**

$$f(a,b) = a\circ\mu(b) - \mu(a\circ b) + \mu(a)\circ b, \quad , \quad \mu : M \to M \text{ linear}$$

$$H^1(M,M) = \frac{\text{1-cocycles}}{\text{1-coboundaries}} = \frac{\text{derivations}}{\text{inner derivations}}$$

$$H^1_J(M,M) = \frac{\text{Jordan 1-cocycles}}{\text{Jordan 1-coboundaries}} = \frac{\text{Jordan derivations}}{\text{inner Jordan derivations}}$$

$$H^2(M,M) = \frac{\text{2-cocycles}}{\text{2-coboundaries}} \quad , \quad H^2_J(M,M) = \frac{\text{Jordan 2-cocycles}}{\text{Jordan 2-coboundaries}}$$

For almost all von Neumann algebras, $H^2(M,M) = 0$. How about $H^2_J(M,M)$?

FD char 0: **Albert** 1947, **Penico** 1951; char $\neq 2$: **Taft** 1957

Two elegant approaches: Jordan classification; Lie algebras
One inelegant approach: solving linear equations

Linear algebra approach-Level 1

Let h be a Hochschild 1-cocycle, that is, a linear map $h : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfying $h(ab) - ah(b) - h(a)b = 0$. To show that there is an element $x \in M_n(\mathbb{C})$ such that $h(a) = xa - ax$, it is enough to prove this with $a \in \{e_{ij}\}$. With

$$x = \sum_{p,q} x_{pq} e_{pq}. \tag{5}$$

and γ_{ijpq} defined by

$$h(e_{ij}) = \sum_{p,q} \gamma_{ijpq} e_{pq}, \tag{6}$$

we arrive at the system of linear vector equations

$$\sum_{p,q} \gamma_{ijpq} e_{pq} = \sum_{p,q} \delta_{qi} x_{pq} e_{pj} - \sum_{p,q} \delta_{jp} x_{pq} e_{iq}. \tag{7}$$

with n^2 unknowns x_{ij}. Then any solution of (7) proves the result.

Linear algebra approach-Level 2

Let h be a Hochschild 2-cocycle, that is, a bilinear map $h : M_n(\mathbb{C}) \times M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfying $ah(b,c) - h(ab,c) + h(a,bc) - h(a,b)c = 0$.
To show that there is a linear transformation $\mu : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that $h(a,b) = \mu(ab) - a\mu(b) - \mu(a)b$, it is enough to prove that this holds with $a, b \in \{e_{ij}\}$, that is

$$h(e_{ij}, e_{kl}) = \delta_{jk}\mu(e_{il}) - e_{ij}\mu(e_{kl}) - \mu(e_{ij})e_{kl}. \tag{8}$$

With $\mu(e_{ij}) = \sum_{k,l} \mu_{ijkl} e_{kl}$ and γ_{ijklpq} defined by $h(e_{ij}, e_{kl}) = \sum_{p,q} \gamma_{ijklpq} e_{pq}$, we arrive at the system of n^6 linear equations

$$\sum_{p,q} \gamma_{ijklpq} e_{pq} = \sum_{p,q} \delta_{jk}\mu_{ilpq} - \sum_{p,q} \delta_{jp}\mu_{klpq} e_{iq} - \sum_{p,q} \delta_{qk}\mu_{ijpq} e_{pl}, \tag{9}$$

with n^4 unknowns μ_{ijkl}. Then any solution of (9) proves (8).

Linear algebra approach-Level 3

Let $M = M_2(L^\infty(\Omega))$ be a finite von Neumann algebra of type I_n with $n = 2$. Let f be a Jordan 2-cocycle, that is, a symmetric bilinear map $f : M \times M \to M$ with

$$f(a^2, ab) + f(a, b)a^2 + f(a, a)(ab) - f(a^2 b, a) - f(a^2, b)a - (f(a, a)b)a = 0. \quad (10)$$

(To save space, ab denotes the Jordan product in the associative algebra M)

To show that there is a linear transformation $\mu : M \to M$ such that

$$f(a, b) = \mu(ab) - a\mu(b) - \mu(a)b, \quad (11)$$

it is enough to prove, for $a, b \in Z(M)$,

$$f(ae_{ij}, be_{kl}) = \delta_{jk}\mu(abe_{il}) - ae_{ij}\mu(be_{kl}) - \mu(ae_{ij})be_{kl}. \quad (12)$$

