An introduction to Leibniz algebras (from calculus to algebra)

Math 199A—Fall 2017
Independent Studies
University of California, Irvine

Bernard Russo

Outline

Part 1: Solvable groups

In many ways, abstract algebra began with the work of Abel and Galois on the solvability of polynomial equations by radicals.

The key idea Galois had was to transform questions about fields and polynomials into questions about finite groups.

For the proof that it is not always possible to express the roots of a polynomial equation in terms of the coefficients of the polynomial using arithmetic expressions and taking roots of elements, the appropriate group theoretic property that arises is the idea of solvability.

Definition

A group G is **solvable** if there is a chain of subgroups

$$\{e\} = H_0 \subset H_1 \subset \cdots H_{n-1} \subset H_n = G$$

such that, for each i, the subgroup H_i is normal in H_{i+1} and the quotient group H_{i+1}/H_i is Abelian.

An Abelian group G is solvable; $\{e\} \subset G$

The symmetric groups S_3 and S_4 are solvable by considering the chains

$$\{e\} \subset A_3 \subset S_3 \text{ and } \{e\} \subset H \subset A_4 \subset S_4,$$

respectively, where $H = \{e, (12)(34), (13)(24); (14)(23)\}$

S_n is not solvable if $n \geq 5$.

This is the group theoretic result we need to show that the roots of the general polynomial of degree n (over a field of characteristic 0) cannot be written in terms of the coefficients of the polynomial by using algebraic operations and extraction of roots.

An alternate definition, more suitable for algebras

If G is a group, let $G^{(0)} = [G, G]$ be the commutator subgroup of G, that is, the set of all finite products of commutators $ghg^{-1}h^{-1}$.

Define $G^{(i)}$ by recursion: $G^{(i+1)} = [G^{(i)}, G^{(i)}]$

We have

- $G \supset G^{(0)} \supset G^{(1)} \supset G^{(2)} \supset \cdots \supset G^{(n)} \supset \cdots$
- $G^{(m+1)}$ is normal in $G^{(m)}$ and $G^{(m)}/G^{(m+1)}$ is Abelian

Lemma: G is solvable if and only if $G^{(n)} = \{e\}$ for some n.

Proposition: A group G with a normal subgroup N is solvable if and only if N and G/N are both solvable.

Theorem: If $n \geq 5$, then S_n is not solvable.

Edward Frenkel, Love and Math, 2013, chapter 7

The vast majority of the numbers that we encounter in everyday life are fractions, or rational numbers. But there are also numbers which are not rational

Since $\sqrt{2}$ is the length of the hypotenuse of a certain right triangle, we know that this number is out there. (It is also one of the solutions to the equation $x^2 = 2$) But it just does not fit in the numerical system of rational numbers

Let's drop $\sqrt{2}$ in the rationals and see what kind of numerical system we obtain. This numerical system has at least two symmetries:

$$\frac{m}{n} + \frac{k}{\ell}\sqrt{2} \longrightarrow \frac{m}{n} + \frac{k}{\ell}\sqrt{2} \quad \text{and} \quad \frac{m}{n} + \frac{k}{\ell}\sqrt{2} \longrightarrow \frac{m}{n} - \frac{k}{\ell}\sqrt{2}$$

If the solutions of any polynomial equations, such as $x^3 - x + 1 = 0$, or $x^3 = 2$, are not rational numbers, then we can adjoint them to the rational numbers.

The resulting numerical systems (called number fields) have symmetries which form a group (called the Galois group of the number field)

What Galois has done was bring the idea of symmetry, intuitively familiar in geometry, to the forefront of number theory

Formulas for solutions of equations of degree 3 and 4 were discovered in the early 16th century. Prior to Galois, mathematicians had been trying to find a formula for the solutions of an equation of degree 5 for almost 300 years, to no avail

The question of describing the Galois group turns out to be much more tractable than the question of writing an explicit formula for the solutions

Galois was able to show that a formula for solutions in terms of radicals (square roots, cube roots, and so on) exists if and only if the corresponding Galois group had a particular attribute, which is not present for degree 5 or higher.

