Anti-C*-algebras
(Joint work with Robert Pluta)

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Main Theorem 1

The **normed standard embedding** $\mathcal{A}(M)$ of a **C*-ternary ring** M is the direct sum of a C*-algebra and a semisimple Banach algebra $\mathcal{B}$ with a continuous involution and a bounded approximate identity. The algebra $\mathcal{B}$ cannot be renormed to be a C*-algebra. We call $\mathcal{B}$ an **anti-C*-algebra**.

Main Theorem 2

C*-ternary rings and anti-C*-algebras are **semisimple**. In particular, TROs and anti-TROs are semisimple

Main Reference 1

R. Pluta, B. Russo, *Anti-C*-algebras*, ArXiv 2305.1227 (July 2023)

Main Reference 2

R. Pluta, B. Russo, *Ternary Operator Categories*, J. Math. Anal. Appl. **505** (2022)

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1. C*-algebra. $\|xx^*\| = \|x\|^2$

abstractly

{C*-algebras} $\subset$ {Banach *-algebras} $\subset$ {Banach algebras}

concretely

{C*-algebras} $\subset$ {operator algebras} $\subset$ {operator spaces}

Gelfand-Naimark Theorem I

Commutative C*-algebras correspond to locally compact Hausdorff spaces

Gelfand-Naimark Theorem II

Arbitrary C*-algebras correspond to norm closed self-adjoint subalgebras of $B(H)$

2. Ternary Rings of Operators

Let H and K be complex Hilbert spaces. Denote by $\mathcal{B}(H,K)$ the set of all bounded linear operators from H to K, and write $\mathcal{B}(H)$ for $\mathcal{B}(H,H)$. Consider $\mathcal{B}(H,K)$ as a Banach space with the usual operator norm and additional algebraic structure given by *ternary product* $(x,y,z) \mapsto xy^*z$, so that for every $x,y,z,u,v \in \mathcal{B}(H,K)$ we have:

$$xy^*(zu^*v) = x(uz^*y)^*v = (xy^*z)u^*v$$

$$\|xy^*z\| \leq \|x\| \, \|y\| \, \|z\| \quad \text{and} \quad \|xx^*x\| = \|x\|^3 .$$

A Banach subspace X of $\mathcal{B}(H,K)$ is called a *TRO* (**ternary ring of operators**) if $xy^*z \in X$ for every choice of $x,y,z \in X$. A TRO $X \subseteq \mathcal{B}(H,K)$ is called a *W^*-TRO* if it is weak* closed (equivalently, weak operator closed, or strong operator closed) in $\mathcal{B}(H,K)$. A TRO that is dual as a Banach space is a W^*-TRO, and every W^*-TRO has a unique Banach space predual, up to isometry (Effros-Ozawa-Ruan, 2001)

TROs are studied extensively in the book by Blecher and LeMerdy 2004, where we can find the following on page 351:

> *Around 1999, interest in TROs picked up with the important paper Effros-Ozawa-Ruan, Duke MJ, 2001. As evidenced by the number of recent papers using them, it seems that TRO and C*-module methods are playing an increasingly central role in operator space theory at the present time.*

Let X be a TRO contained in $\mathcal{B}(H, K)$. The left C^*-algebra of X, denoted by C is the C^*-subalgebra of $\mathcal{B}(K)$ generated by elements of the form xy^* with $x, y \in X$. Similarly, the right C^*-algebra of X, denoted by D, is the C^*-subalgebra of $\mathcal{B}(H)$ generated by elements of the form y^*z with $y, z \in X$ (C and D need not be unital algebras). The connection between C and D is made via the **linking C^*-algebra** of X, defined as $A_X = \left[\begin{smallmatrix} C & X \\ X^* & D \end{smallmatrix}\right]$, where $X^* = \{x^* : x \in X\}$ is the space of adjoints of elements of X.

3. C*-ternary rings

A **C*-ternary ring** was introduced by Zettl (Adv. in Math. 1983) as a complex Banach space $(Z, \|\cdot\|)$ with a ternary operation $(\cdot,\cdot,\cdot) : Z \times Z \times Z \to Z$ which is linear in the outer variables and conjugate linear in the middle variable, associative in the sense that

$$(((v, w, x), y, z) = (v, (y, x, w), z) = (v, w, (x, y, z)),$$

and for which $\|(x, y, z)\| \le \|x\|\|y\|\|z\|$ and $\|(x, x, x)\| = \|x\|^3$. In addition, if Z is a dual Banach space, it is called a **W*-ternary ring**.

Examples

TROs (xy^*z), anti-TROs ($-xy^*z$), abelian C*-algebras ($[fgh]_\pm = \pm f\overline{g}h$)

Ω compact Hausdorff, Ω_1 and open and closed subset;
$C(\Omega_1) \oplus C(\Omega - \Omega_1)$ with $[fgh] = f\overline{g}h$ on Ω_1 and $[fgh] = -f\overline{g}h$ on $\Omega - \Omega_1$

Gelfand-Naimark theorem for a C*-ternary ring Z (Adv. Math., Zettl, 1983)

(i) Z **is the direct sum of two C*-ternary subrings** Z_+ **and** Z_-**which are respectively isometrically isomorphic and isometrically anti-isomorphic to TROs**

(ii) The decomposition is unique: if Z_1 and Z_2 are C*-ternary subrings of Z with $Z = Z_1 \oplus Z_2$, Z_1 isomorphic to a TRO, and Z_2 anti-isomorphic to a TRO, then $Z_+ = Z_1$, $Z_- = Z_2$.