With $\mu(ae_{ij}) = \sum_{k,l} \mu_{ijkl}(a)e_{kl}$ and $\gamma_{ijklpq}(a,b) \in Z(M)$ defined by

$$f(ae_{ij}, be_{kl}) = \sum_{p,q} \gamma_{ijklpq}(a,b)e_{pq}, \tag{13}$$

we arrive at the system of n^6 linear vector equations with $3n^4$ unknowns $\mu_{ijkl}(ab), \mu_{ijkl}(a), \mu_{ijkl}(b)$

$$\begin{aligned} 2\sum_{p,q} \gamma_{ijklpq}(a,b)e_{pq} &= \delta_{jk} \sum_{p,q} \mu_{ilpq}(ab)e_{pq} \\ &- \sum_{p,q} a(\delta_{jp}\mu_{klpq}(b)e_{iq} + \delta_{iq}\mu_{klpq}(b)e_{pj}) \\ &- \sum_{p,q} b(\delta_{qk}\mu_{ijpq}(a)e_{pl} + \delta_{lp}\mu_{ijpq}(a)e_{kq}). \end{aligned} \tag{14}$$

Then any solution of (14) proves (12) and hence (11).

Some properties of Jordan 2-cocycles

Proposition

Every symmetric Hochschild 2-cocycle is a Jordan 2-cocycle.

Recall that every Jordan derivation on a semisimple Banach algebra is a derivation (Sinclair). If every Jordan 2-cocycle was a Hochschild 2-cocycle, GAME OVER

Proposition Let M be a von Neumann algebra.

(a) Let $f : M \times M \to M$ be defined by $f(a,b) = a \circ b$. Then f is a Jordan 2-cocycle with values in M, which is not a Hochschild 2-cocycle unless M is commutative.

(b) If M is finite with trace tr, then $f : M \times M \to M_*$ defined by $f(a,b)(x) = \mathrm{tr}\,((a \circ b)x)$ is a Jordan 2-cocycle with values in M_* which is not a Hochschild 2-cocycle unless M is commutative.

Proposition

Let f be a Jordan 2-cocycle on the von Neumann algebra M. Then $f(1,x) = xf(1,1)$ for every $x \in M$ and $f(1,1)$ belongs to the center of M.

Recall the definition of Jordan 2-cocycle

$$f(a^2, a \circ b) + f(a, b) \circ a^2 + f(a, a) \circ (a \circ b) \tag{15}$$

$$-f(a^2 \circ b, a) - f(a^2, b) \circ a - (f(a, a) \circ b) \circ a = 0$$

Proof of part (a) of the second proposition

Let $f(a, b) = a \circ b$. The equation (15) reduces to

$$a^2 \circ (a \circ b) + (a \circ b) \circ a^2 + a^2 \circ (a \circ b) - (a^2 \circ b) \circ a - (a^2 \circ b) \circ a - (a^2 \circ b) \circ a,$$

which is zero by the Jordan axiom, so f is a Jordan 2-cocycle.
If this f were a Hochschild 2-cocycle, we would have

$$c(a \circ b) - (ca) \circ b + c \circ (ab) - (c \circ a)b = 0,$$

which reduces to $[[c, b], a] = 0$ and therefore $[M, M] \subset Z(M)$ (the center of M). Since $M = Z(M) + [M, M]$, M is commutative. This proves (a).

4. Algebras of unbounded operators

Murray-von Neumann algebras

Physics considerations demand that the Hamiltonian of a quantum system will, in general, correspond to an unbounded operator on a Hilbert space H. These unbounded operators will not lie in a von Neumann algebra, but they may be affiliated with the von Neumann algebra corresponding to the quantum system. In general, unbounded operators do not behave well with respect to addition and multiplication. However, as noted by Murray and von Neumann in 1936, for the finite von Neumann algebras, their families of affiliated operators do form a *-algebra. Thus, it natural to study derivations of algebras that include such unbounded operators.

A closed densely defined operator T on a Hilbert space H is affiliated with a von Neumann algebra R when $UT = TU$ for each unitary operator U in R', the commutant of R. If operators S and T are affiliated with R, then S + T and ST are densely defined, preclosed and their closures are affiliated with R. Such algebras are referred to as **Murray-von Neumann algebras**.

If R is a finite von Neumann algebra, we denote by $A_f(R)$ its associated Murray-von Neumann algebra. It is natural to conjecture that every derivation of $A_f(R)$ should be inner. Kadison and Liu proved in 2014 that extended derivations of $A_f(R)$ (those that map R into R) are inner.

They also proved that each derivation of $A_f(R)$ with R a factor of type II_1 that maps $A_f(R)$ into R is 0.