Mark Ronan, Symmetry and the Monster, 2006, chapter 2

In Paris on the evening of 29 May 1832 the young French mathematician Évariste Galois wrote a letter he knew would be the last of his life

Though his fame as a revolutionary was transient, his mathematics was timeless: Galois groups are common currency in mathematics today.

As a young man of 20, he joined the ranks of the immortals. How is this possible?

When Galois was held back by the headmaster against his father's will, the effect was devastating and the 15 year old started rejecting everything but mathematics.

The conflict between Galois' father and the headmaster was part of a wider political problem.

Galois' main ideas concerned the solution of algebraic equations

The quadratic formula is ancient, first discovered by the Babylonians in about 1800 BCE, nearly 4000 years ago (They wrote in words rather than symbols)

A general method for dealing with equations of degree 3 had to wait nearly 3000 years until Omar Khayyám (1048–1134), the famous Persian mathematician and astronomer (better known for his poetry), devised a geometric method

A numerical formula was found 400 years later during the Italian Renaissance for equations of degree 3 and 4. In the early 16th century four Italian mathematicians moved algebra into a new era

In retrospect, these four were men of genius and "constituted the most singular team in the whole history of science."

However, no one could find a recipe for solving equations of degree 5

In 1799, no less an authority than Gauss wrote “Since the works of many geometers left very little hope of ever arriving at the resolution of the general equation algebraically, it appears increasingly likely that this resolution is impossible and contradictory”

That same year, this “conjecture” was “confirmed” and published in a 500 page book. However, this work was never fully accepted

The matter was finally settled in 1824 when a young Norwegian mathematician, Niels Hendrik Abel (1802–1829) produced an independent proof

Abel showed that there were some equations of degree 5 whose solutions could not be extracted using square roots, cube roots, fourth and fifth roots, etc

But the problem was to decide which equations could be solved in this way, and which couldn’t. This set the stage for the entrance of Évariste Galois, who died even younger than Abel.

Galois measured the amount of symmetry between the various solutions to a given equation and used it in an imaginative new way

Galois' ideas for using symmetry were profound and far-reaching, but none of this was fully understood at the time, and political events were overtaking his work

Rejected by the academic establishment, rejected by the state, rejected in romance, and losing the father he loved, there remained only the republican ideals to satisfy his anger

Galois: if a body was needed, it should be his. He would arrange a duel and a riot would take place at his funeral. (The dual took place, but not the riot at his funeral)

Galois' death at 20 achieved nothing for the revolution. For mathematics, however, his achievements will last forever

Part 2: Solvable and nilpotent non associative algebras

K. Meyberg, Lectures on algebras and triple systems, 1972

For any algebra A, define $A^{(0)} = A$, $A^{(k+1)} = A^{(k)}A^{(k)}$, so that

$$A = A^{(0)} \supset A^{(1)} \supset A^{(2)} \supset \cdots A^{(k)} \supset \cdots$$

A is **solvable** if $A^{(k)} = \{0\}$ for some k. (Notation: If B, C are subsets of A, BC denotes the subspace spanned by products bc where $b \in B, c \in C$.)

- Subalgebras and homomorphic images of solvable algebras are solvable.
- If I is an ideal in A, then A is solvable if and only if I and A/I are solvable.
- Every algebra contains a largest solvable ideal, called the **solvable radical**.

For any algebra A, define $A^0 = A$, $A^{k+1} = A^k A$, so that

$$A = A^0 \supset A^1 \supset A^2 \supset \cdots A^k \supset \cdots$$

A is **nilpotent** if $A^k = \{0\}$ for some k.