(iii) There exists one, and only one, operator $T : Z \to Z$ with
- $T^2 = I$;
- $T((x,y,z)) = (Tx,y,z) = (x,Ty,z) = (x,y,Tz)$
- $(Z, T \circ (x,y,z))$ is a C*-ternary ring which is isomorphic to a TRO.

A linear bijection $\varphi : Z_1 \to Z_2$ between two C*-ternary rings $(Z_1, (\cdot,\cdot,\cdot)_1)$ and $(Z_2, (\cdot,\cdot,\cdot)_2)$ is an **isomorphism** if $\varphi((x,y,z)_1) = (\varphi(x), \varphi(y), \varphi(z))_2$ and an **anti-isomorphism** if $\varphi((x,y,z)_1) = -(\varphi(x), \varphi(y), \varphi(z))_2$.

Some ideas from the proof (first: two standard definitions)

Let A be a Banach algebra. A **right Banach A-module** M consists of a Banach space M and a bilinear map $M \times A \ni (x, a) \mapsto x \cdot a \in M$ such that

1. $x \cdot (ab) = (x \cdot a) \cdot b$
2. $\|x \cdot a\| \leq \|x\|\|a\|$

Let A be a C*-algebra. A **Hilbert A-module** consists of a right Banach A-module M and a conjugate bilinear[a] map $\langle \cdot, \cdot \rangle : M \times M \to A$ such that

1. $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$
2. $\langle x \cdot a, y \rangle = \langle x, y \rangle a$
3. $\langle x, y \rangle^* = \langle y, x \rangle$

and the norm satisfies $\|x\|_M = \|\langle x, x \rangle\|^{1/2}$.

[a] linear in the first variable

The TRO $B(H, K)$ is a Hilbert $B(H)$ module

Theorem 2.6 (Zettl) (every Hilbert module is a TRO)

There is a linear isometry U of a Hilbert A-module M onto a TRO $R \subset B(H, K)$ such that

1. $U(x \cdot a) = U(x)\pi(a)$ for $x \in M, a \in A$
2. $U(y)^* U(x) = \pi(\langle x, y \rangle)$ for $x, y \in M$

Proof sketch

Let M be a Hilbert A-module for some C*-algebra A. Let $(\pi_\varphi, \mathcal{A}_\varphi)$ be the cyclic representation of A on the Hilbert space $\mathcal{A}_\varphi$ corresponding to the positive functional φ on A via the GNS construction. Then there exists a Hilbert space $\mathcal{B}_\varphi$ and a linear map $U_\varphi : M \to B(\mathcal{A}_\varphi, \mathcal{B}_\varphi)$ with the following properties:

1. $U_\varphi(x \cdot a) = U_\varphi(x)\pi_\varphi(a)$
2. $U_\varphi(y)^* U_\varphi(x) = \pi_\varphi(\langle x, y \rangle)$

$$\text{Now set } H = \oplus_\varphi \mathcal{A}_\varphi \qquad K = \oplus_\varphi \mathcal{B}_\varphi$$

$$U = \oplus_\varphi U_\varphi : M \to B(H, K) \qquad \pi = \oplus_\varphi \pi_\varphi : A \to B(H).$$

Let $(M, (\cdot,\cdot,\cdot))$ be a C*-ternary ring,

$\mathfrak{A} :=$ (the opposite algebra of) $\overline{sp}\ \{(\cdot, x, y) : x, y \in M\} \subset B(M)$ is an algebra with involution $(\cdot, x, y)^* = (\cdot, y, x)$.

(multiplication) $(\cdot, x, y)(\cdot, z, w) = (\cdot, z, w) \circ (\cdot, x, y) = ((\cdot, x, y), z, w)$
$= (\cdot, (z, y, x), w) = (\cdot, x, (y, z, w))$

(C*-property)
Let $a = \sum_i (\cdot, x_i, y_i)$ and let $z \in M$.

$$\begin{aligned} \|a(z)\|^3 &= \|(a(z), a(z), a(z))\| = \|\sum_i ((z, x_i, y_i), a(z), a(z))\| \\ &= \|\sum_i (z, (a(z), y_i, x_i), a(z)) = \|(z, a^*(a(z)), a(z))\| \\ &\leq \|z\|^2 \|aa^*\| \|a(z)\| \end{aligned}$$

so $\|a(z)\|^2 \leq \|aa^*\| \|z\|^2$, $\|a\|^2 \leq \|aa^*\| \leq \|a\| \|a^*\|$ and $\|a\| \leq \|a^*\|$, thus $\|a\|^2 = \|aa^*\|$.