In other words, by requiring that the range of the derivation is in R, the bounded part of $A_f(R)$, allows a noncommutative unbounded version of the well-known Singer-Wermer conjecture.

This extends to the general von Neumann algebra of type II_1

We turn our attention now to commutators in $A_f(N)$, where N is a finite von Neumann algebra. The following theorem is related to the work of Kadison and Liu described above.

Theorem (Liu 2011)

Let N be II_1-factor. If the element b of N is a commutator of self-adjoint elements in $A_f(N)$, then b has trace zero. In particular, the scalar operator $i1 \in N$ is not a commutator of self-adjoint elements in $A_f(N)$.

This has been complemented as follows.

Theorem (Thom 2014)

Let $a, b \in A_f(N)$ where N is a II_1-factor. Assume that either a, b belong to the Haagerup-Schultz algebra (a *-subalgebra of $A_f(N)$) or if $ab \in A_f(N)$ is conjugate to a self-adjoint element. If $[a, b] = \lambda 1$, then $\lambda = 0$.

As for sums of commutators, we have the following theorem.

Theorem (Thom 2014)

Let N be a II_1-factor. There exist $a, b, c, d \in A_f(N)$ such that $1 = [a, b] + [c, d]$.
Every element of $A_f(N)$ is the sum of two commutators.

Algebras of measurable operators

Noncommutative integration theory was initiated by Segal in 1953, who considered new classes of algebras of unbounded operators, in particular the algebra $S(M)$ of all measurable operators affiliated with a von Neumann algebra M.

A study of the derivations on the algebra $S(M)$ was initiated by Ayupov in 2000.

For example, in the commutative case $M = L^\infty(\Omega, \Sigma, \mu)$, $S(M)$ is isomorphic to the algebra $L^0(\Omega)$ of all complex measurable functions and it is shown by Ber-Chillin-Sukochev in 2006 that $L^0(0, 1)$ admits nonzero derivations which are discontinuous in the measure topology.

The study of derivations on various subalgebras of the algebra $LS(M)$ of all locally measurable operators in the general semifinite case was initiated by Alberio-Ayupov-Kudaybergenov in 2007-8, with the most complete results obtained in the type I case, 2009.
(See the 2010 survey by Ayupov-Kudaybergenov.

It was shown by Alberio-Ayupov-Kudaybergenov in 2009 (and by Ber-dePadgter-Sukochev in 2011 for separable predual) that if M is properly infinite and of type I, then every derivation of $LS(M)$ is continuous in the local measure topology $t(M)$ on $LS(M)$. The same holds for M of type III as shown by Ayupov-Kudaybergenov in 2010 and for type II_∞ (Ber-Chillin-Sukochev 2013).

The following illustrates the state of the art.

Theorem (Ber-Chillin-Sukochev 2014)

Let M be any von Neumann algebra. Every derivation on the *-algebra $LS(M)$ continuous with respect to the topology $t(M)$ is inner.

Corollary

If M is a properly infinite von Neumann algebra, then every derivation on $LS(M)$ is inner.

5. Local derivations on JB^*-triples

Linear maps which agree with a derivation at each point are called local derivations. These have been studied in the Banach setting by Kadison 1990, Johnson 2001, Ajupov et. al 2009-11, among others.

Kadison proved that a continuous local derivation from a von Neumann algebra into a Banach dual module is a derivation.

Let X and Y be Banach spaces. A subset $\mathcal{D}$ of the Banach space $B(X, Y)$, of all bounded linear operators from X into Y, is called *algebraically reflexive* in $B(X, Y)$ when it satisfies the property:

$$T \in B(X, Y) \text{ with } T(x) \in \mathcal{D}(x),\ \forall x \in X \Rightarrow T \in \mathcal{D}. \tag{16}$$

Algebraic reflexivity of $\mathcal{D}$ in the space $L(X, Y)$, of all linear mappings from X into Y, a stronger version of the above property not requiring continuity of T, is defined by:

$$T \in L(X, Y) \text{ with } T(x) \in \mathcal{D}(x),\ \forall x \in X \Rightarrow T \in \mathcal{D}. \tag{17}$$

In 1990, Kadison proved that (16) holds if $\mathcal{D}$ is the set $\mathrm{Der}(M, X)$ of all (associative) derivations on a von Neumann algebra M into a dual M-bimodule X.

Johnson extended Kadison's result by establishing that the set $\mathcal{D} = \mathrm{Der}(A, X)$, of all (associative) derivations from a C*-algebra A into a Banach A-bimodule X satisfies (17).

