Associative, Jordan, and triple derivations

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Derivations on matrix algebras

We consider the algebra $M_n(\mathbb{C})$ of all n by n complex matrices.

Matrix units

$E_{kl} = (a_{ij})$ where $a_{ij} = \delta_{(i,j),(k,l)}$

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

PROPERTIES OF MATRIX UNITS

- $\sum_{j=1}^{n} E_{jj} = I$
- $E_{ij}^t = E_{ji}$
- $E_{ij}E_{kl} = \delta_{kl}E_{il}$

THEOREM 1

Let $\delta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a derivation: δ is linear and $\delta(AB) = A\delta(B) + \delta(A)B$. Then there exists a matrix K such that $\delta(X) = XK - KX$ for X in $M_n(\mathbb{C})$.

COROLLARY

$H^1(M_n(\mathbb{C}), M_n(\mathbb{C})) = 0$

PROOF OF THEOREM 1 (from Blackadar book)

$$\begin{aligned} 0 &= \delta(1) = \delta(E_{11} + E_{22}) = \delta(E_{11}) + \delta(E_{22}) \\ &= \delta(E_{11}E_{11}) + \delta(E_{21}E_{12}) \\ &= E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} + E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} \\ &= E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) + \delta(E_{11})E_{11} + \delta(E_{21})E_{12}. \end{aligned}$$

Let $K = E_{11}\delta(E_{11}) + E_{21}\delta(E_{12}) = -\delta(E_{11})E_{11} - \delta(E_{21})E_{12}$. Then

- $KE_{11} = -\delta(E_{11})E_{11} \quad , \quad E_{11}K = E_{11}\delta(E_{11})$
- $KE_{12} = -\delta(E_{11})E_{12} \quad , \quad E_{12}K = E_{11}\delta(E_{12})$
- $KE_{21} = -\delta(E_{21})E_{11} \quad , \quad E_{21}K = E_{21}\delta(E_{11})$
- $KE_{22} = -\delta(E_{21})E_{12} \quad , \quad E_{22}K = E_{21}\delta(E_{12})$

- $E_{11}K - KE_{11} = E_{11}\delta(E_{11}) + \delta(E_{11})E_{11} = \delta(E_{11}E_{11}) = \delta(E_{11})$
- $E_{12}K - KE_{12} = E_{11}\delta(E_{12}) + \delta(E_{11})E_{12} = \delta(E_{11}E_{12}) = \delta(E_{12})$
- $E_{21}K - KE_{21} = E_{21}\delta(E_{11}) + \delta(E_{21})E_{11} = \delta(E_{21}E_{11}) = \delta(E_{21})$
- $E_{22}K - KE_{22} = E_{21}\delta(E_{12}) + \delta(E_{21})E_{12} = \delta(E_{21}E_{12}) = \delta(E_{22})$ Q.E.D.

Another proof of Theorem 1 (Kadison and Ringrose Acta Math 1972)

DEFINITION

Let ρ be a linear transformation on M_2. We define linear transformations σ_1 and σ_2 on M_2 by

$$\sigma_1(A) = E_{11}\rho(E_{11}A) + E_{21}\rho(E_{12}A)$$

and

$$\sigma_2(A) = E_{12}\rho(E_{21}A) + E_{22}\rho(E_{22}A)$$

LEMMA 1

$\sigma_1(A) = A\sigma_1(I)$ and $\sigma_2(A) = A\sigma_2(I)$

We only need σ_1 or σ_2, not both. We'll go with σ_1.

PROOF OF LEMMA 1

$E_{11}AE_{11} = c_{11}E_{11}$, $E_{12}AE_{11} = c_{21}E_{11}$, $E_{11}AE_{21} = c_{12}E_{11}$, $E_{12}AE_{21} = c_{22}E_{11}$

- $E_{11}A = E_{11}AE_{11}E_{11} + E_{11}AE_{21}E_{12} = c_{11}E_{11} + c_{12}E_{11}E_{12} = c_{11}E_{11} + c_{12}E_{12}$
- $E_{12}A = E_{12}AE_{11}E_{11} + E_{12}AE_{21}E_{12} = c_{21}E_{11} + c_{22}E_{11}E_{12} = c_{21}E_{11} + c_{22}E_{12}$

- $AE_{11} = E_{11}E_{11}AE_{11} + E_{21}E_{12}AE_{11} = c_{11}E_{11} + c_{21}E_{21}E_{11} = c_{11}E_{11} + c_{21}E_{21}$
- $AE_{21} = E_{11}E_{11}AE_{21} + E_{21}E_{12}AE_{21} = c_{12}E_{11} + c_{22}E_{21}E_{11} = c_{12}E_{21} + c_{22}E_{21}$

$$\begin{aligned}
\sigma_1(A) &= E_{11}\rho(E_{11}A) + E_{21}\rho(E_{12}A) \\
&= E_{11}\rho(c_{11}E_{11} + c_{12}E_{12}) + E_{21}\rho(c_{21}E_{11} + c_{22}E_{12}) \\
&= c_{11}E_{11}\rho(E_{11}) + c_{12}E_{11}\rho(E_{12}) + c_{21}E_{21}\rho(E_{11}) + c_{22}E_{21}\rho(E_{12}) \\
&= (c_{11}E_{11} + c_{21}E_{21})\rho(E_{11}) + (c_{12}E_{11} + c_{22}E_{21})\rho(E_{12}) \\
&= AE_{11}\rho(E_{11}) + AE_{21}\rho(E_{12}) \\
&= A\sigma_1(1) \quad \text{Q.E.D.}
\end{aligned}$$

