Cohomology of Jordan triples via Lie algebras

University of Granada

Bernard Russo Menasche
(A joint work with Cho-Ho Chu)

University of California, Irvine

September 9, 2015

Outline

1. Introduction

A veritable army of researchers took the theory of derivations of operator algebras to dizzying heights—producing a theory of cohomology of operator algebras as well as much information about automorphisms of operator algebras—Richard Kadison 2000

In addition to associative algebras, cohomology groups are defined for Lie algebras and, to some extent, for Jordan algebras.

Since the structures of Jordan derivations and Lie derivations on von Neumann algebras are well understood, and in view of the above quotation, isn't it time to study the higher dimensional non associative cohomology of a von Neumann algebra?

The joint paper on which this talk is based was motivated by this rhetorical question.

In the paper we develop a cohomology theory for Jordan triples, including the infinite dimensional ones, by means of the cohomology of the corresponding TKK Lie algebras.

This is a standard technique in Jordan algebraic theory.

We obtain some preliminary results for von Neumann algebras.

2. Brief survey of cohomology theories

The starting point for the cohomology theory of associative algebras is the paper of Hochschild from 1945. The standard reference of the theory is the book by Cartan and Eilenberg of 1956. Two other useful references are due to Weibel (1994,1995)

- G. Hochschild, *On the cohomology groups of an associative algebra.* Ann. of Math. (2) **46** (1945), 58–67.
- H. Cartan and S. Eilenberg, *Homological algebra.* Princeton University Press, Princeton, N. J., 1956. xv+390 pp.
- C. A. Weibel, *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994. xiv+450 pp.
- C. A. Weibel, *History of homological algebra.* History of topology, 797–836, North-Holland, Amsterdam, 1999.

Shortly after the introduction of cohomology for associative algebras, there appeared a corresponding theory for Lie algebras (Chevalley-Eilenberg 1948). A standard reference is the book of Jacobson 1962. Applications can be found in the books by Fuks 1986 and Knapp 1988.

- Chevalley, Claude; Eilenberg, Samuel Cohomology theory of Lie groups and Lie algebras. Trans. Amer. Math. Soc. 63, (1948). 85–124.
- N. Jacobson, *Lie algebras.* Interscience Tracts in Pure and Applied Mathematics, No. 10 Interscience Publishers (a division of John Wiley & Sons), New York-London 1962 ix+331 pp.
- D. B. Fuks, *Cohomology of infinite-dimensional Lie algebras.* Translated from the Russian by A. B. Sosinski. Contemporary Soviet Mathematics. Consultants Bureau, New York, 1986. xii+339 pp.
- A. W. Knapp, *Lie groups, Lie algebras, and cohomology.* Mathematical Notes, 34. Princeton University Press, Princeton, NJ, 1988. xii+510 pp.

The cohomology theory for Jordan algebras is less well developed than for associative and Lie algebras. A starting point would seem to be the papers of Gerstenhaber in 1964 and Glassman in 1970, which concern arbitrary nonassociative algebras. A study focussed primarily on Jordan algebras is another paper by Glassman in 1970.

- M. Gerstenhaber, *A uniform cohomology theory for algebras.* Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 626–629.
- N. D. Glassman, *Cohomology of nonassociative algebras.* Pacific J. Math. **33** (1970), 617–634.
- N. D. Glassman, *Cohomology of Jordan algebras.* J. Algebra **15** (1970), 167–194.

Two fundamental results appeared earlier, namely, the Jordan analogs of the first and second Whitehead lemmas (1947,1951).

Jordan analog of first Whitehead lemma (Jacobson 1951)

Let J be a finite dimensional semisimple Jordan algebra over a field of characteristic 0 and let M be a J-module. Let f be a linear mapping of J into M such that $f(ab) = f(a)b + af(b)$. Then there exist $v_i \in M, b_i \in J$ such that

$$f(a) = \sum_i ((v_i a)b - v_i(ab_i)).$$

- N. Jacobson, *General representation theory of Jordan algebras.* Trans. Amer. Math. Soc. **70** (1951), 509–530.

Jordan analog of second Whitehead lemma (Albert 1947, Penico 1951)

Let J be a finite dimensional separable[a] Jordan algebra and let M be a J-module. Let f be a bilinear mapping of $J \times J$ into M such that $f(a,b) = f(b,a)$ and

$$f(a^2, ab) + f(a,b)a^2 + f(a,a)ab = f(a^2b, a) + f(a^2, b)a + (f(a,a)b)a$$

Then there exist a linear mapping g from J into M such that

$$f(a,b) = g(ab) - g(b)a - g(a)b$$

[a]Separable, in this context, means that the algebra remains semisimple with respect to all extensions of the ground field. For algebraically closed fields, this is the same as being semisimple

- A. A. Albert, The Wedderburn principal theorem for Jordan algebras, Ann. of Math. vol. 48 (1947) pp. 1–7.
- A. J. Penico, *The Wedderburn principal theorem for Jordan algebras.* Trans. Amer. Math. Soc. **70** (1951). 404–420.

