DERIVATIONS
An introduction to non associative algebra (or, Playing havoc with the product rule)

Part 8
Vanishing Theorems in dimensions 1 and 2

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History of these lectures

- PART I FEBRUARY 8, 2011 **ALGEBRAS; DERIVATIONS**
- PART II JULY 21, 2011 **TRIPLE SYSTEMS; DERIVATIONS**
- PART III FEBRUARY 28, 2012 **MODULES; DERIVATIONS**
- PART IV JULY 26, 2012 **COHOMOLOGY (ASSOCIATIVE ALGEBRAS)**
- PART V OCTOBER 25, 2012 **THE SECOND COHOMOLOGY GROUP**
- PART VI MARCH 7, 2013 **COHOMOLOGY (LIE ALGEBRAS)**
- PART VII JULY 25, 2013 **COHOMOLOGY (JORDAN ALGEBRAS)**
- PART VIII SEPTEMBER 17, 2013 (today)

VANISHING THEOREMS IN DIMENSIONS 1 AND 2
(ASSOCIATIVE ALGEBRAS)

Outline

- Review of Algebras
- Derivations on matrix algebras
- Review of Cohomology
- $H^1(M_2, M_2) = 0$
- $H^2(M_2, M_2) = 0$

Introduction

I will present simple proofs of vanishing of the first and second cohomology groups of an associative algebra, illustrating with the algebra of two by two matrices with matrix multiplication.

The relevant definitions and examples from earlier talks in the series will be reviewed beforehand.

Review of Algebras—Axiomatic approach

AN ALGEBRA IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH TWO BINARY OPERATIONS, CALLED ADDITION AND MULTIPLICATION

ADDITION IS DENOTED BY $a+b$ AND IS REQUIRED TO BE
COMMUTATIVE $a+b=b+a$
AND ASSOCIATIVE $(a+b)+c=a+(b+c)$

MULTIPLICATION IS DENOTED BY ab AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION
$(a+b)c=ac+bc, \quad a(b+c)=ab+ac$

AN ALGEBRA IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)
(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

Table 1 (FASHIONABLE) ALGEBRAS

commutative algebras $ab = ba$

associative algebras $a(bc) = (ab)c$

Lie algebras $a^2 = 0\ ,\ \ (ab)c + (bc)a + (ca)b = 0$

Jordan algebras $ab = ba$, $a(a^2b) = a^2(ab)$

We shall only be concerned with <u>associative</u> algebras in this talk, in fact, only the algebra of two by two matrices under matrix multiplication.

DERIVATIONS ON MATRIX ALGEBRAS

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER
MATRIX ADDITION $A + B$
AND
MATRIX MULTIPLICATION $A \times B$
WHICH IS ASSOCIATIVE BUT NOT COMMUTATIVE.

For the Record:

$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$ $\qquad$ $[a_{ij}] \times [b_{ij}] = [\sum_{k=1}^{n} a_{ik} b_{kj}]$

DEFINITION

A <u>DERIVATION</u> ON $M_n(\mathbf{R})$ WITH <u>RESPECT TO MATRIX MULTIPLICATION</u>
IS A LINEAR PROCESS δ: $\quad \delta(A + B) = \delta(A) + \delta(B)$
WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \times B) = \delta(A) \times B + A \times \delta(B)$$

.

PROPOSITION

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION (WHICH CAN BE NON-ZERO)

THEOREM (Noether,Wedderburn,Hochschild,Jacobson, Kaplansky,Kadison,Sakai)

EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

We shall give a proof of this theorem for $n = 2$ in this talk.