Second proof of Theorem 1 (Kadison-Ringrose)

Let $\sigma(a) = \sum_j e_{j1}\rho(e_{1j}a)$. Then $\sigma(a) = a\sigma(1)$ and $\sigma(1) = \sum_j e_{j1}\rho(e_{1j})$

Let T_0 be the linear transformation from $M_n(\mathbb{C})$ to linear transformations on $M_n(\mathbb{C})$ defined by $T_0(b)(x) = xb - bx$.

Let T_1 be the linear transformation from $L(M_n(\mathbb{C}))$ to bilinear transformations on $M_n(\mathbb{C})$ defined by $T_1 f(a,b) = af(b) - f(ab) + f(a)b$.

Then $T_0(\sigma(1))(x) = x\sigma(1) - \sigma(1)x = \sum_j e_{j1}\rho(e_{1j}x) - \sigma(1)x$

If ρ is a derivation, then $0 = \sum_j e_{j1}(T_1\rho)(e_{1j}, x)$

$= \sum_j e_{j1}(e_{1j}\rho(x) - \rho(e_{1j}x) + \rho(e_{1j})x) = \rho(x) - \sum_j e_{j1}\rho(e_{1j}x) + \sum_j e_{j1}\rho(e_{1j})x$

Thus $x\sigma(1) - \sigma(1)x = \rho(x)$ Q.E.D.

Jordan derivations

DEFINITION

A linear map D on $M_n(\mathbb{C})$ is a Jordan derivation if
$D(ab+ba) = (Da)b + b(Da) + (Db)a + a(Db)$

This is the same as $D(a^2) = (Da)a + a(Da)$

THEOREM 2

Let $D : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a Jordan derivation. Then D is an inner (associative) derivation, that is, there exists a matrix K such that $\delta(X) = XK - KX$ for X in $M_n(\mathbb{C})$.

Since every derivation is a Jordan derivation, Theorem 2 provides a third proof of Theorem 1.

Diagonals

Let $d = \frac{1}{n}\sum_{i,j} e_{ij} \otimes e_{ji}$. Then d is a **diagonal** for $M_n(\mathbb{C})$, that is, $\pi(d) = 1$ and $a \cdot d = d \cdot a$ for all $a \in M_n(\mathbb{C})$, where $\pi(x \otimes y) = xy$, $a \cdot x \otimes y = (ax) \otimes y$ and $x \otimes y \cdot a = x \otimes (ya)$.
Explicitly, $\pi(d) = \frac{1}{n}\sum_{i,j} e_{ij}e_{ji} = 1$, $\frac{1}{n}\sum_{i,j}(ae_{ij}) \otimes e_{ji} = \frac{1}{n}\sum_{i,j} e_{ij} \otimes (e_{ji}a)$

The symmetric nature of d implies $\frac{1}{n}\sum_{i,j}(e_{ij}a) \otimes e_{ji} = \frac{1}{n}\sum_{i,j} e_{ij} \otimes (ae_{ji})$
For any linear transformation D, apply $1 \otimes D$ and then π, to get
$\frac{1}{n}\sum_{i,j}(ae_{ij})D(e_{ji}) = \frac{1}{n}\sum_{i,j} e_{ij}D(e_{ji}a)$ and $\frac{1}{n}\sum_{i,j}(e_{ij}a)D(e_{ji}) = \frac{1}{n}\sum_{i,j} e_{ij}D(ae_{ji})$

Proof of Theorem 2 (Barry Johnson 1996)

Let $D : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a Jordan derivation

Define $x = \frac{1}{n}\sum_{i,j} e_{ij} De_{ji}$. Then

$ax = \frac{1}{n}\sum_{i,j} ae_{ij} De_{ji} = \frac{1}{n}\sum_{i,j} e_{ij} D(e_{ji}a)$

$D(e_{ji}a) + D(ae_{ji}) = e_{ji}Da + (Da)e_{ji} + (De_{ji})a + aDe_{ji}$

$ax = \frac{1}{n}\sum_{i,j} e_{ij}[e_{ji}Da + (D(e_{ji})a + (Da)e_{ji} + aDe_{ji} - D(ae_{ji})]$

$ax = Da + xa + \Delta(a) + 0$, where

$\Delta(a) = \frac{1}{n}\sum_{i,j} e_{ij}(Da)e_{ji}$ (recall that $\frac{1}{n}\sum_{i,j}(e_{ij}a)D(e_{ji}) = \frac{1}{n}\sum_{i,j} e_{ij}D(ae_{ji})$)

Δ is a Jordan derivation with $a\Delta(b) = \Delta(b)a$, that is,
$\frac{1}{n}\sum_{i,j} ae_{ij}(Db)e_{ji} = \frac{1}{n}\sum_{i,j} e_{ij}(Db)e_{ji}a$
Proof: Apply $R_{Db} \otimes 1$, then π to $\frac{1}{n}\sum_{i,j}(ae_{ij}) \otimes e_{ji} = \frac{1}{n}\sum_{i,j} e_{ij} \otimes (e_{ji}a)$