Jordan 2-cocycles

Let M be an associative algebra. A **Hochschild 2-cocycle** is a bilinear map $f : M \times M \to M$ satisfying

$$af(b,c) - f(ab,c) + f(a,bc) - f(a,b)c = 0 \tag{1}$$

EXAMPLE: **Hochschild 2-coboundary**

$$f(a,b) = a\mu(b) - \mu(ab) + \mu(a)b \quad , \qquad \mu : M \to M \text{ linear}$$

A **Jordan 2-cocycle** is a bilinear map $f : M \times M \to M$ satisfying

$$f(a,b) = f(b,a) \text{ (symmetric)}$$

$$\begin{aligned} f(a^2, a\circ b) + f(a,b)\circ a^2 + f(a,a)\circ(a\circ b) \\ -f(a^2\circ b, a) - f(a^2,b)\circ a - (f(a,a)\circ b)\circ a = 0 \end{aligned} \tag{2}$$

EXAMPLE: **Jordan 2-coboundary**

$$f(a,b) = a\circ\mu(b) - \mu(a\circ b) + \mu(a)\circ b, \qquad , \quad \mu : M \to M \text{ linear}$$

$$H^1(M,M) = \frac{\text{1-cocycles}}{\text{1-coboundaries}} = \frac{\text{derivations}}{\text{inner derivations}} \quad (= 0 \text{ for } M \text{ von Neumann})$$

$$H^1_J(M,M) = \frac{\text{Jordan 1-cocycles}}{\text{Jordan 1-coboundaries}} = \frac{\text{Jordan derivations}}{\text{inner Jordan derivations}} \quad (= 0 \text{ for } M \text{ vN})$$

$$H^2(M,M) = \frac{\text{2-cocycles}}{\text{2-coboundaries}} \quad , \quad H^2_J(M,M) = \frac{\text{Jordan 2-cocycles}}{\text{Jordan 2-coboundaries}}$$

For almost all von Neumann algebras, $H^2(M,M) = 0$. How about $H^2_J(M,M)$?

FD char 0: **Albert** 1947, **Penico** 1951; char $\neq 2$: **Taft** 1957

Two approaches: Jordan classification; TKK Lie algebra

- O. Kühn and A. Rosendahl, *Wedderburnzerlegung für Jordan-Paare.* (German. English summary) Manuscripta Math. **24** (1978), no. 4, 403–435.

Examples

Every symmetric Hochschild 2-cocycle is a Jordan 2-cocycle.

Recall that (at least for C*-algebras) every Jordan derivation (Jordan 1-cocycle) is a derivation (Hochschild 1-cocycle)

Let M be an associative algebra. Let $f : M \times M \to M$ be defined by $f(a, b) = a \circ b$. Then f is a Jordan 2-cocycle with values in M, which is not a Hochschild 2-cocycle unless M is commutative.

If M is a von Neumann algebra with faithful normal finite trace tr, then $f : M \times M \to M_*$ defined by $f(a, b)(x) = \mathrm{tr}\,((a \circ b)x)$ is a Jordan 2-cocycle with values in M_* which is not a Hochschild 2-cocycle unless M is commutative.

Conjecture

If M is a von Neumann algebra, then $H^2_J(M, M) = 0$

A study of low dimensional cohomology for quadratic Jordan algebras was initiated by McCrimmon, in 1971.

- K. McCrimmon, *Representations of quadratic Jordan algebras.* Trans. Amer. Math. Soc. **153** (1971), 279–305.

This paper, which is mostly concerned with representation theory, proves, for the only cohomology groups defined, the linearity of the functor H^n:

$$H^n(J, \oplus_i M_i) = \oplus_i H^n(J, M_i), \quad n = 1, 2.$$

Although this paper is about Jordan algebras, the concepts are phrased in terms of the associated triple product $\{abc\} = (ab)c + (cb)a - (ac)b$.

Quadratic Jordan algebras can be considered as a bridge from Jordan algebras to Jordan triple systems

A subsequent paper, by McCrimmon in 1982, which is mostly concerned with compatibility of tripotents in Jordan triple systems, proves versions of the linearity of the functor H^n, $n = 1, 2$, corresponding to the Jordan triple structure.

- K. McCrimmon, *Compatible Peirce decompositions of Jordan triple systems.* Pacific J. Math. **103** (1982), no. 1, 57–102.

However, both papers stop short of defining higher cohomology groups.

3. Jordan triples and TKK Lie algebras

By a *Jordan triple*, we mean a real or complex vector space V, equipped with a Jordan triple product $\{\cdot,\cdot,\cdot\} : V^3 \to V$ which is linear and symmetric in the outer variables, conjugate linear in the middle variable, and satisfies the Jordan triple identity

$$\{x, y, \{a, b, c\}\} = \{\{x, y, a\}, b, c\} - \{a, \{y, x, b\}, c\} + \{a, b, \{x, y, c\}\} \qquad (3)$$

for $a, b, c, x, y \in V$.

Let V be a Jordan triple. A vector space M over the same scalar field is called a *Jordan triple V-module* if it is equipped with three mappings
$\{\cdot,\cdot,\cdot\}_1 : M \times V \times V \to M,\ \{\cdot,\cdot,\cdot\}_2 : V \times M \times V \to M,\ \{\cdot,\cdot,\cdot\}_3 : V \times V \times M \to M$
such that (i) $\{a, b, c\}_1 = \{c, b, a\}_3$;
(ii) $\{\cdot,\cdot,\cdot\}_1$ is linear in the first two variables and conjugate linear in the last variable, $\{\cdot,\cdot,\cdot\}_2$ is conjugate linear in all variables;
(iii) denoting by $\{\cdot,\cdot,\cdot\}$ any of the products $\{\cdot,\cdot,\cdot\}_j$ ($j = 1, 2, 3$), the identity (3) is satisfied whenever one of the above elements is in M and the rest are in V.