Part 3: Solvable and Nilpotent Lie Algebras

J. E. Humphreys, Introduction to Lie algebras and representation theory, 1972
Section 1: Definitions and Basic Concepts

definition, Jacobi identity, isomorphism, subalgebra

End(V), $g\ell(V) = \textbf{End}\,(V)^{-}$, linear Lie algebra

Classical Lie algebras A_k, B_k, C_k, D_k

derivation, inner derivation, adjoint representation

abelian Lie algebras, structure constants

Lie algebras of dimensions 1 and 2

ideal, center, derived algebra (see Part 2)

simple Lie algebra, $sl(2, F)$ is simple

quotient Lie algebra

normalizer and centralizer of a subset

homomorphism, homomorphism theorems

derivation, representation, automorphism group

exponential of a nilpotent derivation, inner automorphism

derived series, solvable Lie algebra (a main actor today)

Example: upper triangular matrices

radical, semisimple Lie algebra

decending central series, nilpotent Lie algebra

Example: strictly upper triangular matrices

ad-nilpotent element, Engel's theorem, flag

- If all elements of a Lie algebra are ad-nilpotent, then the algebra is nilpotent.
- A nilpotent Lie algebra can be represented by strictly upper triangular matrices.

solvable implies common eigenvector

• If a Lie algebra consists only of nilpotent elements, then there is a single vector which is an eigenvector corresponding to eigenvalue 0 for each and every element of the Lie algebra. (implies Engel's theorem)

Lie: solvable implies upper triangular (not necessarily strict)

If a linear Lie algebra is solvable, there is a common eigenvector for very element in the algebra

semisimple endomorphism

Jordan-Chevalley Decomposition

Cartan's criteria

Cartan: trace condition implies solvable

Part 4: Solvable and Nilpotent Leibniz algebras

I. S. Rakhimov, On classification problem of Loday algebras

It is well known that any associative algebra gives rise to a Lie algebra, with bracket $[x, y] := xy - yx$.

In 1990-s J.-L. Loday introduced a non-antisymmetric version of Lie algebras, whose bracket satisfies the Leibniz identity

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]$$

and therefore they have been called Leibniz algebras.

The Leibniz identity combined with antisymmetry, is a variation of the Jacobi identity, hence Lie algebras are antisymmetric Leibniz algebras.

The Leibniz algebras are characterized by the property that the multiplication (called a bracket) from the right is a derivation but the bracket is no longer skew-symmetric as for Lie algebras.

In fact, the definition above is a definition of the right Leibniz algebras, whereas the identity for the left Leibniz algebra is as follows

$$[x,[z,y]] = [[x,z],y] + [z,[x,y]], \text{ for all } x,y,z \in L.$$

The passage from the right to the left Leibniz algebra can be easily done by considering a new product "$[\cdot,\cdot]_{opp}$" on the algebra by "$[x,y]_{opp} = [y,x]$."

Clearly, a Lie algebra is a Leibniz algebra, and conversely, a Leibniz algebra L with property $[x,y] = -[y,x]$, for all $x,y \in L$ is a Lie algebra. Hence, we have an inclusion functor $inc : Lie \longrightarrow Leib$.

This functor has a left adjoint $imr : Leib \longrightarrow Lie$ which is defined on the objects by $L_{Lie} = L/I$, where I is the ideal of L generated by all squares. That is, any Leibniz algebra L gives rise to the Lie algebra L_{Lie}, which is obtained as the quotient of L by relation $[x,x] = 0$.

One has an exact sequence of Leibniz algebras:

$$0 \longrightarrow I \longrightarrow L \longrightarrow L_{Lie} \longrightarrow 0.$$

We consider finite-dimensional algebras L over a field K of characteristic 0 (in fact it is only important that this characteristic is not equal to 2).

A linear transformation d of a Leibniz algebra L is said to be a derivation if $d([x,y]) = [d(x),y] + [x,d(y)]$ for all $x, y \in L$.

Let consider $d_a : L \longrightarrow L$ defined by $d_a(x) = [x,a]$ for $a \in L$. Then the Leibniz identity is written as $d_a([x,y]) = [d_a(x),y] + [x,d_a(y)]$ for any $a, x \in L$ showing that the operator d_a for all $a \in L$ is a derivation on the Leibniz algebra L.

In other words, the right Leibniz algebra is characterized by this property, i.e., any right multiplication operator is a derivation of L. Notice that for the left Leibniz algebras a left multiplication operator is a derivation.

The theory of Leibniz algebras was developed by Loday himself with his coauthors. Mostly they dealt with the (co)homological problems of this class of algebras.

The study of structural properties of Loday algebras has begun after private conversation between Loday and Ayupov in Strasbourg in 1994.

The set of all derivations of L is denoted by $Der(L)$.