Start over with D replaced by Δ
$x_0 = \frac{1}{n}\sum_{i,j} e_{ij}\Delta(e_{ji})$
$ax_0 = \Delta a + x_0 a + \frac{1}{n}\sum_{i,j} e_{ij}\Delta(a)(e_{ji}) = 2\Delta a + x_0 a$

$\Delta a = \frac{1}{2}(ax_0 - x_0 a)$
$Da = ax - xa - \Delta a = a(x - \frac{1}{2}x_0) - (x - \frac{1}{2}x_0)a$
is an inner associative derivation. Q.E.D.

Yet another proof of Theorem 1 (Jacobson 1937)

If δ is a derivation, consider the two representations of $M_n(\mathbb{C})$

$$z \mapsto \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \text{ and } z \mapsto \begin{bmatrix} z & 0 \\ \delta(z) & z \end{bmatrix}$$

The first is a direct sum of two copies of the identity representation; but so is the second, since

$$\begin{bmatrix} 0 & 0 \\ \delta(z) & z \end{bmatrix} \text{ is equivalent to } \begin{bmatrix} 0 & 0 \\ 0 & z \end{bmatrix}$$

so

$$\begin{bmatrix} z & 0 \\ \delta(z) & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}$$

Thus $az = za, \ bz = zb$

$$\delta(z)a = cz - zc \text{ and } \delta(z)b = dz - zd.$$

a and b are multiples of I and can't both be zero. QED

Modules

Let A be an associative algebra. Let us recall that an A-**bimodule** is a vector space X, equipped with two bilinear products $(a,x) \mapsto ax$ and $(a,x) \mapsto xa$ from $A \times X$ to X satisfying the following axioms for every $a, b \in A$ and $x \in X$:

$$a(bx) = (ab)x, \quad a(xb) = (ax)b, \text{ and } (xa)b = x(ab),$$

The space $A \oplus X$ is an associative algebra with respect to the product

$$(a,x)(b,y) := (ab, ay + bx).$$

Let A be a Jordan algebra. A **Jordan** A**-module** is a vector space X, equipped with two bilinear products $(a,x) \mapsto a \circ x$ and $(x,a) \mapsto x \circ a$ from $A \times X$ to X, satisfying:

$$a \circ x = x \circ a, \quad a^2 \circ (x \circ a) = (a^2 \circ x) \circ a, \text{ and,}$$

$$2((x \circ a) \circ b) \circ a + x \circ (a^2 \circ b) = 2(x \circ a) \circ (a \circ b) + (x \circ b) \circ a^2,$$

for every $a, b \in A$ and $x \in X$.

The space $A \oplus X$ is a Jordan algebra with respect to the product

$$(a,x) \circ (b,y) := (a \circ b, a \circ y + b \circ x).$$

Jordan triple system

A complex (resp., real) **Jordan triple** is a complex (resp., real) vector space E equipped with a triple product

$$E \times E \times E \to E \qquad (xyz) \mapsto \{x, y, z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one and satisfying the so-called *"Jordan Identity"*:

$$L(a,b)L(x,y) - L(x,y)L(a,b) = L(L(a,b)x,y) - L(x,L(b,a)y),$$

for all a, b, x, y in E, where $L(x,y)z := \{x,y,z\}$.

The Jordan identity is equivalent to

$$\{a,b,\{c,d,e\}\} = \{\{a,b,c\},d,e\} - \{c,\{b,a,d\},e\} + \{c,d,\{a,b,e\}\},$$

which asserts that the map $iL(a,a)$ is a *triple derivation* (to be defined shortly).

It also shows that the span of the "multiplication" operators $L(x,y)$ is a Lie algebra.

Jordan triple module

Let E be a complex (resp. real) Jordan triple. A **Jordan triple E-module** is a vector space X equipped with three mappings

$$\{.,.,.\}_1 : X \times E \times E \to X, \quad \{.,.,.\}_2 : E \times X \times E \to X$$

$$\text{and } \{.,.,.\}_3 : E \times E \times X \to X$$

in such a way that the space $E \oplus X$ becomes a real Jordan triple with respect to the triple product $\{a_1 + x_1, a_2 + x_2, a_3 + x_3\} =$

$$\{a_1, a_2, a_3\}_E + \{x_1, a_2, a_3\}_1 + \{a_1, x_2, a_3\}_2 + \{a_1, a_2, x_3\}_3.$$

(PS: we don't really need the subscripts on the triple products)

The Jordan identity

$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\}$,
holds whenever exactly one of the elements belongs to X.

In the complex case we have the unfortunate technical requirement that

$\{x, a, b\}_1$ $(=\{b, a, x\}_3)$ is linear in a and x and conjugate linear in b

$\{a, x, b\}_2$ is conjugate linear in a, b, x.

Every (associative) Banach A-bimodule (resp., Jordan Banach A-module) X over an associative Banach algebra A (resp., Jordan Banach algebra A) is a real Banach triple A-module (resp., A-module) with respect to the "*elementary*" product

$$\{a, b, c\} := \frac{1}{2}(abc + cba)$$

(resp., $\{a, b, c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$), where one element of a, b, c is in X and the other two are in A.