Given two elements a, b in a Jordan triple V, we define the *box operator* $a\Box b : V \to V$ by $a\Box b(\cdot) = \{a, b, \cdot\}$.

If M is a Jordan triple V-module, the box operator $a\Box b : V \to V$ can also be considered as a mapping from M to M. Similarly, for $u \in V$ and $m \in M$, the "*box operators*"

$$u\Box m,\ m\Box u : V \longrightarrow M$$

are defined in a natural way as $v \mapsto \{u, m, v\}$ and $v \mapsto \{m, u, v\}$ respectively.

Given $a, b, u \in V$ and $m \in M$, the identity (3) implies

$$[a\Box b, u\Box m] = \{a, b, u\}\Box m - u\Box \{m, a, b\} \tag{4}$$

and

$$[a\Box b, m\Box u] = \{a, b, m\}\Box u - m\Box \{u, a, b\}. \tag{5}$$

We also have $[u\Box m, a\Box b] = \{u, m, a\}\Box b - a\Box \{b, u, m\}$ and similar identity for $[m\Box u, a\Box b]$.

A Jordan triple is called *nondegenerate* if for each $a \in V$, the condition $\{a, a, a\} = 0$ implies $a = 0$.

Given that V is nondegenerate facilitates a simple definition of the TKK Lie algebra $\mathfrak{L}(V) = V \oplus V_0 \oplus V$ of V, with an involution θ, where $V_0 = \{\sum_j a_j \square b_j : a_j, b_j \in V\}$, the Lie product is defined by

$$[(x, h, y), (u, k, v)] = (hu - kx, \, [h, k] + x \square v - u \square y, \, k^\natural y - h^\natural v),$$

and for each $h = \sum_i a_i \square b_i$ in the Lie subalgebra V_0 of $\mathfrak{L}(V)$, the map $h^\natural : V \to V$ is well defined by $h^\natural = \sum_i b_i \square a_i$.

The involution $\theta : \mathfrak{L}(V) \to \mathfrak{L}(V)$ is given by

$$\theta(x, h, y) = (y, -h^\natural, x) \qquad ((x, h, y) \in \mathfrak{L}(V)).$$

Let M_0 be the linear span of $\{u \square m, n \square v : u, v \in V, m, n \in M\}$ in the vector space $L(V, M)$ of linear maps from V to M. [a]

[a] M_0 is the space of *inner structural transformations* $\mathrm{Instrl}\,(V, M)$ (see the 1982 paper of McCrimmon quoted above—more about structural transformations later)

Extending the above commutator products (4) and (5) by linearity, we can define an action of V_0 on M_0 by

$$(h, \varphi) \in V_0 \times M_0 \mapsto [h, \varphi] \in M_0,$$

where $h = \sum a_i \square b_i$ and $\varphi = \sum_j (m_j \square u_j + v_j \square n_j)$.

Given a Lie algebra $\mathfrak{L}$ and a module X over $\mathfrak{L}$, we denote the action of $\mathfrak{L}$ on X by

$$(\ell, x) \in \mathfrak{L} \times X \mapsto \ell.x \in X \text{ so that } [\ell, \ell'].x = \ell'.(\ell.x) - \ell.(\ell'.x).$$

M_0 is a V_0-module of the Lie algebra V_0, that is,

$$[[h, k], \varphi] = [h, [k, \varphi]] - [k, [h, \varphi]].$$

Let V be a Jordan triple and $\mathfrak{L}(V)$ its TKK Lie algebra. Given a triple V-module M, let $\mathcal{L}(M) = M \oplus M_0 \oplus M$ and define the action

$$((a,h,b),(m,\varphi,n)) \in \mathfrak{L}(V) \times \mathcal{L}(M) \mapsto (a,h,b).(m,\varphi,n) \in \mathcal{L}(M)$$

by $$(a,h,b).(m,\varphi,n) = (hm - \varphi a,\ [h,\varphi] + a \square n - m \square b,\ \varphi^\natural b - h^\natural(n)),$$

where, for $h = \sum_i a_i \square b_i$ and $\varphi = \sum_i u_i \square m_i + \sum_j n_j \square v_j$, we have the following natural definitions

$$hm = \sum_i \{a_i, b_i, m\}, \quad \varphi a = \sum_i \{u_i, m_i, a\} + \sum_j \{n_j, v_j, a\},$$

and

$$\varphi^\natural = \sum_i m_i \square u_i + \sum_j v_j \square n_j.$$

Theorem

Let V be a Jordan triple and let $\mathfrak{L}(V)$ be its TKK Lie algebra. Let M be a triple V-module. Then $\mathcal{L}(M)$ is a Lie $\mathfrak{L}(V)$-module.