Michael Mackey gave a talk on this topic at the conference in honor of Cho-Ho Chu's 65th birthday in May 2012 in Hong Kong. He proved that every continuous local derivation on a JBW^*-triple is a derivation, and he suggested some problems, among them whether every local derivation on a JB^*-triple into itself or into a Banach module is automatically continuous, and if so, whether it is a derivation. There are other problems in this area, some involving nonlinear maps. Many of these have now been answered.

Algebraic reflexivity of the set of local triple derivations on a C^*-algebra and on a JB^*-triple have been studied in 2013-14 by Peralta, Polo, Burgos, Garces, Molina.

More precisely, Mackey proves that the set $\mathcal{D} = \mathrm{Der}_t(M)$, of all triple derivations on a JBW^*-triple M satisfies (16).

The result has been supplemented by Burgos, Fernandez-Polo and Peralta who prove that for each JB^*-triple E, the set $\mathcal{D} = \mathrm{Der}_t(E)$ of all triple derivations on E satisfies (17).

In what follows, *algebraic reflexivity* will refer to the stronger version (17) which does not assume the continuity of T.

In 1995, Brešar and Šemrl proved that the set of all (algebra) automorphisms of $B(H)$ is algebraically reflexive whenever H is a separable, infinite-dimensional Hilbert space.

Given a Banach space X. A linear mapping $T : X \to X$ satisfying the hypothesis at (17) for $\mathcal{D} = \text{Aut}(X)$, the set of automorphisms on X, is called a *local automorphism*.

Larson and Sourour showed in 1990 that for every infinite dimensional Banach space X, every surjective local automorphism T on the Banach algebra $B(X)$, of all bounded linear operators on X, is an automorphism.

Motivated by the results of Šemrl in 1997, references witness a growing interest in a subtle version of algebraic reflexivity called *algebraic 2-reflexivity*

A subset $\mathcal{D}$ of the set $\mathcal{M}(X, Y) = Y^X$, of all mappings from X into Y, is called *algebraically 2-reflexive* when the following property holds: for each mapping T in $\mathcal{M}(X, Y)$ such that for each $a, b \in X$, there exists $S = S_{a,b} \in \mathcal{D}$ (depending on a and b), with $T(a) = S_{a,b}(a)$ and $T(b) = S_{a,b}(b)$, then T lies in $\mathcal{D}$.

A mapping $T : X \to Y$ satisfying that for each $a, b \in X$, there exists $S = S_{a,b} \in \mathcal{D}$ (depending on a and b), with $T(a) = S_{a,b}(a)$ and $T(b) = S_{a,b}(b)$ will be called a 2-local $\mathcal{D}$-mapping.

Šemrl establishes that for every infinite-dimensional separable Hilbert space H, the sets $\mathrm{Aut}(B(H))$ and $\mathrm{Der}(B(H))$, of all (algebra) automorphisms and associative derivations on $B(H)$, respectively, are algebraically 2-reflexive in $\mathcal{M}(B(H)) = \mathcal{M}(B(H), B(H))$.

Ayupov and Kudaybergenov proved in 2012 that the same statement remains true for general Hilbert spaces.

Actually, the set Hom(A), of all homomorphisms on a general C*-algebra A, is algebraically 2-reflexive in the Banach algebra $B(A)$, of all bounded linear operators on A, and the set *-Hom(A), of all *-homomorphisms on A, is algebraically 2-reflexive in the space $L(A)$, of all linear operators on A (Peralta 2014).

In recent contributions (2014), Burgos, Fernandez-Polo and Peralta prove that the set Hom(M) (respectively, Hom$_t(M)$), of all homomorphisms (respectively, triple homomorphisms) on a von Neumann algebra (respectively, on a JBW*-triple) M, is an algebraically 2-reflexive subset of $\mathcal{M}(M)$, while Ayupov and Kudaybergenov establish that set Der(M) of all derivations on M is algebraically 2-reflexive in $\mathcal{M}(M)$.

We recall that every C*-algebra A can be equipped with a ternary product of the form

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a).$$

When A is equipped with this product it becomes a JB*-triple. A linear mapping $\delta : A \to A$ is said to be a *triple derivation* when it satisfies the (triple) Leibnitz rule:

$$\delta\{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}.$$

According to the standard notation, 2-local $\text{Der}_t(A)$-mappings from A into A are called *2-local triple derivations*.

Theorem (Kudaybergenov-Oikhberg-Peralta-Russo 2014)

Every (not necessarily linear nor continuous) 2-local triple derivation on an arbitrary von Neumann algebra M is a triple derivation (hence linear and continuous), equivalently, $\text{Der}_t(M)$ is algebraically 2-reflexive in $\mathcal{M}(M)$.

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