Due to the operation of commutation of linear operators $Der(L)$ is a Lie algebra.

The automorphism group $Aut(L)$ of the algebra L also can be naturally defined.

If the field K is $\mathbb{R}$ or $\mathbb{C}$, then the automorphism group is a Lie group and the Lie algebra $Der(L)$ is its Lie algebra.

One can consider $Aut(L)$ as an algebraic group (or as a group of K-points of an algebraic group defined over the field K).

Let L be any right Leibniz algebra. Consider a subspace spanned by elements of the form $[x, x]$ for all possible $x \in L$ denoted by I : $I = Span_K\{[x, x]|\ x \in L\}$.

In fact, I is a two-sided ideal in L. The product $[L, I]$ is equal to 0 due to the Leibniz identity. The fact that it is a right ideal follows from the identity

$$[[x, x], y] = [[y, y] + x, [y, y] + x] - [x, x] \text{ in } L.$$

For a non Lie Leibniz algebra L the ideal I always is different from the L.

The quotient algebra L/I is a Lie algebra. Therefore I can be viewed as a "non-Lie core" of the L.

The ideal I can also be described as the linear span of all elements of the form $[x, y] + [y, x]$. The quotient algebra L/I is called the liezation of L and it could be denoted by L_{Lie}.

Let us now consider the centers of Leibniz algebras. Since there is no commutativity there are left and right centers, which are given by

$$Z^l(L) = \{x \in L | [x, L] = 0\} \text{ and } Z^r(L) = \{x \in L | [L, x] = 0\},$$

respectively.

Both these centers can be considered for the left and right Leibniz algebras. For the right Leibniz algebra L the right center $Z^r(L)$ is an ideal, moreover it is two-sided ideal (since $[L, [x, y]] = -[L, [y, x]]$) but the $Z^l(L)$ not necessarily be a subalgebra even.

For the left Leibniz algebra it is exactly opposite.

In general, the left and right centers are different; even they may have different dimensions.

Obviously, $I \subset Z^r(L)$. Therefore $L/Z^r(L)$ is a Lie algebra, which is isomorphic to the Lie algebra $ad(L)$ mentioned above.

Many notions in the theory of Lie algebras may be naturally extended to Leibniz algebras.

For example, the solvability is defined by analogy to the definition of the derived series:

$$D^n(L) : D^1(L) = [L, L],\ D^{k+1}(L) = [D^k(L), D^k(L)],\ k = 1, 2, ...$$

A Leibniz algebra is said to be solvable if its derived series terminates.

It is easy to verify that the sum of solvable ideals in a Leibniz algebra also is a solvable ideal. Therefore, there exists a largest solvable ideal R containing all other solvable ideals.

Naturally, it is called the radical of Leibniz algebra. Since the ideal I of a Leibniz algebra L is abelian it is contained in the radical R of L.

The notion of nilpotency also can be defined by using the decreasing central series

$$C^n(L) : C^1(L) = [L, L],\ C^{k+1}(L) = [L, C^k(L)],\ k = 1, 2, 3, ... \text{ of } L.$$

Despite of a certain lack of symmetry of the definition (multiplication by L only from the right) members of this series are two-sided ideals, moreover, a simple observation shows that the inclusion $[C^p(L), C^q(L)] \subset C^{p+q}(L)$ is implied.

Leibniz algebra is called nilpotent if its central series terminates. As it is followed from the definition that the centers (left and right) for nilpotent Leibniz algebras are nontrivial. The following proposition can be easily proved.

Proposition

Any Leibniz algebra L has a maximal nilpotent ideal (containing all nilpotent ideals of L).

The nilradical of a Leibniz algebra L is maximal nilpotent ideal in L (which exists by the Proposition).

Due to this definition, the nilradical is a characteristic ideal, i.e. it remains invariant under all automorphisms of the Leibniz algebra L.

Obviously, it is contained in the radical of the Leibniz algebra and it equals to the nilradical of the solvable radical of L. The nilradical contains left center, as well as the ideal I.