Review of Cohomology

NOTATION

n is a positive integer, $n = 1, 2, \cdots$
f is a function of n variables
F is a function of $n+1$ variables ($n+2$ variables?)
$x_1, x_2, \cdots, x_{n+1}$ belong to an algebra A
$f(y_1, \ldots, y_n)$ and $F(y_1, \cdots, y_{n+1})$ also belong to A

The basic formula of homological algebra

$F(x_1, \ldots, x_n, x_{n+1}) =$
$x_1 f(x_2, \ldots, x_{n+1})$
$-f(x_1x_2, x_3, \ldots, x_{n+1})$
$+f(x_1, x_2x_3, x_4, \ldots, x_{n+1})$
$- \cdots$
$\pm f(x_1, x_2, \ldots, x_nx_{n+1})$
$\mp f(x_1, \ldots, x_n)x_{n+1}$

HIERARCHY

$x_1, x_2, \ldots, x_n$ are points (or vectors)
f and F are functions—they take points to points
T, defined by $T(f) = F$ is a transformation—takes functions to functions
points $x_1, \ldots, x_{n+1}$ and $f(y_1, \ldots, y_n)$ will belong to an algebra A
functions f will be either constant, linear or multilinear (hence so will F)
transformation T is linear

SHORT FORM OF THE FORMULA

$$(Tf)(x_1, \ldots, x_n, x_{n+1})$$

$$= x_1 f(x_2, \ldots, x_{n+1})$$

$$+ \sum_{j=1}^{n} (-1)^j f(x_1, \ldots, x_j x_{j+1}, \ldots, x_{n+1})$$

$$+ (-1)^{n+1} f(x_1, \ldots, x_n) x_{n+1}$$

FIRST CASES

$\underline{n = 0}$
If f is any constant function from A to A, say, $f(x) = b$ for all x in A, where b is a fixed element of A, we have, consistent with the basic formula, a linear function $T_0(f)$:

$$T_0(f)(x_1) = x_1 b - b x_1$$

$\underline{n = 1}$
If f is a linear function from A to A, then $T_1(f)$ is a bilinear function

$$T_1(f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$$

$\underline{n = 2}$
If f is a bilinear function from $A \times A$ to A, then $T_2(f)$ is a trilinear function

$$T_2(f)(x_1, x_2, x_3) =$$

$$x_1 f(x_2, x_3) - f(x_1 x_2, x_3) + f(x_1, x_2 x_3) - f(x_1, x_2) x_3$$

FIRST COHOMOLOGY GROUP

Kernel and Image of a linear transformation

$G : X \to Y$
Since X and Y are vector spaces, they are in particular, commutative groups.
Kernel of G (also called **nullspace** of G) is
$\ker G = \{x \in X : G(x) = 0\}$
This is a subgroup of X
Image of G is
$\text{im}\, G = \{G(x) : x \in X\}$
This is a subgroup of Y

$\underline{G = T_0}$
$X = A$ (the algebra)
$Y = L(A)$ (all linear transformations on A)
$T_0(f)(x_1) = x_1 b - bx_1$
$\ker T_0 = \{b \in A : xb - bx = 0 \text{ for all } x \in A\}$ (center of A)
$\text{im}\, T_0 =$ the set of all linear maps of A of the form $x \mapsto xb - bx$,
in other words, the set of all inner derivations of A
$\ker T_0$ is a subgroup of A
$\text{im}\, T_0$ is a subgroup of $L(A)$

$\underline{G = T_1}$
$X = L(A)$ (linear transformations on A)
$Y = L^2(A)$ (bilinear transformations on $A \times A$)
$T_1(f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$
$\ker T_1 = \{f \in L(A) : T_1 f(x_1, x_2) = 0 \text{ for all } x_1, x_2 \in A\} =$ the set of all derivations of A
$\text{im } T_1 =$ the set of all bilinear maps of $A \times A$ of the form

$$(x_1, x_2) \mapsto x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2,$$

for some linear function $f \in L(A)$.
$\ker T_1$ is a subgroup of $L(A)$
$\text{im } T_1$ is a subgroup of $L^2(A)$