4. Cohomology of Lie algebras with (or without) involution

Given an involutive Lie algebra $(\mathfrak{L}, \theta)$, an $(\mathfrak{L}, \theta)$-*module* is a (left) $\mathfrak{L}$-module $\mathfrak{M}$, equipped with an involution $\widetilde{\theta} : \mathfrak{M} \to \mathfrak{M}$ satisfying

$$\widetilde{\theta}(\ell.\mu) = \theta(\ell).\widetilde{\theta}(\mu) \qquad (\ell \in \mathfrak{L}, \mu \in \mathfrak{M}).$$

A k-linear map $\psi : \mathfrak{L}^k \to \mathfrak{M}$ is called θ-*invariant* if

$$\psi(\theta x_1, \cdots, \theta x_k) = \widetilde{\theta}\psi(x_1, \cdots, x_k) \quad \text{for} \quad (x_1, \cdots, x_k) \in \mathfrak{L} \times \cdots \times \mathfrak{L}.$$

Let $(\mathfrak{L},\theta)$ be an involutive Lie algebra and $\mathfrak{M}$ an $(\mathfrak{L},\theta)$-module. We define

$$A^0(\mathfrak{L},\mathfrak{M}) = \mathfrak{M} \text{ and } A^0_\theta(\mathfrak{L},\mathfrak{M}) = \{\mu \in \mathfrak{M} : \widetilde{\theta}\mu = \mu\}.$$

For $k = 1, 2, \ldots$, we let

$$A^k(\mathfrak{L},\mathfrak{M}) = \{\psi : \mathfrak{L}^k \to \mathfrak{M} \mid \psi \text{ is k-linear and alternating}\} \quad \text{and}$$

$$A^k_\theta(\mathfrak{L},\mathfrak{M}) = \{\psi \in A^k(\mathfrak{L},\mathfrak{M}) : \mid \psi \text{ is } \theta\text{-invariant}\}.$$

For $k = 0, 1, 2, \ldots$, we define the *coboundary operator* in the usual way: $d_k : A^k(\mathfrak{L},\mathfrak{M}) \to A^{k+1}(\mathfrak{L},\mathfrak{M})$ by $d_0 m(x) = x.m$ and for $k \geq 1$,

$$\begin{aligned}(d_k\psi)(x_1,\ldots,x_{k+1}) &= \sum_{\ell=1}^{k+1}(-1)^{\ell+1}x_\ell.\psi(x_1,\ldots,\widehat{x}_\ell,\ldots,x_{k+1}) \\ &\quad + \sum_{1\leq i<j\leq k+1}(-1)^{i+j}\psi([x_i,x_j],\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_{k+1})\end{aligned}$$

where the symbol $\widehat{z}$ indicates the omission of z.

The restriction of d_k to the subspace $A^k_\theta(\mathfrak{L},\mathfrak{M})$, which by an abuse of notation, we still denote by d_k, has range in $A^k_\theta(\mathfrak{L},\mathfrak{M})$. Also, as usual, we have $d_k d_{k-1} = 0$ for $k = 1, 2, \ldots$ and two cochain complexes

$$\begin{array}{ccccccc} A^0(\mathfrak{L},\mathfrak{M}) & \xrightarrow{d_0} & A^1(\mathfrak{L},\mathfrak{M}) & \xrightarrow{d_1} & A^2(\mathfrak{L},\mathfrak{M}) & \xrightarrow{d_2} & \cdots \\ A^0_\theta(\mathfrak{L},\mathfrak{M}) & \xrightarrow{d_0} & A^1_\theta(\mathfrak{L},\mathfrak{M}) & \xrightarrow{d_1} & A^2_\theta(\mathfrak{L},\mathfrak{M}) & \xrightarrow{d_2} & \cdots . \end{array}$$

As usual, we define the *k-th cohomology group of $\mathfrak{L}$ with coefficients in* $\mathfrak{M}$ to be

$$H^k(\mathfrak{L},\mathfrak{M}) = \ker d_k / d_{k-1}(A^{k-1}(\mathfrak{L},\mathfrak{M})) = \ker d_k / \operatorname{im} d_{k-1}$$

for $k = 1, 2, \ldots$ and define $H^0(\mathfrak{L},\mathfrak{M}) = \ker d_0$.

We define the *k-th involutive cohomology group of $(\mathfrak{L},\theta)$ with coefficients in an $(\mathfrak{L},\theta)$-module* $\mathfrak{M}$ to be the quotient

$$H^k_\theta(\mathfrak{L},\mathfrak{M}) = \ker d_k / d_{k-1}(A^{k-1}_\theta(\mathfrak{L},\mathfrak{M})) = \ker d_k / \operatorname{im} d_{k-1}$$

for $k = 1, 2, \ldots$ and define $H^0_\theta(\mathfrak{L},\mathfrak{M}) = \ker d_0 \subset H^0(\mathfrak{L},\mathfrak{M})$.

5. Cohomology of Jordan triples

Let V be a Jordan triple and let $\mathfrak{L}(V) = V \oplus V_0 \oplus V$ be its TKK Lie algebra with the involution $\theta(a, h, b) = (b, -h^\natural, a)$. Given a V-module M, we have shown that $\mathfrak{L}(M) = M \oplus M_0 \oplus M$ is an $\mathfrak{L}(V)$-module. We define an induced involution $\widetilde{\theta} : \mathfrak{L}(M) \to \mathfrak{L}(M)$ by

$$\widetilde{\theta}(m, \varphi, n) = (n, -\varphi^\natural, m)$$

for $(m, \varphi, n) \in M \oplus M_0 \oplus M$.

Lemma

$\mathfrak{L}(M)$ is an $(\mathfrak{L}(V), \theta)$-module, that is, we have $\widetilde{\theta}(\ell.\mu) = \theta(\ell).\widetilde{\theta}(\mu)$ for $\ell \in \mathfrak{L}(V)$ and $\mu \in \mathfrak{L}(M)$.