The dual space, E^*, of a complex (resp., real) Jordan Banach triple E is a complex (resp., real) triple E-module with respect to the products:

$$\{a, b, \varphi\}(x) = \{\varphi, b, a\}(x) := \varphi\{b, a, x\} \tag{1}$$

and

$$\{a, \varphi, b\}(x) := \overline{\varphi\{a, x, b\}}, \tag{2}$$

$\forall x \in X, a, b \in E, \varphi \in E^*$. (the ''usual" adjoint action)

Derivations

Let X be a Banach A-bimodule over an (associative) Banach algebra A. A linear mapping $D : A \to X$ is said to be a **derivation** if $D(ab) = D(a)b + aD(b)$, for every a, b in A. For emphasis we call this a **binary (or associative) derivation**.

We denote the set of all continuous binary derivations from A to X by $\mathcal{D}_b(A, X)$.

When X is a Jordan Banach module over a Jordan Banach algebra A, a linear mapping $D : A \to X$ is said to be a **derivation** if $D(a \circ b) = D(a) \circ b + a \circ D(b)$, for every a, b in A. For emphasis we call this a **Jordan derivation**.

We denote the set of continuous Jordan derivations from A to X by $\mathcal{D}_J(A, X)$.

In the setting of Jordan Banach triples, a **triple** or **ternary derivation** from a (real or complex) Jordan Banach triple, E, into a Banach triple E-module, X, is a conjugate linear mapping $\delta : E \to X$ satisfying

$$\delta \{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}, \tag{3}$$

for every a, b, c in E.

We denote the set of all continuous ternary derivations from E to X by $\mathcal{D}_t(E, X)$.

Inner derivations

Let X be a Banach A-bimodule over an associative Banach algebra A. Given x_0 in X, the mapping $D_{x_0} : A \to X$, $D_{x_0}(a) = x_0 a - a x_0$ is a bounded (associative or binary) derivation. Derivations of this form are called **inner**.

The set of all inner derivations from A to X will be denoted by $\mathcal{I}nn_b(A, X)$.

When x_0 is an element in a Jordan Banach A-module, X, over a Jordan Banach algebra, A, for each $b \in A$, the mapping $\delta_{x_0,b} : A \to X$,

$$\delta_{x_0,b}(a) := (x_0 \circ a) \circ b - (b \circ a) \circ x_0, \ (a \in A),$$

is a bounded derivation. Finite sums of derivations of this form are called **inner**.

The set of all inner Jordan derivations from A to X is denoted by $\mathcal{I}nn_J(A, X)$

Let E be a complex (resp., real) Jordan triple and let X be a triple E-module. For each $b \in E$ and each $x_0 \in X$, we conclude, via the main identity for Jordan triple modules, that the mapping $\delta = \delta(b, x_0) : E \to X$, defined by

$$\delta(a) = \delta(b, x_0)(a) := \{b, x_0, a\} - \{x_0, b, a\} \quad (a \in E), \tag{4}$$

is a ternary derivation from E into X. Finite sums of derivations of the form $\delta(b, x_0)$ are called **inner triple derivations**.

The set of all inner ternary derivations from E to X is denoted by $\mathcal{I}nn_t(E, X)$.

Our proof of Theorem 3 below uses the following proposition, due to Ho,Peralta, and Russo 2012.

Proposition

Let A be a unital Banach *-algebra equipped with the ternary product given by $\{a, b, c\} = \frac{1}{2}\,(ab^*c + cb^*a)$ and the Jordan product $a \circ b = (ab + ba)/2$.

(a) Let D be an element in $\mathcal{I}nn_b(A, A)$, that is, $D = \mathrm{ad}\, a$ for some a in A. Then D is a *-derivation whenever $a^* = -a$. Conversely, if D is a *-derivation, then $a^* = -a + z$ for some z in the center of A.

(b) $\mathcal{D}_t(A, A) = \mathcal{D}_J^*(A, A) + \mathcal{I}nn_t(A, A)$.

Theorem 3

Let M be any von Neumann algebra. Then

- Every Jordan derivation of M is an inner Jordan derivation.
- Every triple derivation of M is an inner triple derivation.

Proof of Theorem 3 (Pluta-R)

To prove the second statement, it suffices, by Proposition (b), to show that $\mathcal{D}_J^*(M,M) \subset \mathcal{I}nn_t(M,M)$. Suppose δ is a self-adjoint Jordan derivation of M. By Theorem 2, δ is an inner associative derivation so by Proposition (a), $\delta(x) = ax - xa$ where $a^* + a = z$ is a self adjoint element of the center of M. [a] Since $M = Z(M) + [M,M]$, where $Z(M)$ denotes the center of M, we can write $a = z' + \sum_j [b_j + ic_j, b_j' + ic_j']$, where b_j, b_j', c_j, c_j' are self adjoint elements of M and $z' \in Z(M)$. It follows that
$0 = a^* + a - z = (z')^* + z' - z + 2i\sum_j([c_j, b_j'] + [b_j, c_j'])$ so that $\sum_j([c_j, b_j'] + [b_j, c_j'])$ belongs to the center of M. We now have

$$\delta = \mathrm{ad}\, a = \mathrm{ad}\, \sum_j ([b_j, b_j'] - [c_j, c_j']) \tag{5}$$

and therefore a direct calculation shows that δ is equal to the inner triple derivation

$$\sum_j \left(L(b_j, 2b_j') - L(2b_j', b_j) - L(c_j, 2c_j') + L(2c_j', c_j) \right).$$

[a] You would need Theorem 1 in situations more general than $M_n(\mathbb{C})$.