Linear representation (sometimes referred as module) of a Leibniz algebra is a vector space V, equipped with two actions (left and right) of the Leibniz algebra L

$$[\cdot,\cdot] : L \times V \longrightarrow V \text{ and } [\cdot,\cdot] : V \times L \longrightarrow V,$$

such that the identity

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]]$$

is true whenever one (any) of the variables is in V, and the other two in L, i.e.,

- $[x,[y,v]] = [[x,y],v] + [y,[x,v]]$;
- $[x,[v,y]] = [[x,v],y] + [v,[x,y]]$;
- $[v,[x,y]] = [[v,x],y] + [x,[v,y]]$.

Note that the concept of representations of Lie algebras and Leibniz algebras are different.

Therefore, such an important theorem in the theory of Lie algebras, as the Ado theorem on the existence of faithful representation in the case of Leibniz algebras was proved much easier and gives a stronger result.

It is because the kernel of the Leibniz algebra representation is the intersection of kernels (in general,different one's) of right and left actions, in contrast to representations of Lie algebras, where these kernels are the same.

Therefore, an faithful representation of Leibniz algebras can be obtained easier than faithful representation of the case of Lie algebras.

Proposition (Barnes 2013)

Any Leibniz algebra has a faithful representation of dimension no more than $dim(L) + 1$.

Levi theorem for Leibniz algebras (Barnes 2012)

For a Leibniz algebra L there exists a subalgebra S (which is a semisimple Lie algebra), which gives the decomposition $L = S \dot{+} R$, where R is the radical.

Barnes has given the non-uniqueness of the subalgebra S (the minimum dimension of Leibniz algebra in which this phenomena appears is 6). It is known that in the case of Lie algebras the semi-simple Levi factor is unique up to conjugation.

Proposition (Lie Theorem for solvable Leibniz algebras)

A solvable right Leibniz algebra L over $\mathbb{C}$ has a complete flag of subspaces which is invariant under the right multiplication. In other words, all linear operators r_x of right multiplications can be simultaneously reduced to triangular form.

Proposition

Let R be the radical of a Leibniz algebra L, and N be its nilradical. Then $[L, R] \subset N$.

Two corollaries

- One has $[R, R] \subset N$. In particular, $[R, R]$ is nilpotent.
- Leibniz algebra L is solvable if and only if $[L, L]$ is nilpotent.

Nilpotent Leibniz algebras

Let L be a Leibniz algebra. define

$$L^1 = L, \; L^{k+1} = [L^k, L], \; k \geq 1.$$

Clearly,

$$L^1 \supseteq L^2 \supseteq \cdots$$

A Leibniz algebra L is said to be a nilpotent, if there exists $s \in \mathbb{N}$, such that

$$L^1 \supset L^2 \supset ... \supset L^s = \{0\}.$$

A Leibniz algebra L is said to be a filiform, if $dimL^i = n - i$, where $n = dimL$ and $2 \leq i \leq n$.

The class of filiform Leibniz algebras in dimension n over K we denote by $Lb_n(\mathbb{K})$. There is a breaking $Lb_n(\mathbb{C})$ into three subclasses:

$$Lb_n(\mathbb{C}) = FLb_n \cup SLb_n \cup TLb_n.$$

Engel's theorem for Leibniz algebras

If all operators r_x of right multiplication for the right Leibniz algebra L are nilpotent, then the algebra L is nilpotent. In particular, for right multiplications there is a common eigenvector with zero eigenvalue. In some basis the matrices of all r_x have upper-triangular form.

Two corollaries

- The normalizer (left - for the left Leibniz algebra) of some subalgebra M in a nilpotent Leibniz algebra L is not equal to the subalgebra M (it strictly contains M).
- A subspace $V \subset L$ generates a Leibniz algebra if and only if $V + [L, L] = L$.

It is interesting to note that not all properties of nilpotent Lie algebras, even a simple and well-known one's, hold for the case of Leibniz algebras.

For example, there is a simple statement for nilpotent Lie algebras of dimension 2 or more: "the codimension of the commutant is more or equal to 2".

For Leibniz algebras it is not true (though not only for nilpotent, but for all solvable Leibniz algebras we have $codim_L[L, L] > 0$).