We will construct cohomology groups of a Jordan triple V with coefficients in a V-module M using the cohomology groups of $\mathfrak{L}(V)$ with coefficients $\mathfrak{L}(M)$.

Let V be a Jordan triple. V is identified as the subspace $\{(v,0,0) : v \in V\}$ of the TKK Lie algebra $\mathfrak{L}(V)$.

For a triple V-module M, there is a natural embedding of M into $\mathfrak{L}(M) = M \oplus M_0 \oplus M$ given by $\iota : m \in M \mapsto (m,0,0) \in \mathfrak{L}(M)$ and we will identify M with $\iota(M)$.

We denote by $\iota_p : \mathfrak{L}(M) \to \iota(M)$ the natural projection $\iota_p(m,\varphi,n) = (m,0,0)$.

We define $A^0(V,M) = M$ and for $k = 1,2,\ldots$, we denote by $A^k(V,M)$ the vector space of all alternating k-linear maps $\omega : V^k = \overbrace{V \times \cdots \times V}^{k-\text{times}} \to M$.

To motivate the definition of an extension $\mathfrak{L}_k(\omega) \in A^k(\mathfrak{L}(V), \mathfrak{L}(M))$ of an element $\omega \in A^k(V, M)$, for $k \geq 1$, we first consider the case $k = 1$ and note that $\omega \in A^1(V, M)$ is a Jordan triple derivation if and only if

$$\omega \circ (a \square b) - (a \square b) \circ \omega = \omega(a) \square b + a \square \omega(b).$$

Let us call a linear transformation $\omega : V \to M$ *extendable* if the following condition holds:

$$\sum_i a_i \square b_i = 0 \Rightarrow \sum_i (\omega(a_i) \square b_i + a_i \square \omega(b_i)) = 0.$$

Thus a Jordan triple derivation is extendable, and if ω is any extendable transformation in $A^1(V, M)$, then the map

$$\mathfrak{L}_1(\omega)(x_1 \oplus a_1 \square b_1 \oplus y_1) := (\,\omega(x_1),\, \omega(a_1) \square b_1 + a_1 \square \omega(b_1),\ \omega(y_1)\,)$$

is well defined and extends linearly to an element $\mathfrak{L}_1(\omega) \in A^1(\mathfrak{L}(V), \mathfrak{L}(M))$, in which case we call $\mathfrak{L}_1(\omega)$ the *Lie extension* of ω on the Lie algebra $\mathfrak{L}(V)$.

For $k > 1$, given a k-linear mapping $\omega : V^k \to M$, we say that ω is *extendable* if it satisfies the following condition under the assumption $\sum_i u_i \square v_i = 0$:

$$\sum_i (\omega(u_i, a_2, \dots, a_k) \square (v_i + b_2 + \dots + b_k))$$

$$+ \sum_i ((u_i + a_2 + \dots + a_k) \square \omega(v_i, b_2, \dots, b_k)) = 0,$$

for all $a_2, \dots, a_k, b_2, \dots, b_k \in V$.

For an extendable ω, we can unambiguously define a k-linear map $\mathfrak{L}_k(\omega) : \mathfrak{L}(V)^k \to \mathfrak{L}(M)$ as the linear extension (in each variable) of

$$\mathfrak{L}_k(\omega)(x_1 \oplus a_1 \square b_1 \oplus y_1,\ x_2 \oplus a_2 \square b_2 \oplus y_2,\ \cdots,\ x_k \oplus a_k \square b_k \oplus y_k) =$$

$$(\omega(x_1, \dots, x_k), \sum_{j=1}^{k} \omega(a_1, \dots, a_k) \square b_j + \sum_{j=1}^{k} a_j \square \omega(b_1, \dots, b_k), \omega(y_1, \dots, y_k)).$$

We call $\mathfrak{L}_k(\omega)$ the *Lie extension* of ω and $\mathfrak{L}_k(\omega) \in A^k(\mathfrak{L}(V), \mathfrak{L}(M))$.

We thus have the following extension map[a] on the subspace $A^k(V, M)'$ of extendable maps in $A^k(V, M)$:

$$\mathfrak{L}_k : \omega \in A^k(V, M)' \mapsto \mathfrak{L}_k(\omega) \in A^k(\mathfrak{L}(V), \mathfrak{L}(M)).$$

[a]Note that $\mathfrak{L}_k(\omega) \in A^k_\theta(\mathfrak{L}(V), \mathfrak{L}(M))$ if and only if k is odd.

Conversely, given $\psi \in A^k(\mathfrak{L}(V), \mathfrak{L}(M))$ for $k = 1, 2, \ldots$, one can define an alternating map

$$J^k(\psi) : V^k \to M$$

by

$$J^k(\psi)(x_1, \ldots, x_k) = \iota_p \psi(\,(x_1, 0, 0),\ \ldots\ , (x_k, 0, 0)\,)$$

for $(x_1, \ldots, x_k) \in V^k$.

We define $J^0 : \mathfrak{L}(M) \to \iota(M) \approx M = A^0(V, M)$ by

$$J^0(m, \varphi, n) = (m, 0, 0) \qquad ((m, \varphi, n) \in \mathfrak{L}(M)).$$

We call $J^k(\psi)$ the *Jordan restriction* of ψ in $A^k(V, M)$.