Proof of the first statement

We have just shown that a self adjoint Jordan derivation δ of M has the form (5). Then another direct calculation shows that δ is equal to the inner Jordan derivation

$$4\sum_j \left(L(b_j)L(b_j') - L(b_j')L(b_j) - L(c_j)L(c_j') + L(c_j')L(c_j)\right).$$

If δ is any Jordan derivation, so are δ^* and $i\delta$, so δ is an inner Jordan derivation.

Details

Let $\delta = \mathrm{ad}\,[b, b']$. Then
$\delta(x) = (bb' - b'b)x - x(bb' - b'b) = bb'x - b'bx - xbb' + xb'b$

$(L(b, 2b') - L(2b', b))(x) = (b(2b')^*x + x(2b')^*b)/2 - (2b'(b')^*x + xb^*2b')/2 =$
$bb'x - b'bx - xbb' + xb'b$

$4(L(b)L(b') - L(b')L(b))(x) = 4b \circ (b' \circ x) - 4b' \circ (b \circ x) =$
$b(b'x + xb') + (b'x + xb')b - b'(bx + xb) - (bx + xb)b' = bb'x - b'bx - xbb' + xb'b$

Ternary Weak Amenability (Ho-Peralta-R)

Proposition

Let A be a unital Banach *-algebra equipped with the ternary product given by $\{a, b, c\} = \frac{1}{2}\,(ab^*c + cb^*a)$ and the Jordan product $a \circ b = (ab + ba)/2$. Then

$$\mathcal{D}_t(A, A^*) \subset \mathcal{D}_J^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*).$$

Proposition

Every commutative (real or complex) C*-algebra A is **ternary weakly amenable**, that is $\mathcal{D}_t(A, A^*) = \mathcal{I}nn_t(A, A^*)$ ($\neq 0$ btw).

Proposition

The C*-algebra $A = M_n(\mathbb{C})$ is ternary weakly amenable (Hochschild 1945) and **Jordan weakly amenable** (Jacobson 1951).

Question

Is $C_0(X, M_n(\mathbb{C}))$ ternary weakly amenable?

Negative results

Proposition

The C*-algebra $A = K(H)$ of all compact operators on an infinite dimensional Hilbert space H is **not** ternary weakly amenable.

Proposition

The C*-algebra $A = B(H)$ of all bounded operators on an infinite dimensional Hilbert space H is **not** ternary weakly amenable.

Non algebra results

Theorem

Let H and K be two complex Hilbert spaces with $\dim(H) = \infty > \dim(K)$. Then the rectangular complex Cartan factor of type I, $B(H, K)$, and all its real forms are **not** ternary weakly amenable. (triple product: $\{xyz\} = (xy^*z + zy^*x)/2$)

Theorem

Every commutative (real or complex) JB*-triple (**def:** $\|\{xxx\}\| = \|x\|^3$ and $L(x, x)$ hermitian positive) E is **approximately ternary weakly amenable**, that is, $\mathcal{I}nn_t(E, E^*)$ is a norm-dense subset of $\mathcal{D}_t(E, E^*)$.

Commutative Jordan Gelfand Theory (Kaup,Friedman-R)

Given a commutative (complex) JB*-triple E, there exists a principal $\mathbb{T}$-bundle $\Lambda = \Lambda(E)$, i.e. a locally compact Hausdorff space Λ together with a continuous mapping $\mathbb{T} \times \Lambda \to \Lambda$, $(t, \lambda) \mapsto t\lambda$ such that $s(t\lambda) = (st)\lambda$, $1\lambda = \lambda$ and $t\lambda = \lambda \Rightarrow t = 1$, satisfying that E is JB*-triple isomorphic to

$$\mathcal{C}_0^{\mathbb{T}}(\Lambda) := \{f \in \mathcal{C}_0(\Lambda) : f(t\lambda) = tf(\lambda), \forall t \in \mathbb{T}, \lambda \in \Lambda\}.$$

Normal ternary weak amenability

Corollary

Let M be a von Neumann algebra and consider the submodule $M_* \subset M^*$. Then

$$\mathcal{D}_t(M, M_*) = \mathcal{I}nn_b^*(M, M_*) \circ * + \mathcal{I}nn_t(M, M_*).$$

Note

L^∞ is ternary weakly amenable and **normally ternary weakly amenable**, that is, $\mathcal{D}_t(L^\infty, L^1) = \mathcal{I}nn_t(L^\infty, L^1)$.

Question

Is $L^\infty \otimes M_n(\mathbb{C})$ normally ternary weakly amenable?