For example, two-dimensional Leibniz algebra $L = span\{e_1, e_2\}$, with $[e_1, e_1] = e_2$ is nilpotent, but its commutant has codimension 1. This is due to the fact that its liezation is one-dimensional.

But for one-dimensional Lie algebras above mentioned statement is incorrect.

Corollary

If Leibniz algebra L is nilpotent and $codim_L([L, L]) = 1$, then the algebra L is generated by one element.

So for $codim_L([L, L]) = 1$ a nilpotent Leibniz algebra is a kind of "cyclic". The study of such nilpotent algebras is the specifics of the theory of Leibniz algebras; Lie algebra has no analogue. Such cyclic L can be explicitly described. The minimal number of generators of a Leibniz algebra L equals $dim\ L/[L, L]$.

Classification of complex Leibniz algebras

Two-dimensional Leibniz algebras have been given by Loday

J.-L. Loday, Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *L'Ens. Math.*, 39, 1993, 269–293 (in French).

In dimension three there are fourteen isomorphism classes (5 parametric family of orbits and 9 single orbits), the list can be found in

J.M. Casas, M.A. Insua, M. Ladra, S. Ladra, An algorithm for the classification of 3-dimensional complex Leibniz algebras, *Linear Algebra and Appl.*, 436, 2012, 3747–3756.
and

I.M. Rikhsiboev, I.S. Rakhimov, Classification of three-dimensional complex Leibniz algebras, *International Journal of Modern Physics, Conference Series* (IJMPCS), 1450(1), 2012, 358–362.

There is no simple Leibniz algebra in dimension three.

Starting dimension four there are partial classifications. The list of isomorphisms classes of four-dimensional nilpotent Leibniz algebras has been given in

S. ALBEVERIO, B.A. OMIROV, I.S. RAKHIMOV, Classification of 4-dimensional nilpotent complex Leibniz algebras, *Extracta Mathematicae*, 21(3), 2006, 197–210.

Papers on classification of low-dimensional complex solvable Leibniz algebras:
• E.M. CAÑETE, A.KH. KHUDOYBERDIYEV, The classification of 4-dimensional Leibniz algebras, *Linear Algebra and Appl.*, 439(1), 2013, 273–288.,
• J.M CASAS, M. LADRA, B.A. OMIROV, I.A. KARIMJANOV, Classification of solvable Leibniz algebras with naturally graded filiform nil-radical, *Linear Algebra and Appl.*, 438(7), 2013, 2973–3000.,
• J.M. CASAS, M. LADRA, B.A. OMIROV, I.A. KARIMJANOV, Classification of solvable Leibniz algebras with null-filiform nilradical, *Linear and Multilinear Algebra,* 61(6), 2013, 758–774,
• A.KH. KHUDOYBERDIYEV, M. LADRA, B.A. OMIROV, On solvable Leibniz algebras whose nilradical is a direct sum of null-filiform algebras, *Linear and Multilinear Algebra*, 62(9), 2014, 1220–1239,
• A.KH. KHUDOYBERDIYEV, I.S. RAKHIMOV, S.K. SAID HUSAIN, On classification of 5-dimensional solvable Leibniz algebras, *Linear algebra and Appl.*, 457, 2014, 428–454.

The notion of filiform Leibniz algebra was introduced by Ayupov and Omirov in

SH.A. AYUPOV, B.A. OMIROV, On some classes of nilpotent Leibniz algebras, *Sibirsk. Mat. Zh.*, 42(1), 2001, 18–29 (in Russian) (English translation in Siberian Math. J., 42(1), 2001, 15 - 24).

According to Ayupov-Gómez-Omirov theorem, the class of all filiform Leibniz algebras is split into three subclasses which are invariant with respect to the action of the general linear group. One of these classes contains the class of filiform Lie algebras.

There is a classification of the class of filiform Lie algebras in small dimensions (Gómez-Khakimdjanov) and there is a classification of filiform Lie algebras admitting a non trivial Malcev Torus (Goze-Khakimdjanov) in

M. GOZE, YU.B. KHAKIMDJANOV, Sur les algèbres de Lie nilpotentes admettant un tore de dérivations, *Manuscripta Math.*, 1994, 84, 115–124 (in French).