$\underline{G = T_2}$
$X = L^2(A)$ (bilinear transformations on $A \times A$)
$Y = L^3(A)$ (trilinear transformations on $A \times A \times A$)
$T_2(f)(x_1, x_2, x_3) = x_1 f(x_2, x_3)) - f(x_1 x_2, x_3) + f(x_1 x_2, x_3) - f(x_1, x_2) x_3$
$\ker T_2 = \{f \in L^2(A) : T_2 f(x_1, x_2, x_3) = 0 \text{ for all } x_1, x_2, x_3 \in A\}$
im $T_2 =$ the set of all trilinear maps of $A \times A \times A$ of the form

$$(x_1, x_2, x_3) \mapsto x_1 f(x_2, x_3)) - f(x_1 x_2, x_3) + f(x_1 x_2, x_3) - f(x_1, x_2) x_3$$

for some bilinear function $f \in L(A)$.
ker T_2 is a subgroup of $L^2(A)$
im T_2 is a subgroup of $L^3(A)$

$L^0(A) \xrightarrow{T_0} L(A) \xrightarrow{T_1} L^2(A) \xrightarrow{T_2} L^3(A) \cdots$
FACTS: $T_1 \circ T_0 = 0$
$T_2 \circ T_1 = 0$
$\ldots$
$T_{n+1} \circ T_n = 0$
$\ldots$

Therefore

im $T_n \subset$ ker $T_{n+1} \subset L^n(A)$
and therefore
im T_n is a subgroup of ker T_{n+1}

TERMINOLOGY

im $T_{n-1} =$ the set of n-**coboundaries**
ker $T_n =$ the set of n-**cocycles**
and therefore
every n-coboundary is an n-cocycle.

im $T_0 \subset$ ker T_1
says
Every inner derivation (1-coboundary) is a derivation (1-cocycle).

im $T_1 \subset$ ker T_2
says
for every linear map f, the bilinear map Fdefined by

$$F(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$$

(2-coboundary) satisfies the equation

$$x_1 F(x_2, x_3) - F(x_1 x_2, x_3) + F(x_1, x_2 x_3) - F(x_1, x_2) x_3 = 0$$

for every $x_1, x_2, x_3 \in A$ (2-cocycle).

The cohomology groups of A are defined as the quotient groups

$$H^n(A) = \frac{\ker T_n}{\text{im } T_{n-1}} = \frac{n\text{-cocycles}}{n\text{-coboundaries}} \qquad (n = 1, 2, \ldots)$$

Thus

$$H^1(A) = \frac{\ker T_1}{\text{im } T_0} = \frac{\text{1-cocycles}}{\text{1-coboundaries}} = \frac{\text{derivations}}{\text{inner derivations}}$$

$$H^2(A) = \frac{\ker T_2}{\text{im } T_1} = \frac{\text{2-cocycles}}{\text{2-coboundaries}} = \frac{\text{null extensions}}{\text{split null extensions}}$$

The theorem that every derivation of $M_n(\mathbf{R})$ is inner (that is, of the form δ_a for some $a \in M_n(\mathbf{R})$, Theorem 1 below for $n = 2$) can now be restated as:
"the cohomology group $H^1(M_n(\mathbf{R}))$ is the trivial one element group"

The theorem that every null extension of $M_n(\mathbf{R})$ is a split null extension (Corollary 2 of Theorem 2 below for $n = 2$) can be stated as:
"the cohomology group $H^2(M_n(\mathbf{R}))$ is the trivial one element group"

$H^1(M_2, M_2) = 0$

Matrix units

Let $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

LEMMA

- $E_{11} + E_{22} = I$
- $E_{ij}^t = E_{ji}$
- $E_{ij}E_{kl} = \delta_{kl}E_{il}$

THEOREM 1

Let $\delta : M_2 \to M_2$ be a derivation: δ is linear and $\delta(AB) = A\delta(B) + \delta(A)B$. Then there exists a matrix K such that $\delta(X) = XK - KX$ for X in M_2.