Main results: Pluta-R 2013

Theorem

If M is a properly infinite factor, then the real vector space of triple derivations of M into M_*, modulo the norm closure of the inner triple derivations, has dimension 1.

$$\mathcal{D}_t(M, M_*)/\overline{\mathcal{I}nn_t(M, M_*)} \sim \mathbb{R}$$

Theorem

If M is a von Neumann algebra, then M is finite if and only if every triple derivation of M into M_* is approximated in norm by inner triple derivations.

$$\mathcal{D}_t(M, M_*) = \overline{\mathcal{I}nn_t(M, M_*)}$$

compare: Bunce-Pashcke,Haagerup 1983

If M is a von Neumann algebra, then every derivation of M into M_* is inner.

$$\mathcal{D}_b(M, M_*)/\mathcal{I}nn_b(M, M_*) = 0$$

Nonassociative algebras

Table 1 (FASHIONABLE) ALGEBRAS

commutative algebras $ab = ba$

associative algebras $a(bc) = (ab)c$

Lie algebras $a^2 = 0$, $(ab)c + (bc)a + (ca)b = 0$

Jordan algebras $ab = ba$, $a(a^2b) = a^2(ab)$

In the rest of this talk we shall mainly be concerned with associative algebras, in fact, primarily the algebra of n by n matrices under matrix multiplication.

Review of Cohomology (associative algebras)

NOTATION

n is a positive integer, $n = 1, 2, \cdots$
f is a function of n variables
F is a function of $n + 1$ variables ($n + 2$ variables?)
$x_1, x_2, \cdots, x_{n+1}$ belong to an algebra A
$f(y_1, \ldots, y_n)$ and $F(y_1, \cdots, y_{n+1})$ also belong to A

The basic formula of homological algebra

$F(x_1, \ldots, x_n, x_{n+1}) =$
$x_1 f(x_2, \ldots, x_{n+1})$
$-f(x_1 x_2, x_3, \ldots, x_{n+1})$
$+f(x_1, x_2 x_3, x_4, \ldots, x_{n+1})$
$- \cdots$
$\pm f(x_1, x_2, \ldots, x_n x_{n+1})$
$\mp f(x_1, \ldots, x_n) x_{n+1}$

HIERARCHY

$x_1, x_2, \ldots, x_n$ are points (or vectors)
f and F are functions—they take points to points
T, defined by $T(f) = F$ is a transformation—takes functions to functions
points $x_1, \ldots, x_{n+1}$ and $f(y_1, \ldots, y_n)$ will belong to an algebra A
functions f will be either constant, linear or multilinear (hence so will F)
transformation T is linear

SHORT FORM OF THE FORMULA

$$(Tf)(x_1, \ldots, x_n, x_{n+1})$$

$$= x_1 f(x_2, \ldots, x_{n+1})$$

$$+ \sum_{j=1}^{n} (-1)^j f(x_1, \ldots, x_j x_{j+1}, \ldots, x_{n+1})$$

$$+(-1)^{n+1} f(x_1, \ldots, x_n) x_{n+1}$$

FIRST CASES

$\underline{n = 0}$

If f is any constant function from A to A, say, $f(x) = b$ for all x in A, where b is a fixed element of A, we have, consistent with the basic formula, a linear function $T_0(f)$:

$$T_0(f)(x_1) = x_1 b - b x_1$$

$\underline{n = 1}$

If f is a linear function from A to A, then $T_1(f)$ is a bilinear function

$$T_1(f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$$

$\underline{n = 2}$

If f is a bilinear function from $A \times A$ to A, then $T_2(f)$ is a trilinear function

$$T_2(f)(x_1, x_2, x_3) =$$

$$x_1 f(x_2, x_3) - f(x_1 x_2, x_3) + f(x_1, x_2 x_3) - f(x_1, x_2) x_3$$

FIRST COHOMOLOGY GROUP

Kernel and Image of a linear transformation

$G : X \to Y$
Since X and Y are vector spaces, they are in particular, commutative groups.
Kernel of G (also called **nullspace** of G) is
$\ker G = \{x \in X : G(x) = 0\}$
This is a subgroup of X
Image of G is
$\text{im}\, G = \{G(x) : x \in X\}$
This is a subgroup of Y

$\underline{G = T_0}$
$X = A$ (the algebra)
$Y = L(A)$ (all linear transformations on A)
$T_0(f)(x_1) = x_1 b - b x_1$
$\ker T_0 = \{b \in A : xb - bx = 0 \text{ for all } x \in A\}$ (center of A)
$\text{im}\, T_0 =$ the set of all linear maps of A of the form $x \mapsto xb - bx$,
in other words, the set of all inner derivations of A
$\ker T_0$ is a subgroup of A
$\text{im}\, T_0$ is a subgroup of $L(A)$

$\underline{G = T_1}$
$X = L(A)$ (linear transformations on A)
$Y = L^2(A)$ (bilinear transformations on $A \times A$)
$T_1(f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$
ker $T_1 = \{f \in L(A) : T_1 f(x_1, x_2) = 0$ for all $x_1, x_2 \in A\} =$ the set of all derivations of A
im $T_1 =$ the set of all bilinear maps of $A \times A$ of the form

$$(x_1, x_2) \mapsto x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2,$$

for some linear function $f \in L(A)$.
ker T_1 is a subgroup of $L(A)$
im T_1 is a subgroup of $L^2(A)$