The other two of the three classes come out from naturally graded non-Lie filiform Leibniz algebras. For this case the isomorphism criteria have been given in

J.R. GÓMEZ, B.A. OMIROV, *On classification of complex filiform Leibniz algebras*, 2006, http://front.math.ucdavis.edu/, arXiv: 0612735 v1 [math.RA]..

A method of classification of filiform Leibniz algebras based on algebraic invariants has been developed in

I.S. RAKHIMOV, U.D. BEKBAEV, On isomorphisms and invariants of finite dimensional complex filiform Leibniz algebras, *Communications in Algebra*, 38(12), 2010, 4705–4738.

Then the method has been implemented to low-dimensional cases in

I.S. RAKHIMOV, S.K. SAID HUSAIN, On isomorphism classes and invariants of a subclass of low dimensional complex filiform Leibniz algebras, *Linear and Multilinear Algebra*, 59(2), 2011, 205–220,

and

I.S. RAKHIMOV, S.K. SAID HUSAIN, Classification of a subclass of low-dimensional complex filiform Leibniz algebras, *Linear and Multilinear Algebra*, 59(3), 2011, 339–354.

The third class that comes out from naturally graded filiform Lie algebras, has been treated in the paper

B.A. OMIROV, I.S. RAKHIMOV, On Lie-like filiform Leibniz algebras, *Bull. Aust. Math. Soc.*, 79, 2009, 391–404.

Then the classifications of some subclasses and low-dimensional cases of this class have been given in

I.S. RAKHIMOV, M.A. HASSAN, Isomorphisms and invariants of low-dimensional filiform Leibniz algebras arising from filiform graded Lie algebras, *Bull. Malaysian Math. Sci. Soc.*, 2011, 34(3), 475–485.

I.S. RAKHIMOV, M.A. HASSAN, On one-dimensional central extension of a filiform Lie algebra, *Bull. Aust. Math. Soc.*, 84, 2011, 205-224.

I.S. RAKHIMOV, M.A. HASSAN, On isomorphism criteria for Leibniz central extensions of a linear deformation of μ_n, *International Journal of Algebra and Computations*, 21(5), 2011, 715–729.

In
Sh.A. Ayupov, B.A. Omirov, On some classes of nilpotent Leibniz algebras, *Sibirsk. Mat. Zh.*, 42(1), 2001, 18–29 (in Russian) (English translation in Siberian Math. J., 42(1), 2001, 15 - 24).
he authors split the class of complex filiform Leibniz algebras obtained from naturally graded filiform non-Lie Leibniz algebras into two disjoint classes.

If we add here the class of filiform Leibniz algebras appearing from naturally graded filiform Lie algebras then the final result can be written as follows.

Theorem

Any $(n+1)$-dimensional complex non-Lie filiform Leibniz algebra L admits a basis $\{e_0, e_1, e_2, ..., e_n\}$ such that L has a table of multiplication one of the following form (unwritten product are supposed to be zero)

$$FLb_{n+1} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1 \\ [e_0, e_1] = \alpha_3 e_3 + \alpha_4 e_4 + ... + \alpha_{n-1} e_{n-1} + \theta e_n, & \\ [e_j, e_1] = \alpha_3 e_{j+2} + \alpha_4 e_{j+3} + ... + \alpha_{n+1-j} e_n, & 1 \leq j \leq n-2 \end{cases}$$

(Theorem continues on next page)

$$SLb_{n+1} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 2 \le i \le n-1 \\ [e_0, e_1] = \beta_3 e_3 + \beta_4 e_4 + ... + \beta_n e_n, & \\ [e_1, e_1] = \gamma e_n, & \\ [e_j, e_1] = \beta_3 e_{j+2} + \beta_4 e_{j+3} + ... + \beta_{n+1-j} e_n, & 2 \le j \le n-2 \end{cases}$$

$$TLb_{n+1} =$$

$$\begin{cases} [e_0, e_0] = e_n, & \\ [e_1, e_1] = \alpha e_n, & \\ [e_i, e_0] = e_{i+1}, & 1 \le i \le n-1 \\ [e_0, e_1] = -e_2 + \beta e_n, & \\ [e_0, e_i] = -e_{i+1}, & 2 \le i \le n-1 \\ [e_i, e_j] = -[e_j, e_i] \in lin < e_{i+j+1}, e_{i+j+2}, \ldots, e_n >, & 1 \le i \le n-3, \\ & 2 \le j \le n-1-i \\ [e_{n-i}, e_i] = -[e_i, e_{n-i}] = (-1)^i \delta e_n, & 1 \le i \le n-1 \end{cases} \quad \text{where}$$