M is a right Banach $\mathfrak{A}$-module via $x \cdot a = a(x)$.
$\alpha : M \times M \to A$ is defined by $\alpha(z, y) = (\cdot, y, z)$.
$M_+ = \{x \in M : \alpha(x, x) \geq 0\}$ is sub-C*-ternary ring isomorphic to a TRO.
$M_- = \{x \in M : \alpha(x, x) \leq 0\}$ is anti-isomorphic to a TRO.
$M = M_+ \oplus M_-$

This proves

Proposition 3.2 (Zettl)

If $(M, (\cdot, \cdot, \cdot))$ is a C*-ternary ring, then there exist $\mathfrak{A}$, α such that

1. $\mathfrak{A}$ is a C*-algebra, M is a right Banach $\mathfrak{A}$-module
2. $\alpha : M \times M \to \mathfrak{A}$ is conjugate linear, $\|\alpha\| \leq 1$ and
   $\alpha(x \cdot a, y) = \alpha(x, y)a \quad , \quad \alpha(x, y)^* = \alpha(y, x)$
3. $(x, y, z) = x \cdot \alpha(z, y)$
4. $\overline{sp}\ \alpha(M, M) = \mathfrak{A}$

4. Associative triple systems. Standard Embedding

The following construction is due to Ottmar Loos (1972) and is central. The complications due to taking direct sums, which were not necessary in Zettl's work, are unavoidable since the module actions defined below make essential use of the second component and are critical to the proof of a key proposition.

A vector space V with a trilinear map $m : V \times V \times V \to V$ with $m(x, y, z)$, called the triple product, and denoted by (x, y, z) is called an **associative triple system** if it satisfies

$$(x, y, (z, u, v)) = ((x, y, z), u, v) = (x, (u, z, y), v)$$

for all elements $x, y, z, u, v \in V$. If the base field is the complex numbers(usually the case), the triple product is assumed to be conjugate linear in the middle variable.

If M is an associative triple system, triple product denoted by $[hgf]$, let

$$E(M) = \mathrm{End}\,(M) \oplus [\overline{\mathrm{End}\,(M)}]^{op} \subset \mathrm{End}\,(M \oplus M)$$

where the notation $\overline{V}$ for a complex vector space means that the scalar multiplication in $\overline{V}$ is $(\lambda, v) \in \mathbb{C} \times V \mapsto \lambda \circ v = \bar{\lambda} v$.

We shall often denote the products in $E(M)^{op}$ and in $[\mathrm{End}\,(M)]^{op}$ by[a] $X \circ Y = YX$. Explicitly, for $A = (A_1, A_2)$, $A' = (A'_1, A'_2) \in E(M)$,

$$AA' = (A_1, A_2)(A'_1, A'_2) = (A_1 A'_1, A_2 \circ A'_2) = (A_1 A'_1, A'_2 A_2),$$

and for $B = (B_1, B_2)$ and $B' = (B'_1, B'_2)$ belonging to $E(M)^{op}$,

$$\begin{aligned} B \circ B' &= (B_1, B_2) \circ (B'_1, B'_2) = (B'_1, B'_2)(B_1, B_2) \\ &= (B'_1 B_1, B'_2 \circ B_2) = (B'_1 B_1, B_2 B'_2). \end{aligned}$$

[a] not to be confused with composition

Involutions, that is, conjugate linear anti-isomorphisms of order 2, are defined on $E(M)$ by

$$A = (A_1, A_2) \mapsto \overline{A} = \overline{(A_1, A_2)} = (A_2, A_1),$$

so that $\overline{AA'} = \overline{A'}\,\overline{A}$, and

$$\overline{\lambda A} = \overline{(\lambda A_1, \lambda \circ A_2)} = \overline{(\lambda A_1, \overline{\lambda} A_2)} = (\overline{\lambda} A_2, \lambda A_1) = (\overline{\lambda} A_2, \overline{\lambda} \circ A_1) = \overline{\lambda}\ \overline{A};$$

Similarly on $E(M)^{op}$ by

$$B = (B_1, B_2) \mapsto \overline{B} = \overline{(B_1, B_2)} = (B_2, B_1),$$

so that $\overline{B \circ B'} = \overline{B'} \circ \overline{B}$ and

$$\overline{\lambda B} = \overline{(\lambda B_1, \lambda \circ B_2)} = \overline{(\lambda B_1, \overline{\lambda} B_2)} = (\overline{\lambda} B_2, \lambda B_1) = (\overline{\lambda} B_2, \overline{\lambda} \circ B_1) = \overline{\lambda}\ \overline{B}.$$

For $g, h \in M$, define (left and right "multiplication" operators)

$$L(g,h) = [gh\cdot] \in \mathrm{End}(M), \quad R(g,h) = [\cdot hg] \in \overline{\mathrm{End}(M)}^{op},$$

$$\ell(g,h) = (L(g,h), L(h,g)) = ([gh\cdot],[hg\cdot]) \in E(M)$$

and

$$r(g,h) = (R(h,g), R(g,h)) = ([\cdot gh],[\cdot hg]) \in E(M)^{op}.$$

Next, define

$$L_0 = L_0(M) = \mathrm{span}\,\{\ell(g,h) : g,h \in M\} \subset E(M)$$

and

$$R_0 = R_0(M) = \mathrm{span}\,\{r(g,h) : g,h \in M\} \subset E(M)^{op}.$$

The next three lemmas follow straightforwardly from the above construction. Their statements have their origins in Loos, Manuscripta Math 1972 and are reproduced in Meyberg's lecture notes of 1972: *Lectures on algebras and triple systems*, University of Virginia 227 pp. 1972. Available at **Jordan theory preprint archive**.