$\underline{G = T_2}$
$X = L^2(A)$ (bilinear transformations on $A \times A$)
$Y = L^3(A)$ (trilinear transformations on $A \times A \times A$)
$T_2(f)(x_1, x_2, x_3) = x_1 f(x_2, x_3)) - f(x_1 x_2, x_3) + f(x_1 x_2, x_3) - f(x_1, x_2)x_3$
ker $T_2 = \{f \in L^2(A) : T_2 f(x_1, x_2, x_3) = 0 \text{ for all } x_1, x_2, x_3 \in A\}$
im $T_2 =$ the set of all trilinear maps of $A \times A \times A$ of the form

$$(x_1, x_2, x_3) \mapsto x_1 f(x_2, x_3)) - f(x_1 x_2, x_3) + f(x_1 x_2, x_3) - f(x_1, x_2)x_3$$

for some bilinear function $f \in L(A)$.
ker T_2 is a subgroup of $L^2(A)$
im T_2 is a subgroup of $L^3(A)$

$L^0(A) \xrightarrow{T_0} L(A) \xrightarrow{T_1} L^2(A) \xrightarrow{T_2} L^3(A) \cdots$
FACTS: $T_1 \circ T_0 = 0$
$T_2 \circ T_1 = 0$
$\ldots$
$T_{n+1} \circ T_n = 0$
$\ldots$

Therefore

im $T_n \subset$ ker $T_{n+1} \subset L^n(A)$
and therefore
im T_n is a subgroup of ker T_{n+1}

TERMINOLOGY

im $T_{n-1} =$ the set of n-**coboundaries**
ker $T_n =$ the set of n-**cocycles**
and therefore
every n-coboundary is an n-cocycle.

im $T_0 \subset$ ker T_1
says
Every inner derivation (1-coboundary) is a derivation (1-cocycle).

im $T_1 \subset$ ker T_2
says
for every linear map f, the bilinear map Fdefined by

$$F(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1)x_2$$

(2-coboundary) satisfies the equation

$$x_1 F(x_2, x_3) - F(x_1 x_2, x_3) + F(x_1, x_2 x_3) - F(x_1, x_2)x_3 = 0$$

for every $x_1, x_2, x_3 \in A$ (2-cocycle).

The cohomology groups of A are defined as the quotient groups

$$H^n(A) = \frac{\ker T_n}{\operatorname{im} T_{n-1}} = \frac{n\text{-cocycles}}{n\text{-coboundaries}} \qquad (n = 1, 2, \ldots)$$

Thus

$$H^1(A) = \frac{\ker T_1}{\operatorname{im} T_0} = \frac{\text{1-cocycles}}{\text{1-coboundaries}} = \frac{\text{derivations}}{\text{inner derivations}}$$

$$H^2(A) = \frac{\ker T_2}{\operatorname{im} T_1} = \frac{\text{2-cocycles}}{\text{2-coboundaries}} = \frac{\text{null extensions}}{\text{split null extensions}}$$

The theorem that every derivation of $M_n(\mathbb{R})$ is inner (that is, of the form δ_a for some $a \in M_n(\mathbb{R})$, Theorem 1) can now be restated as:
"the cohomology group $H^1(M_n(\mathbb{R}))$ is the trivial one element group"

The theorem that every null extension of $M_n(\mathbb{R})$ is a split null extension (Corollary 2 of Theorem 4 below for $n = 2$) can be stated as:
"the cohomology group $H^2(M_n(\mathbb{R}))$ is the trivial one element group"

$H^2(M_2, M_2) = 0$

DEFINITION

Let ρ be a linear transformation on M_2. We define linear transformations σ_1 and σ_2 on M_2 by

$$\sigma_1(A) = E_{11}\rho(E_{11}A) + E_{21}\rho(E_{12}A)$$

and

$$\sigma_2(A) = E_{12}\rho(E_{21}A) + E_{22}\rho(E_{22}A)$$

recall LEMMA 1

$\sigma_1(A) = A\sigma_1(I)$ and $\sigma_2(A) = A\sigma_2(I)$

We only need σ_1 or σ_2, not both. We'll go with σ_1.

DEFINITION

Let f be a bilinear transformation on $M_2 \times M_2$. We define bilinear transformations τ_1 and τ_2 on $M_2 \times M_2$ by

$$\tau_1(A, B) = E_{11} f(E_{11} A, B) + E_{21} f(E_{12} A, B)$$

and

$$\tau_2(A, B) = E_{12} f(E_{21} A, B) + E_{22} f(E_{22} A, B)$$

LEMMA 2

$\tau_1(A, B) = A\tau_1(I, B)$ and $\tau_2(A, B) = A\tau_2(I, B)$

We only need τ_1 or τ_2, not both. We'll go with τ_1.

PROOF OF LEMMA 2

For B fixed, let $\rho(A) = f(A, B)$ and apply LEMMA 1 to this ρ. Namely, set $\sigma(A) = E_{11}\rho(E_{11}A) + E_{21}\rho(E_{12}A)$. Then $\sigma(A) = \tau_1(A, B)$. By LEMMA 1, $\sigma(A) = A\sigma(1)$ and $\tau_1(A, B) = \sigma(A) = A\sigma(1) = A\tau_1(1, B)$. Q.E.D.