$[\cdot,\cdot]$ is the multiplication in L and $\delta \in \{0,1\}$ for odd n and $\delta = 0$ for even n.

Part 5: Semisimple Leibniz algebras

The quotient of a Leibniz algebra with respect to the ideal I generated by squares is a Lie algebra and I itself can be regarded as a module over this Lie algebra.

There are results on description of such a Leibniz algebras with a fixed quotient Lie algebra. The case $L/I = sl_2$ has been treated in

B.A. Omirov, I.S. Rakhimov, R. Turdibaev, On description of Leibniz algebras corresponding to sl_2, *Algebras and Representation Theory*, 16(5), 2013, 1507–1519.

In

L.M. Camacho, S. Gómez-Vidal, B.A. Omirov, *Leibniz algebras with associated Lie algebras* $sl_2 \dot{+} R$ $(dimR = 2)$, arXiv: 111.4631v1, [math. RA], 2011.

the authors describe Leibniz algebras L with $L/I = sl_2 \dot{+} R$, where R is solvable and $dimR = 2$.

When $L/I = sl_2 \dot{+} R$ with $dimR = 3$ the result has been given in

I.S. Rakhimov, I.M. Rikhsiboev, A.Kh. Khudoyberdiyev, I.A. Karimjonov, Description of some classes of Leibniz algebras, *Linear Algebra and Appl.*, 437, 2012, 2209–2227.

All these results are based on the classical result on description of irreducible representations of the simple Lie algebra sl_2.

Unfortunately, the decomposition of a semisimple Leibniz algebra into direct sum of simple ideals is not true. Here an example from

L.M. Camacho, S. Gómez-Vidal, B.A. Omirov, I.A. Karimdjanov, Leibniz algebras whose semisimple part is related to sl_2, *Bull. Malaysian Math. Sci. Soc.*, 2015 (to appear)

supporting this claim. Let L be a complex Leibniz algebra satisfying the following conditions

(a) $L/I \cong sl_2^1 \oplus sl_2^2$;

(b) $I = I_{1,1} \oplus I_{1,2}$ such that $I_{1,1}, I_{1,2}$ are irreducible sl_2^1-modules and $dim I_{1,1} = dim I_{1,2}$;

(c) $I = I_{2,1} \oplus I_{2,2} \oplus ... \oplus I_{2,m+1}$ such that $I_{2,k}$ are irreducible sl_2^2-modules with $1 \leq k \leq m+1$.

Then there is a basis $\{e_1, f_1, h_1, e_2, f_2, h_2, x_0^1, x_1^1, x_2^1, ..., x_m^1, x_0^2, x_1^2, x_2^2, ..., x_m^2\}$ such that the table of multiplication of L in this basis is represented as follows:

$$L \cong \quad \begin{cases} [e_i, h_i] = -[h_i, e_i] = 2e_i, & \\ [e_i, f_i] = -[f_i, e_i] = h_i, & \\ [h_i, f_i] = -[f_i, h_i] = 2f_i, & \\ [x_k^i, h_1] = (m-2k)x_k^i, & 0 \leq k \leq m, \\ [x_k^i, f_1] = x_{k+1}^i, & 0 \leq k \leq m-1, \\ [x_k^i, e_1] = -k(m+1-k)x_{k-1}^i, & 1 \leq k \leq m, \\ [x_j^1, e_2] = [x_j^2, h_2] = x_j^2, & \\ [x_j^1, h_2] = [x_j^2, f_2] = -x_j^1, & \end{cases}$$

with $1 \leq i \leq 2$ and $0 \leq j \leq m$. The algebra L cannot be represented as a direct sum of simple Leibniz algebras.