COROLLARY

$H^1(M_2, M_2) = 0$

PROOF OF THEOREM 1

$$\begin{aligned} 0 &= \delta(1) = \delta(E_{11} + E_{22}) = \delta(E_{11}) + \delta(E_{22}) \\ &= \delta(E_{11}E_{11}) + \delta(E_{21}E_{12}) \\ &= E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} + E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} \\ &= E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) + \delta(E_{11})E_{11} + \delta(E_{21})E_{12}. \end{aligned}$$

Let $K = E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) = -\delta(E_{11})E_{11} - \delta(E_{21})E_{12}$. Then

- $KE_{11} = -\delta(E_{11})E_{11}$, $E_{11}K = E_{11}\delta(E_{11})$
- $KE_{12} = -\delta(E_{11})E_{12}$, $E_{12}K = E_{11}\delta(E_{12})$
- $KE_{21} = -\delta(E_{21})E_{11}$, $E_{21}K = E_{21}\delta(E_{11})$
- $KE_{22} = -\delta(E_{21})E_{12}$, $E_{22}K = E_{21}\delta(E_{12})$

- $E_{11}K - KE_{11} = E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} = \delta(E_{11}E_{11}) = \delta(E_{11})$
- $E_{12}K - KE_{12} = E_{11}\delta(E_{12}) + \delta(E_{11})E_{12} = \delta(E_{11}E_{12}) = \delta(E_{12})$
- $E_{21}K - KE_{21} = E_{21}\delta(E_{11}) + \delta(E_{21})E_{11} = \delta(E_{21}E_{11}) = \delta(E_{21})$
- $E_{22}K - KE_{22} = E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} = \delta(E_{21}E_{12}) = \delta(E_{22})$ Q.E.D.

PROOF OF THEOREM 1

$$\begin{aligned} 0 &= \delta(1) = \delta(E_{11} + E_{22}) = \delta(E_{11}) + \delta(E_{22}) \\ &= \delta(E_{11}E_{11}) + \delta(E_{21}E_{12}) \\ &= E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} + E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} \\ &= E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) + \delta(E_{11})E_{11} + \delta(E_{21})E_{12}. \end{aligned}$$

Let $K = E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) = -\delta(E_{11})E_{11} - \delta(E_{21})E_{12}$. Then

- $KE_{11} = -\delta(E_{11})E_{11}$, $E_{11}K = E_{11}\delta(E_{11})$
- $KE_{12} = -\delta(E_{11})E_{12}$, $E_{12}K = E_{11}\delta(E_{12})$
- $KE_{21} = -\delta(E_{21})E_{11}$, $E_{21}K = E_{21}\delta(E_{11})$
- $KE_{22} = -\delta(E_{21})E_{12}$, $E_{22}K = E_{21}\delta(E_{12})$

- $E_{11}K - KE_{11} = E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} = \delta(E_{11}E_{11}) = \delta(E_{11})$
- $E_{12}K - KE_{12} = E_{11}\delta(E_{12}) + \delta(E_{11})E_{12} = \delta(E_{11}E_{12}) = \delta(E_{12})$
- $E_{21}K - KE_{21} = E_{21}\delta(E_{11}) + \delta(E_{21})E_{11} = \delta(E_{21}E_{11}) = \delta(E_{21})$
- $E_{22}K - KE_{22} = E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} = \delta(E_{21}E_{12}) = \delta(E_{22})$ Q.E.D.

PROOF OF THEOREM 1

$$\begin{aligned} 0 &= \delta(1) = \delta(E_{11} + E_{22}) = \delta(E_{11}) + \delta(E_{22}) \\ &= \delta(E_{11}E_{11}) + \delta(E_{21}E_{12}) \\ &= E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} + E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} \\ &= E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) + \delta(E_{11})E_{11} + \delta(E_{21})E_{12}. \end{aligned}$$

Let $K = E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) = -\delta(E_{11})E_{11} - \delta(E_{21})E_{12}$. Then