Lemma 1

With the above notation

(i) $R(f,g)L(h,k) = L(h,k)R(f,g)$ [a]

(ii) $\ell(g,h)\ell(g',h') = \ell([ghg'],h') = \ell(g,[h'g'h])$ [b]

(iii) $r(g,h)\circ r(g',h') = r(g,[hg'h']) = r([g'hg],h')$, where, as indicated, the product on the left is taken in $E(M)^{op}$.

(iv) $L_0(M)$ is a *-subalgebra of $E(M)$ and $R_0(M)$ is a *-subalgebra of $E(M)^{op}$.

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[a] This is needed in the proof of the bimodule statements in Lemma 2.
[b] $L_0(M)$ is an algebra

Lemma 2

Let $A = (A_1, A_2) \in E(M)$, $B = (B_1, B_2) \in E(M)^{op}$, and $f \in M$.

(i) M is a left $E(M)$-module via $(A, f) \mapsto A \cdot f = A_1 f$, and a right $E(M)^{op}$-module via $(f, B) \mapsto f \cdot B = B_1 f$, and an (L_0, R_0^{op})-bimodule.[a]

(ii) $\overline{M}$ is a left $E(M)^{op}$-module via $(B, \overline{f}) \mapsto B \cdot \overline{f} = \overline{B_2 f}$, and a right $E(M)$-module via $(\overline{f}, A) \mapsto \overline{f} \cdot A = \overline{A_2 f}$, and an (R_0^{op}, L_0)-bimodule.[b]

[a] Recall that $R_0(M) \subset E(M)^{op}$
[b] op. cit.

(Recall that $\overline{M}$ denotes the vector space M with the element f denoted by $\overline{f}$ and with scalar multiplication defined by $(\lambda, \overline{f}) \mapsto \lambda \circ \overline{f} = \overline{\overline{\lambda} f}$. In addition, $\overline{M}$ is considered as an associative triple system with the triple product $[xyz]_{\overline{M}} = [zyx]_M$)

Let $A = (A_1, A_2) \in E(M)$, $B = (B_1, B_2) \in E(M)^{op}$, and $f \in M$.

We have the module properties

$$(AA') \cdot f = A \cdot (A' \cdot f) \text{ and } f \cdot (B \circ B') = (f \cdot B) \cdot B',$$

$$\overline{f} \cdot (AA') = (\overline{f} \cdot A) \cdot A' \text{ and } (B \circ B') \cdot \overline{f} = B \cdot (B' \cdot \overline{f}).$$

And if $A \in L_0(M), B \in R_0(M)$, then (bimodule property)

$$(A \cdot f) \cdot B = A \cdot (f \cdot B) \text{ and } (B \cdot \overline{f}) \cdot A = B \cdot (\overline{f} \cdot A),$$

Given an associative triple system M, let

$$\mathcal{A}_0 = \mathcal{A}_0(M) = L_0(M) \oplus M \oplus \overline{M} \oplus R_0(M)$$

and write the elements a of $\mathcal{A}$ as matrices

$$a = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \in \begin{bmatrix} L_0(M) & M \\ \overline{M} & R_0(M) \end{bmatrix}$$

Define multiplication and involution in $\mathcal{A}_0$ by

$$aa' = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \begin{bmatrix} A' & f' \\ \overline{g'} & B' \end{bmatrix} = \begin{bmatrix} AA' + \ell(f, g') & A \cdot f' + f \cdot B' \\ \overline{g} \cdot A' + B \cdot \overline{g'} & r(g, f') + B \circ B' \end{bmatrix} \qquad (1)$$

(the product $B \circ B'$ taken in $R_0(M) \subset E(M)^{op}$) and

$$a^\sharp = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}^\# = \begin{bmatrix} \overline{A} & g \\ \overline{f} & \overline{B} \end{bmatrix}. \qquad (2)$$

Lemma 3

$\mathcal{A}_0(M)$ is an associative *-algebra and for $f, g, h \in M$,

$$\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & g \\ 0 & 0 \end{bmatrix}^\# \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & [fgh] \\ 0 & 0 \end{bmatrix}.$$

We refer to $\mathcal{A}_0(M)$ as the **standard embedding** of M.

The map $f \overset{\phi}{\mapsto} \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}$ is a triple isomorphism of M into $\mathcal{A}_0(M)$, the latter considered as an associative triple system with triple product $ab^{\#}c$, that is, $\phi([fgh]) = \phi(f)\phi(g)^{\sharp}\phi(h)$.

The correspondence $M \to \mathcal{A}_0(M)$ is a functor from the category of associative triple systems and triple homomorphisms to the category of associative *-algebras and *-homomorphisms.