We now define the two classes of cohomology groups for a Jordan triple V with coefficients M. First, $H^k(V,M)$, and then $H^k_\theta(V,M)$:

$$\begin{array}{ccccccc}
A^0(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_0} & A^1(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_1} & A^2(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_2} & \cdots \\
\downarrow J^0 & & \downarrow J^1 & & \downarrow J^2 & & \cdots \\
M = A^0(V,M) & & A^1(V,M) & & A^2(V,M) & & \cdots
\end{array}$$

For $k = 0, 1, 2, \ldots$, the *k-th cohomology groups* $H^k(V,M)$ are defined by

$$\begin{aligned}
H^0(V,M) &= J^0(\ker d_0) = J^0\{(m,\varphi,n) : (u,h,v).(m,\varphi,n) = 0,\ \forall (u,h,v) \in \mathfrak{L}(V)\\
&= \{m \in M : m \square v = 0,\ \forall v \in V\} = \{0\}
\end{aligned}$$

$$H^k(V,M) = Z^k(V,M)/B^k(V,M) \qquad (k = 1, 2, \ldots)$$

$$Z^k(V,M) = J^k(Z^k(\mathfrak{L}(V),\mathfrak{L}(M))), \quad Z^k(\mathfrak{L}(V),\mathfrak{L}(M)) = \ker d_k$$

$$B^k(V,M) = J^k(B^k(\mathfrak{L}(V),\mathfrak{L}(M))), \quad B^k(\mathfrak{L}(V),\mathfrak{L}(M)) = \mathrm{im}\, d_{k-1}.$$

$$
\begin{array}{ccccccc}
A^0_\theta(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_0} & A^1_\theta(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_1} & A^2_\theta(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_2} & \cdots \\
\downarrow J^0 & & \downarrow J^1 & & \downarrow J^2 & & \cdots \\
M = A^0(V,M) & & A^1(V,M) & & A^2(V,M) & & \cdots
\end{array}
$$

For $k = 0, 1, 2 \ldots$, the *k-th involutive cohomology groups* $H^k_\theta(V, M)$ are

$$H^0(V, M) = J^0(\ker d_0) = \{0\}$$

$$H^k_\theta(V, M) = Z^k_\theta(V, M)/B^k_\theta(V, M) \qquad (k = 1, 2, \ldots)$$

$$Z^k_\theta(V, M) = J^k(Z^k_\theta(\mathfrak{L}(V), \mathfrak{L}(M))), \quad Z^k_\theta(\mathfrak{L}(V), \mathfrak{L}(M))) = \ker d_k|_{A^k_\theta(\mathfrak{L}(V),\mathfrak{L}(M))}$$

$$B^k_\theta(V, M) = J^k(B^k_\theta(\mathfrak{L}(V), \mathfrak{L}(M))), B^k_\theta(\mathfrak{L}(V), \mathfrak{L}(M)) = d_{k-1}(A^{k-1}_\theta(\mathfrak{L}(V), \mathfrak{L}(M))).$$

We call elements in $H^k(V, M)$ the *Jordan triple k-cocycles*, and the ones in $H^k_\theta(V, M)$ the *involutive Jordan triple k-cocycles*. Customarily, elements in $B^k(V, M)$ and $B^k_\theta(V, M)$ are called the *coboundaries*.

Some immediate consequences of the construction

Let V be a Jordan triple with TKK Lie algebra $(\mathfrak{L}(V), \theta)$. If the k-th Lie cohomology group $H^k(\mathfrak{L}(V), \mathfrak{L}(M))$ vanishes, then $H^k(V, M) = \{0\}$ and $H^k_\theta(V, M) = \{0\}$.

From Whitehead's lemmas for semisimple Lie algebras, we have

Let V be a finite dimensional Jordan triple with semisimple TKK Lie algebra $\mathfrak{L}(V)$. Then for any finite dimensional V-module M, we have $H^1(V, M) = H^2(V, M) = \{0\}$.

In particular, if V is a finite dimensional semisimple Jordan triple system, then every derivation of V into M is inner. (Meyberg notes)

In fact, we have $H^k(\mathfrak{L}(V), \mathfrak{L}(M)) = \{0\}$ for all $k \geq 3$ if $\mathfrak{L}(M)$ is a nontrivial irreducible module over $\mathfrak{L}(V)$.

6.1 Examples of Jordan triple cocycles

If $\omega \in A^2(V, M)$ is extendable with $\mathfrak{L}_2(\omega) \in Z^2(\mathfrak{L}(V), \mathfrak{L}(M))$, then $\omega = 0$.

Proof

For $x, y, z \in V$,

$$\begin{array}{rcl} 0 & = & d_2\mathfrak{L}_2(\omega)((x,0,0),(y,0,0),(0,0,z)) \\ & = & (x,0,0)\cdot(\mathfrak{L}(\omega)((y,0,0),(0,0,z) - (y,0,0)\cdot(\mathfrak{L}(\omega)((x,0,0),(0,0,z) \\ & & +(0,0,z)\cdot(\mathfrak{L}(\omega)((x,0,0),(y,0,0) - \mathfrak{L}(\omega)([(x,0,0),(y,0,0],(0,0,z)) \\ & & +\mathfrak{L}(\omega)([(x,0,0),(0,0,z],(y,0,0)) - \mathfrak{L}(\omega)([(y,0,0),(0,0,z],(x,0,0)) \\ & = & -(0,\omega(x,y)\,\square\, z,0), \end{array}$$

hence $\omega(x,y)\,\square\, z = 0$ for all x, y, z and $\omega = 0$.