- $KE_{11} = -\delta(E_{11})E_{11} \quad , \quad E_{11}K = E_{11}\delta(E_{11})$
- $KE_{12} = -\delta(E_{11})E_{12} \quad , \quad E_{12}K = E_{11}\delta(E_{12})$
- $KE_{21} = -\delta(E_{21})E_{11} \quad , \quad E_{21}K = E_{21}\delta(E_{11})$
- $KE_{22} = -\delta(E_{21})E_{12} \quad , \quad E_{22}K = E_{21}\delta(E_{12})$

- $E_{11}K - KE_{11} = E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} = \delta(E_{11}E_{11}) = \delta(E_{11})$
- $E_{12}K - KE_{12} = E_{11}\delta(E_{12}) + \delta(E_{11})E_{12} = \delta(E_{11}E_{12}) = \delta(E_{12})$
- $E_{21}K - KE_{21} = E_{21}\delta(E_{11}) + \delta(E_{21})E_{11} = \delta(E_{21}E_{11}) = \delta(E_{21})$
- $E_{22}K - KE_{22} = E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} = \delta(E_{21}E_{12}) = \delta(E_{22})$ Q.E.D.

$H^2(M_2, M_2) = 0$

DEFINITION

Let ρ be a linear transformation on M_2. We define linear transformations σ_1 and σ_2 on M_2 by

$$\sigma_1(A) = E_{11}\rho(E_{11}A) + E_{21}\rho(E_{12}A)$$

and

$$\sigma_2(A) = E_{12}\rho(E_{21}A) + E_{22}\rho(E_{22}A)$$

LEMMA 1

$\sigma_1(A) = A\sigma_1(I)$ and $\sigma_2(A) = A\sigma_2(I)$

We only need σ_1 or σ_2, not both. We'll go with σ_1.

PROOF OF LEMMA 1

$E_{11}AE_{11} = c_{11}E_{11}$, $E_{12}AE_{11} = c_{21}E_{11}$, $E_{11}AE_{21} = c_{12}E_{11}$, $E_{12}AE_{21} = c_{22}E_{11}$

- $E_{11}A = E_{11}AE_{11}E_{11} + E_{11}AE_{21}E_{12} = c_{11}E_{11} + c_{12}E_{11}E_{12} = c_{11}E_{11} + c_{12}E_{12}$
- $E_{12}A = E_{12}AE_{11}E_{11} + E_{12}AE_{21}E_{12} = c_{21}E_{11} + c_{22}E_{11}E_{12} = c_{21}E_{11} + c_{22}E_{12}$

- $AE_{11} = E_{11}E_{11}AE_{11} + E_{21}E_{12}AE_{11} = c_{11}E_{11} + c_{21}E_{21}E_{11} = c_{11}E_{11} + c_{21}E_{21}$
- $AE_{21} = E_{11}E_{11}AE_{21} + E_{21}E_{12}AE_{21} = c_{12}E_{11} + c_{22}E_{21}E_{11} = c_{12}E_{21} + c_{22}E_{21}$

$$
\begin{aligned}
\sigma_1(A) &= E_{11}\rho(E_{11}A) + E_{21}\rho(E_{12}A) \\
&= E_{11}\rho(c_{11}E_{11} + c_{12}E_{12}) + E_{21}\rho(c_{21}E_{11} + c_{22}E_{12}) \\
&= c_{11}E_{11}\rho(E_{11}) + c_{12}E_{11}\rho(E_{12}) + c_{21}E_{21}\rho(E_{11}) + c_{22}E_{21}\rho(E_{12}) \\
&= (c_{11}E_{11} + c_{21}E_{21})\rho(E_{11}) + (c_{12}E_{11} + c_{22}E_{21})\rho(E_{12}) \\
&= AE_{11}\rho(E_{11}) + AE_{21}\rho(E_{12}) \\
&= A\sigma_1(1) \quad \text{Q.E.D.}
\end{aligned}
$$

DEFINITION

Let f be a bilinear transformation on $M_2 \times M_2$. We define bilinear transformations τ_1 and τ_2 on $M_2 \times M_2$ by

$$\tau_1(A,B) = E_{11}f(E_{11}A,B) + E_{21}f(E_{12}A,B)$$

and

$$\tau_2(A,B) = E_{12}f(E_{21}A,B) + E_{22}f(E_{22}A,B)$$

LEMMA 2

$\tau_1(A,B) = A\tau_1(I,B)$ and $\tau_2(A,B) = A\tau_2(I,B)$

We only need τ_1 or τ_2, not both. We'll go with τ_1.