If the associative triple system M is a normed space, and $\|[hgf]\| \le \|f\|\|g\|\|h\|$, then the **normed standard embedding** of M, denoted by $\mathcal{A}(M)$, is defined in the same way but with R_0 and L_0 replaced by their closures $R(M)$ and $L(M)$ in $B(M \oplus M)$[a]. In this case, the modules defined above are continuous modules, and Banach modules if M is a Banach space.

[a] $M \oplus M$ with a fixed norm, for example $\|x \oplus y\| = (\|x\|^2 + \|y\|^2)^{1/2}$

5. More on C*-ternary rings

Let $(M, [\cdot,\cdot,\cdot])$ be a C*-ternary ring. M being a normed associative triple system, it is a left $L(M)$-Banach module via $L(M) \times M \ni (A, f) \mapsto A \cdot f = A_1 f \in M$ and a right $R(M)^{op}$-Banach module via $M \times R(M) \ni (f, B) \mapsto f \cdot B = B_1 f \in M$, and that

$$\mathcal{A} = \begin{bmatrix} L(M) & M \\ \overline{M} & R(M) \end{bmatrix} = \{a = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} : A \in L(M), B \in R(M), f, g \in M\},$$

is an algebra with multiplication (1) and involution (2) (see page 20).

Recall that the C*-algebra $\mathfrak{A}$ in Zettl's Proposition 3.2 on page 12 is the closed span of $\{[\cdot gh] : g, h \in M\}$ and thus it is *-isomorphic to $R(M)$ via the map $\mathfrak{A} \ni B_1 \mapsto \sigma(B_1) = (B_1, B_1^*) \in R(M)$. Similarly $\tau : \mathfrak{B} \to L(M)$ is the *-isomorphism $A_1 \mapsto (A_1, A_1^*)$, where $\mathfrak{B}$ is the close span of $\{[gh\cdot] : g, h \in M\}$. The C*-ternary ring M is thus both a Banach (L, R^{op})-bimodule and a Banach $(\mathfrak{B}, \mathfrak{A})$-bimodule.

The following proposition plays a key role in the rest of this talk.

Proposition

Let M be a C*-ternary ring. With the above notation, we have

(i) $R(M)$ is a C*-algebra with the norm from $B(M \oplus M)$. (Isomorphic to Zettl's algebra $\mathfrak{A}$)

(ii) M is a right Banach $R(M)^{op}$-module. (Lemma 2 on page 18)

(iii) (just a technical tool) With $\langle f|g\rangle = \langle f|g\rangle_M : M \times M \to R(M)$ defined by $\langle f|g\rangle = r(g, f) = ([\cdot gf], [\cdot fg])$, we have

$$\langle f \cdot B|g\rangle = \langle f|g\rangle \circ B.$$

(iv) If M is a right $R(M)^{op}$-Hilbert module, then $\mathcal{A}$ can be normed to be a C*-algebra. (Sketch on next page)

Proof sketch of (iv)

The map $\pi : \mathcal{A} \to B(M \oplus R)$ to the bounded operators on the right R^{op}-Hilbert module $M \oplus R$ defined, for $a = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \in \mathcal{A}$ by

$$\pi(a) \begin{bmatrix} f' \\ B' \end{bmatrix} = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \begin{bmatrix} f' \\ B' \end{bmatrix} = \begin{bmatrix} A \cdot f' + f \cdot B' \\ r(g, f') + B \circ B' \end{bmatrix}, \qquad (3)$$

is an injective *-homorphism. Letting $\|a\| = \|\pi(a)\|$ turns $\mathcal{A}$ into a C*-algebra. (Mimic of a proof in Blecher-LeMerdy book, 8.1.17)

(notation) R is a right R^{op}-Hilbert module, via
$R \times R^{op} \ni (B', B) \mapsto B \cdot B' = B' \circ B \in R$,
and $M \oplus R$ is a right R^{op}-Hilbert module, via

$$(M \oplus R) \times R^{op} \ni ((f, B'), B) \mapsto (f, B') \cdot B = (f \cdot B, B' \circ B) \in M \oplus R.$$

$$\text{and } \|(f, B)\|_{M \oplus R} = (\|f\|_M^2 + \|B\|_R^2)^{1/2}$$

Some auxiliary results

If M is a C*-ternary ring with decomposition $M = M_+ \oplus M_-$, then M_+ is a right $R(M)^{op}$-Hilbert module. Therefore, $\mathcal{A}(M_+)$ is a C*-algebra.

A surjective homomorphism between C*-ternary rings is contractive.

Let $\varphi : M \to N$ be a surjective homomorphism between C*-ternary rings M and N. Then φ extends to a *-homomorphism $\mathcal{A}(\varphi) : \mathcal{A}(M) \to \mathcal{A}(N)$

6. Anti-C*-algebras

We begin by recalling the construction of the linking algebra of a TRO (see page 6). This will motivate our introduction of an anti-C*-algebra.