We have seen in the first example that there are no non-zero extendable elements $\omega \in Z^2(V,M)$ with $\mathfrak{L}_2(\omega) \in Z^2(\mathfrak{L}(V),\mathfrak{L}(M))$. The next example examines this phenomenon for extendable $\omega \in A^3(V,M)$ with $\mathfrak{L}_3(\omega) \in Z^3_\theta(\mathfrak{L}(V),\mathfrak{L}(M))$.

For $a,b \in V$ and $m \in M$, $[a,b] := a\Box\, b - b\Box\, a$ and $[m,a] := m\Box\, a - a\Box\, m$.

Let ω be an extendable element of $A^3(V,M)$. Then its Lie extension $\mathfrak{L}_3(\omega)$ is a Lie 3-cocycle in $A^3_\theta(\mathfrak{k}(V),\mathfrak{k}(M))$ if and only if ω satisfies the following three conditions for all $a,b,c,d,x,y,z \in V$

$$[a,b]\omega(x,y,z) = \omega([a,b]x,y,z) + \omega(x,[a,b]y,z) + \omega(x,y,[a,b]z) \tag{6}$$

$$[\omega(a,b,c),d] = [\omega(d,b,c),a] = [\omega(a,b,d),c] = [\omega(a,d,c),b] \tag{7}$$

$$[\omega(x,y,[a,b]z),c] = 0. \tag{8}$$

Note that (6)-(8) involve 5,4 and 6 variables respectively.

Let $\omega \in A^3(V, M)$ be extendable and let $\psi = d_3\mathfrak{L}_3(\omega)$ (ψ is θ-invariant since 3 is odd). Write $X_j = (x_j, a_j \Box b_j - b_j \Box a_j, x_j) \in \mathfrak{k}(V)$ as $X_j = (x_j, 0, x_j) + (0, [a_j, b_j], 0)$. By the alternating character of ψ, it is a Lie 3-cocycle, that is, $\psi(X_1, X_2, X_3, X_4) = 0$ for $X_j \in \mathfrak{k}(V)$, if and only if the following five equations hold for $a_i, b_i, x_i \in V$.

$$\psi((x_1, 0, x_1), (x_2, 0, x_2), (x_3, 0, x_3), (x_4, 0, x_4)) = 0, \tag{9}$$

$$\psi((x_1, 0, x_1), (x_2, 0, x_2), (x_3, 0, x_3), (0, [a_4, b_4], 0)) = 0, \tag{10}$$

$$\psi((x_1, 0, x_1), (x_2, 0, x_2), (0, [a_3, b_3], 0), (0, [a_4, b_4], 0)) = 0, \tag{11}$$

$$\psi((x_1, 0, x_1), (0, [a_2, b_2], 0), (0, [a_3, b_3], 0), (0, [a_4, b_4], 0)) = 0, \tag{12}$$

$$\psi((0, [a_1, b_1], 0), (0, [a_2, b_2], 0), (0, [a_3, b_3], 0), (0, [a_4, b_4], 0)) = 0. \tag{13}$$

Since (6)-(8) involve 5,4 and 6 variables respectively, and (9)-(13) involve 4,5,6,7,8 variables respectively, there is an additional amount of redundancy in (9)-(13).

6.2 Examples of TKK Lie algebras

Let A be a unital associative algebra with Lie product the commutator $[x, y] = xy - yx$, Jordan product the anti-commutator $x \circ y = (xy + yx)/2$ and Jordan triple product $\{xyz\} = (xy^*z + zy^*x)/2$ if A has an involution. Denote by $Z(A)$ the center of A and by $[A, A]$ the set of finite sums of commutators.

Suppose $Z(A) \cap [A, A] = \{0\}$. Then the mapping $(x, a \square b, y) \mapsto \begin{bmatrix} ab & x \\ y & -ba \end{bmatrix}$ is an isomorphism of the TKK Lie algebra $\mathfrak{L}(A)$ onto the Lie subalgebra

$$\left\{ \begin{bmatrix} u + \sum[v_i, w_i] & x \\ y & -u + \sum[v_i, w_i] \end{bmatrix} : u, x, y, v_i, w_i \in A \right\} \qquad (14)$$

of the Lie algebra $M_2(A)$ with the commutator product.

- K. Meyberg, *Lectures on algebras and triple systems.* The University of Virginia, Charlottesville, Va., 1972. v+226 pp.
- M. Koecher, *Imbedding of Jordan algebras in Lie algebras.* I, Amer. J. Math **8**9, no. 3 (1967), 787–816.

Proposition

Let V be a finite von Neumann algebra. Then $\mathfrak{L}(V)$ is isomorphic to the Lie algebra $[M_2(V), M_2(V)]$.

In a properly infinite von Neumann algebra, the assumption $Z(A) \cap [A, A] = \{0\}$ fails since $A = [A, A]$.

This assumption also fails in the Murray-von Neumann algebra of measurable operators affiliated with a factor of type II_1 (according to a preprint of Thom in 2013).

For a finite factor of type I_n, the proposition states that the classical Lie algebras $sl(2n, \mathbb{C})$ of type A are TKK Lie algebras.

Similarly, the TKK Lie algebra of a Cartan factor of type 3 on an n-dimensional Hilbert space is the classical Lie algebra $sp(2n, \mathbb{C})$ of type C (Meyberg notes p.131)

More examples of TKK Lie algebras can be found in the book of Chu 2012 (section 1.4) and the lecture notes of Koecher 1969 (chapter III).