PROOF OF LEMMA 2

For B fixed, let $\rho(A) = f(A,B)$ and apply LEMMA 1 to this ρ. Namely, set $\sigma(A) = E_{11}\rho(E_{11}A) + E_{21}\rho(E_{12}A)$. Then $\sigma(A) = \tau_1(A,B)$. By LEMMA 1, $\sigma(A) = A\sigma(1)$ and $\tau_1(A,B) = \sigma(A) = A\sigma(1) = A\tau_1(1,B)$. Q.E.D.

THEOREM 2

Let f be a 2-cocycle: f is bilinear and

$$T_2 f(A,B,C) = Af(B,C) - f(AB,C) + f(A,BC) - f(A,B)C = 0$$

for all A, B, C in M_2. Then there exists a linear transformation ξ on M_2 such that $T_1\xi = f$, that is, f is a 2-coboundary.

COROLLARY 1

$H^2(M_2, M_2) = 0$

COROLLARY 2

It E is any associative algebra containing an ideal J such that E/J is isomorphic to M_2 (E is then said to be an **extension** of M_2), then there is a subalgebra B of E such that $E = B \oplus M_2$ (E is a **split extension**) [a]

[a] There is always a subspace B such that $E = B \oplus M_2$

PROOF OF THEOREM 2

Define a bilinear map $\tau(A,B) = E_{11}f(E_{11}A,B) + E_{21}f(E_{12}A,B)$ and then define a linear map $\xi(B) = \tau(1,B)$. Now just verify that $T_1(\xi) = f$. Q.E.D.

DETAILS

$$
\begin{aligned}
T_1\xi(A,B) &= A\xi(B) - \xi(AB) + \xi(A)B \\
&= A\tau(1,B) - \tau(1,AB) + \tau(1,A)B \\
&= \tau(A,B) - \tau(1,AB) + \tau(1,A)B \\
&= E_{11}f(E_{11}A,B) + E_{21}f(E_{12}A,B) \\
&- E_{11}f(E_{11},AB) - E_{21}f(E_{12},AB) \\
&+ E_{11}f(E_{11},A)B + E_{21}f(E_{12},A)B
\end{aligned}
$$

$$T_2f(E_{11},A,B) = E_{11}f(A,B) - f(E_{11}A,B) + f(E_{11},AB) - f(E_{11},A)B = 0$$

$$T_2f(E_{12},A,B) = E_{12}f(A,B) - f(E_{12}A,B) + f(E_{12},AB) - f(E_{12},A)B = 0$$

$$0 = E_{11} T_2 f(E_{11}, A, B) + E_{21} T_2 f(E_{12}, A, B)$$

$$\begin{aligned} 0 &= E_{11}[E_{11} f(A,B) - f(E_{11}A, B) + f(E_{11}, AB) - f(E_{11}, A)B] \\ &+ E_{21}[E_{12} f(A,B) - f(E_{12}A, B) + f(E_{12}, AB) - f(E_{12}, A)B] \end{aligned}$$

FROM THE PRECEDING PAGE

$$\begin{aligned} T_1\xi(A,B) &= E_{11} f(E_{11}A, B) + E_{21} f(E_{12}A, B) - E_{11} f(E_{11}, AB) \\ &- E_{21} f(E_{12}, AB) + E_{11} f(E_{11}, A)B + E_{21} f(E_{12}, A)B \end{aligned}$$

Add these two equations to get

$$T_1\xi(A,B) = E_{11} f(A,B) + E_{22} f(A,B) = f(A,B) \quad \text{Q.E.D. (again)}$$