Remark

Let $M \subset B(H, K)$ be a ternary ring of operators, so that M is a C*-ternary ring with the triple product $[xyz] := xy^*z$. Then the standard embedding $\mathcal{A}_0(M)$ of M is a pre-C*-algebra, which is *-isomorphic to a dense *-subalgebra of the linking C*-algebra A_M of M.

Hence $\mathcal{A}(M)$ is isomorphic to A_M if M is a TRO. (Details on the next page.)

A_M is the closure of $\begin{bmatrix} MM^* & M \\ M^* & M^*M \end{bmatrix} =$

$$\left\{ \begin{bmatrix} \sum_i x_i y_i^* & z \\ w^* & \sum_j u_j^* v_j \end{bmatrix} : x_i, y_i, u_j, v_j, z, w \in M \right\} \subset \begin{bmatrix} B(K) & B(H,K) \\ B(K,H) & B(H) \end{bmatrix}$$

with (matrix) multiplication and involution

$$\begin{bmatrix} \alpha & z \\ w^* & \beta \end{bmatrix} \times \begin{bmatrix} \alpha' & z' \\ w'^* & \beta' \end{bmatrix} = \begin{bmatrix} \alpha\alpha' + zw'^* & \alpha z' + z\beta' \\ w^*\alpha' + \beta w'^* & w^* z' + \beta\beta' \end{bmatrix},$$

$$\begin{bmatrix} \alpha & z \\ w^* & \beta \end{bmatrix}^* = \begin{bmatrix} \alpha^* & w \\ z^* & \beta^* \end{bmatrix}, \tag{4}$$

which is *-isomorphic to $\mathcal{A}_0(M)$ via the map

$$\mathcal{A}_0(M) \ni \begin{bmatrix} \ell(x,y) & z \\ \overline{w} & r(u,v) \end{bmatrix} \mapsto \begin{bmatrix} xy^* & z \\ w^* & u^* v \end{bmatrix} \in A_M. \tag{5}$$

Proposition

Let $M \subset B(H,K)$ be an **anti-TRO**, that is, as a set, M is equal to a sub-TRO of $B(H,K)$, and it is considered as a C*-ternary ring with the triple product $[xyz] := -xy^*z$. Then the standard embedding $\mathcal{A}_0(M)$ is *-isomorphic to the *-algebra $\mathcal{B}_0 =$

$$\left\{ \begin{bmatrix} \sum_i x_i y_i^* & z \\ w^* & \sum_j u_j^* v_j \end{bmatrix} : x_i, y_i, u_j, v_j, z, w \in M \right\} \subset \begin{bmatrix} B(K) & B(H,K) \\ B(K,H) & B(H) \end{bmatrix}$$

with multiplication, involution, and map (latter two same as (4) and (5))

$$\begin{bmatrix} \alpha & z \\ w^* & \beta \end{bmatrix} \cdot \begin{bmatrix} \alpha' & z' \\ w'^* & \beta' \end{bmatrix} = \begin{bmatrix} -\alpha\alpha' + zw'^* & -\alpha z' - z\beta' \\ -w^*\alpha' - \beta w'^* & w^*z' - \beta\beta' \end{bmatrix}, \tag{6}$$

$$\begin{bmatrix} \alpha & z \\ w^* & \beta \end{bmatrix}^* = \begin{bmatrix} \alpha^* & w \\ z^* & \beta^* \end{bmatrix}, \tag{7}$$

$$\mathcal{A}_0(M) \ni \begin{bmatrix} \ell(x,y) & z \\ \overline{w} & r(u,v) \end{bmatrix} \mapsto \begin{bmatrix} xy^* & z \\ w^* & u^*v \end{bmatrix} \in \mathcal{B}_0.$$

Definition

By an **anti-C*-algebra** is meant a Banach algebra of the form $\mathcal{A}(M)$, for some anti-TRO M.

matrix multiplication

$$\begin{bmatrix} + + & + + \\ + + & + + \end{bmatrix}$$

anti-C*-algebra multiplication

$$\begin{bmatrix} - + & - - \\ - - & + - \end{bmatrix}$$

Example

In the above proposition, if $M = \mathbb{C}$ with triple product $-x\overline{y}z$, then $\mathcal{A}_0(M)$ is equal to $M_2(\mathbb{C})$ as a linear space and is an associative *-algebra, with multiplication and involution given by (6) and (7) respectively. This anti-C*-algebra $(M_2(\mathbb{C}), \cdot)$ has a unit element $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and like its counterpart $(M_2(\mathbb{C}), \times)$ ($\times$ denoting matrix multiplication), it has a trivial center and no nonzero proper two-sided ideals. An element

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if the "determinant" $ad + bc \neq 0$.

Remark

Since the anti-C*-algebra $(M_2(\mathbb{C}), \cdot)$ is central and simple, according to Wedderburn's theorem, it is isomorphic to the C*-algebra $(M_2(\mathbb{C}), \times)$. However, this isomorphism cannot be a *-isomorphism, since in that case it would contradict the last statement in Main Theorem 1 (restated on page 32).

Extended Example

Any finite dimensional anti-C*-algebra is semisimple (see a theorem below) and hence isomorphic, but not *-isomorphic to a C*-algebra. It follows that if M is an infinite direct sum of finite dimensional anti-TROs, then $\mathcal{A}(M)$ is an infinite dimensional anti-C*-algebra which is isomorphic, but not *-isomorphic, to a C*-algebra.