- C-H. Chu, *Jordan structures in geometry and analysis*. Cambridge Tracts in Math. **190**, Cambridge Univ. Press, Cambridge, 2012.
- M. Koecher, *An elementary approach to bounded symmetric domains*, Lecture Notes, Rice University, 1969.

7. Structural transformations

Let V be a Jordan triple and M a triple V-module. A mapping $\omega : V \to M$ is called an *inner triple derivation* if it is of the form

$$\omega = \sum_{i=1}^{k} (m_i \square v_i - v_i \square m_i) \in M_0$$

for some $m_1, \ldots, m_k \in M$ and $v_1, \ldots, v_k \in V$. Note that $\omega^\natural = -\omega$ and $(0, \omega, 0) \in \mathfrak{k}(M)$.

$B^1_\theta(V, M)$ coincides with the space of inner triple derivations from V to M.

$Z^1_\theta(V, M)$ coincides with the set of triple derivations of V

The first involutive cohomology group $H^1_\theta(V, M) = Z^1_\theta(V, M)/B^1_\theta(V, M)$ is the space of triple derivations modulo the inner triple derivations of V into M. This will be generalized shortly.

A (conjugate-) linear transformation $S : V \to M$ is said to be a *structural transformation* if there exists a (conjugate-) linear transformation $S^* : V \to M$ such that

$$S\{xyx\} + \{x(S^*y)x\} = \{xySx\}$$

$$S^*\{xyx\} + \{x(Sy)x\} = \{xyS^*x\}.$$

By polarization, this property is equivalent to

$$S\{xyz\} + \{x(S^*y)z\} = \{zySx\} + \{xySz\}$$

$$S^*\{xyz\} + \{x(Sy)z\} = \{zyS^*x\} + \{xyS^*z\}.$$

- A triple derivation D is a a structural transformation S with $S^* = -S$.
- The space of *inner structural transformations* coincides with the space M_0.
- Triple derivations which are inner structural transformations are inner
- If ω is a structural transformation, then $\omega - \omega^*$ is a triple derivation
- If ω is a triple derivation, $i\omega$ is a structural transformation, inner if ω is inner.

Proposition

Let ψ be a Lie derivation of $\mathfrak{L}(V)$ into $\mathfrak{L}(M)$. Then

(i) $J(\psi) : V \to M$ is a structural transformation with $(J\psi)^* = -J\psi'$ where $\psi' = \widetilde{\theta}\psi\theta$.

(ii) If ψ is θ-invariant, then $\psi' = \psi$ and $J\psi$ is a triple derivation.

(iii) If ψ is an inner derivation then $J\psi$ is an inner structural transformation. In particular, if ψ is a θ-invariant inner derivation then $J\psi$ is an inner triple derivation.

Conversely, let ω be a structural transformation.

(iv) The mapping $D = \frac{1}{2}\mathfrak{L}_1(\omega - \omega^*) : \mathfrak{L}(V) \to \mathfrak{L}(M)$ defined by

$$D(x, a \square b, y) = \frac{1}{2}(\omega(x) - \omega^*(x), \omega(a) \square b - a \square \omega^*(b) - \omega^*(a) \square b + a \square \omega(b), \omega(y)$$

is a derivation of the Lie algebra $\mathfrak{L}(V)$ into $\mathfrak{L}(M)$.

(v) D is θ-invariant if and only if ω is a triple derivation, that is, $\omega^* = -\omega$.

(vi) If ω is an inner structural transformation then D is an inner derivation. In particular, if ω is an inner triple derivation then D is a θ-invariant inner derivation.

The following theorem provides some significant infinite dimensional examples of Lie algebras in which every derivation is inner.

Theorem

Let V be a von Neumann algebra considered as a Jordan triple system with the triple product $\{xyz\} = (xy^*z + zy^*x)/2$. Then every structural transformation on V is an inner structural transformation. Hence, every derivation of the TKK Lie algebra $\mathfrak{L}(V)$ is inner.

- Pluta, Robert; Russo, Bernard Triple derivations on von Neumann algebras. Studia Math. 226 (2015), no. 1, 57–73.

MUCHAS GRACIAS

Let S be a structural transformation on the von Neumann algebra V and to avoid cumbersome notation, denote S^* by $\overline{S}$. From the defining equations, $\overline{S}(1) = S(1)^*$, and if $S(1) = 0$, then S is a Jordan derivation.

For an arbitrary structural transformation S, write $S = S_0 + S_1$ where $S_0 = S - 1 \square \overline{S}(1)$ is therefore a Jordan derivation and $S_1 = 1 \square \overline{S}(1)$ is an inner structural transformation.

By the theorem of Sinclair 1970, S_0 is a derivation and by the theorems of Kadison and Sakai 1966, S_0 is an inner derivation, say $S_0(x) = ax - xa$ for some $a \in V$.

By well known structure of the span of commutators in von Neumann algebras due to Pearcy-Topping, Halmos, Halpern, Fack-de la Harpe, and others $a = z + \sum [c_i, d_i]$, where $c_i, d_i \in V$ and z belongs to the center of V. It follows that

$$S_0 = 2 \sum_i c_i \square d_i^* - 2 \sum_i d_i \square c_i^*$$

and is therefore also an inner structural transformation. The second statement follows from the Proposition.