A Wedderburn isomorphism $\phi : (M_2(\mathbb{C}), \cdot) \to (M_2(\mathbb{C}), \times)$, given by $\phi(E_{ij}) = \sum_{p,q} a_{ijpq} E_{pq}$, satisfies, for all $p, s, i, j, \ell \in \{1, 2\}$, the 32 (nonlinear) equations with 16 unknowns a_{ijpq}

$$\sum_q a_{ijpq} a_{j\ell qs} = \epsilon(ij\ell) a_{i\ell ps},$$

where $\epsilon(ij\ell)$ is defined by $\phi(E_{ij}) \times \phi(E_{j\ell}) = \phi(E_{ij} \cdot E_{j\ell}) = \epsilon(ij\ell) E_{i\ell}$.

The multiplication table for $(M_2(\mathbb{C}), \cdot)$ is

	E_{11}	E_{12}	E_{21}	E_{22}
E_{11}	$-E_{11}$	$-E_{12}$	0	0
E_{12}	0	0	E_{11}	$-E_{12}$
E_{21}	$-E_{21}$	E_{22}	0	0
E_{22}	0	0	$-E_{21}$	$-E_{22}$

Main Theorem 1 (restated)

The normed standard embedding $\mathcal{A}(M)$ of a C*-ternary ring M is the direct sum of a C*-algebra and a semisimple Banach algebra $\mathcal{B}$ with a continuous involution and a bounded approximate identity. The Banach algebra $\mathcal{B}$ cannot be renormed to be a C*-algebra.

Recall that the normed standard embedding $\mathcal{A}(M)$ of a C*-ternary ring is a *-algebra which can be normed to be a C*-algebra if M is an R^{op}-Hilbert module by using a *-isomorphism π into $B(M \oplus R)$, with $M \oplus R$ considered as a Hilbert R^{op}-module.

If M is not a Hilbert R^{op}-module, then $M \oplus R$ is just a Banach space (under any convenient ℓ_2^p-norm). In this case however, the proof that π is an injective homomorphism and the proof of completeness is essentially the same as the proof in the Hilbert module case. The semisimplicity is stated as our Main Theorem 2. Since $\mathcal{B} = \mathcal{A}(M_-)$, it cannot be a C*-algebra since this would imply that M_- is a TRO, contradicting Zettl's Gelfand Naimark theorem.

(**Digression**) The following theorem summarizes some results which characterize TROs in the context of C*-ternary rings. The equivalence of (i) and (v) in Neal-Russo, PJM 2003, answered a question posed by Zettl in 1983, namely, "characterizing the C*-ternary rings which yield $T = I$", equivalently, that are isomorphic to a TRO. Conditions (iii) and (iv) provide two new answers to Zettl's question.

Theorem

Let M be a C*-ternary ring. The following are equivalent:

(i) M is isomorphic as a C*-ternary ring to a TRO.

(ii) $M_- = 0$.

(iii) M is a right $R(M)^{op}$-Hilbert module.

(iv) $\mathcal{A}(M)$ can be normed to be a C*-algebra.

(v) M is a JB*-triple under the triple product

$$\{abc\} = \frac{[abc] + [bca]}{2}.$$

Main Theorem 2 (restated)

C*-ternary rings and anti-C*-algebras are **semisimple**. In particular, TROs and anti-TROs are semisimple

Recall that the Jacobson radical of an associative algebra A is the ideal consisting of the set of elements $x \in A$ which are quasi-invertible in every homotope A_u of A, that is, for every $u \in A$, there exists $y \in A$ (depending on x and u) such that $y - x = xuy = yux$.

Definition

The (Jacobson) **radical** Rad M of an associative triple system, such as a C*-ternary ring M, is the set of elements $x \in M$ which are quasi-invertible in every **homotope** M_u of M, that is, for every $u \in M$, there exists $y \in M$ such that

$$y - x = [yux] = [xuy].$$

A C*-ternary ring is said to be **semisimple** if its radical is 0.

Theorem

If M is a C*-ternary ring, and $\mathcal{A}(M)$ its normed standard embedding, then

(i) $\text{Rad } \mathcal{A} = [\text{Rad } L(M)] \oplus [\text{Rad } M] \oplus [\overline{\text{Rad } M}] \oplus [(\text{Rad } R(M)]$

(ii) Rad M is an ideal in M

(iii) $\mathcal{A}(M)$ is semisimple if and only if M is semisimple.

Corollary

If M is a TRO, then M is semisimple (since $\mathcal{A}(M)$ is a C*-algebra).

Proposition

Let $M \subset B(H, K)$ be an anti-TRO, that is, as a linear space, M is equal to a sub-TRO M' of $B(H, K)$, and it is considered as a C*-ternary ring with the triple product $[xyz] := -xy^*z$. Then Rad $M =$ Rad M'. Thus, an anti-TRO is semisimple.

Some ideas from the proof (of the Proposition)

Definition

A *left ideal* (resp. *right ideal*) I in an associative triple system M is a linear subspace which satisfies $[MMI] \subset I$ (resp. $[IMM] \subset I$). An *ideal* is both a left ideal and a right ideal which satisfies $[MIM] \subset I$. (In a C*-ternary ring, a subspace which is both a left ideal and a right ideal automatically satisifes $[MIM] \subset I$.)

According to Myung, Korean Math Soc. 1975, for any associative triple system N, with ternary product denoted $[xyz]$, Rad $N =$

$$\{x \in N : \text{ the principal ideal } [xNN] \text{ or } [NNx] \text{ is quasi-regular in } N\}. \quad (8)$$

Quasi-regular for the right ideal $[xNN]$, which is equivalent to quasi-regularity for left ideals, means that

$$N = \{u - [vyu] : u, y \in N, v \in [xNN]\}, \quad (9)$$

7. Ideals

There is a one to one correspondence between closed ideals in a TRO M and closed ideals in the C*-algebra $R(M) \sim \mathfrak{A}$.

This was proved directly in Bunce-Timoney, QJ Math 2013. This result was extended to C*-ternary rings in Abadie-Ferraro, Adv. Op. Thy 2017 by using the TRO $M_+ \oplus M_-^{op}$ and appealing to the Raeburn-Williams book, *Morita equivalence and continuous trace C*-algebras*, 1998. It can also be proved by using the TRO $M_+ \oplus M_-^{op}$ and appealing instead to the result of Bunce-Timoney

(in general, if $(M, [\cdot,\cdot,\cdot])$ is a C*-ternary ring, then M^{op} denotes the C*-ternary ring $(M, -[\cdot,\cdot,\cdot])$).

What about ideals in $\mathcal{A}(M)$?

We begin by modifying some earlier notation. Recall that if M is a C*-ternary ring, $\ell(f,g)$, for $f,g \in M$, is the element of $B(M \oplus M)$ defined by $\ell(f,g)(x,y) = ([fgx],[gfy])$. To emphasize the dependence on M we denote $\ell(f,g)$ by $\ell_M(f,g)$. Thus

$$L(M) := \overline{sp}\{\ell_M(f,g) : f,g \in M\} \subset B(M \oplus M).$$

Let I be a closed ideal in the C*-ternary ring M. Since I is also a C*-ternary ring,

$$L(I) := \overline{sp}\{\ell_I(f,g) : f,g \in I\} \subset B(I \oplus I).$$

We define $\tilde{L}(I) = \overline{sp}\{\ell_M(f,g) : f,g \in I\} \subset L(M) \subset B(M \oplus M).$

Note that $\ell_I(f,g) = \ell_M(f,g)|_{I \oplus I}$.

Similarly $\tilde{R}(I) = \overline{sp}\{r_M(f,g) : f,g \in I\} \subset R(M) \subset B(M \oplus M).$

Proposition

Let I be a closed ideal in the C*-ternary ring M.

(a) $\tilde{L}(I)$ is a closed two-sided ideal in the C*-algebra $L(M)$, and the map ϕ defined as $\phi(\ell_M(f,g)) = \ell_I(f,g)$, for $f,g \in I$, extends to a contractive *-homomorphism of $\tilde{L}(I)$ onto $L(I)$.

(b) $\tilde{R}(I)$ is a closed two-sided ideal in the C*-algebra $R(M)$, and the map ψ defined as $\psi(r_M(f,g)) = r_I(f,g)$, for $f,g \in I$, extends to a contractive *-homomorphism of $\tilde{R}(I)$ onto $R(I)$.

We next define $\tilde{\mathcal{A}}(I) = \begin{bmatrix} \tilde{L}(I) & I \\ \overline{I} & \tilde{R}(I) \end{bmatrix} \subset \mathcal{A}(M) = \begin{bmatrix} L(M) & M \\ \overline{M} & R(M) \end{bmatrix}$.

Proposition

Let I be a closed ideal in a C*-ternary ring M. Then $\tilde{\mathcal{A}}(I)$ is a closed self-adjoint ideal in $\mathcal{A}(M)$. The map $I \mapsto \tilde{\mathcal{A}}(I)$ is **injective** from closed ideals of M to closed self-adjoint two-sided ideals of $\mathcal{A}(M)$.

8. Some Questions

Question 1

Which C*-algebras can appear as the linking algebra of a TRO? Which semisimple Banach algebras with approximate identities and with continuous involution can appear as anti-C*-algebras?

Question 2

In what sense is the decomposition $\mathcal{A}(M) =$ C*-algebra $\oplus$ anti-C*-algebra unique? In particular, if M and N are C*-ternary rings, with *-isomorphic normed standard embeddings, does it follow that M and N are isomorphic?

Question 3

If M is a C*-ternary ring, is the map $I \mapsto \tilde{\mathcal{A}}(I)$ from closed ideals of M to closed self-adjoint two-sided ideals of $\mathcal{A}(M)$ surjective? In the special case that M is a TRO (and hence $\mathcal{A}(M)$ is a C*-algebra), this has been proved in Skeide, Algebra, Representation Theory, 2022.

thank you