THEOREM 4

Let f be a 2-cocycle: f is bilinear and

$$T_2 f(A,B,C) = Af(B,C) - f(AB,C) + f(A,BC) - f(A,B)C = 0$$

for all A, B, C in M_2. Then there exists a linear transformation ξ on M_2 such that $T_1\xi = f$, that is, f is a 2-coboundary.

COROLLARY 1

$H^2(M_2, M_2) = 0$

COROLLARY 2

It E is any associative algebra containing an ideal J such that E/J is isomorphic to M_2 (E is then said to be an **extension** of M_2), then there is a subalgebra B of E such that $E = B \oplus M_2$ (E is a **split extension**) [a]

[a] There is always a subspace B such that $E = B \oplus M_2$

PROOF OF THEOREM 4
(Kadison and Ringrose, Acta Math 1972)

Define a bilinear map $\tau(A,B) = E_{11}f(E_{11}A,B) + E_{21}f(E_{12}A,B)$ and then define a linear map $\xi(B) = \tau(1,B)$. Now just verify that $T_1(\xi) = f$. Q.E.D.

$$\begin{aligned} T_1\xi(A,B) &= A\xi(B) - \xi(AB) + \xi(A)B \\ &= A\tau(1,B) - \tau(1,AB) + \tau(1,A)B \\ &= \tau(A,B) - \tau(1,AB) + \tau(1,A)B \\ &= E_{11}f(E_{11}A,B) + E_{21}f(E_{12}A,B) \\ &- E_{11}f(E_{11},AB) - E_{21}f(E_{12},AB) \\ &+ E_{11}f(E_{11},A)B + E_{21}f(E_{12},A)B \end{aligned}$$

$$T_2f(E_{11},A,B) = E_{11}f(A,B) - f(E_{11}A,B) + f(E_{11},AB) - f(E_{11},A)B = 0$$

$$T_2f(E_{12},A,B) = E_{12}f(A,B) - f(E_{12}A,B) + f(E_{12},AB) - f(E_{12},A)B = 0$$

$$0 = E_{11}T_2 f(E_{11}, A, B) + E_{21}T_2 f(E_{12}, A, B)$$

$$\begin{aligned} 0 &= E_{11}[E_{11}f(A,B) - f(E_{11}A,B) + f(E_{11},AB) - f(E_{11},A)B] \\ &+ E_{21}[E_{12}f(A,B) - f(E_{12}A,B) + f(E_{12},AB) - f(E_{12},A)B] \end{aligned}$$

FROM THE PRECEDING PAGE

$$\begin{aligned} T_1\xi(A,B) &= E_{11}f(E_{11}A,B) + E_{21}f(E_{12}A,B) - E_{11}f(E_{11},AB) \\ &- E_{21}f(E_{12},AB) + E_{11}f(E_{11},A)B + E_{21}f(E_{12},A)B \end{aligned}$$

Add these two equations to get

$$T_1\xi(A,B) = E_{11}f(A,B) + E_{22}f(A,B) = f(A,B) \quad \text{Q.E.D. (again)}$$

Some miscellaneous facts
(M is a module)

- $H^1(\mathcal{C}) = 0$, $H^2(\mathcal{C}) = 0$

- $H^1(\mathcal{C}, M) = 0$, $H^2(\mathcal{C}, M) = 0$
(Kamowitz 1962)

- $H^n(M_k(\mathbb{R}), M) = 0 \ \forall n \geq 1, k \geq 2$

- $H^n(A) = H^{n-1}(A, L(A))$ for $n \geq 2$
(Hochschild 1945)

EXTENSIONS

Let A be an algebra. Let M be another algebra which contains an ideal I and let $g : M \to A$ be a homomorphism.
In symbols,
$I \stackrel{\subsetneq}{\to} M \stackrel{g}{\to} A$
This is called an **extension of** A **by** I if

- $\ker g = I$
- $\operatorname{im} g = A$

It follows that M/I is isomorphic to A

EXAMPLE 1

Let A be an algebra.
Define an algebra $M = A \oplus A$ to be the set $A \times A$ with addition
$(a, x) + (b, y) = (a + b, x + y)$
and product
$(a, x)(b, y) = (ab, xy)$

- $\{0\} \times A$ is an ideal in M
- $(\{0\} \times A)^2 \neq 0$
- $g : M \to A$ defined by $g(a, x) = a$ is a homomorphism
- M is an extension of $\{0\} \times A$ by A.

EXAMPLE 2

Let A be an algebra and let $h \in \ker T_2 \subset L^2(A)$.
Recall that this means that for all $x_1, x_2, x_3 \in A$,
$x_1 f(x_2, x_3) - f(x_1 x_2, x_3)$
$+f(x_1, x_2 x_3) - f(x_1, x_2) x_3 = 0$
Define an algebra M_h to be the set $A \times A$ with addition
$(a, x) + (b, y) = (a + b, x + y)$
and the product
$(a, x)(b, y) = (ab, ay + xb + h(a, b))$
Because $h \in \ker T_2$, this algebra is
ASSOCIATIVE!
whenever A is associative.

THE PLOT THICKENS

- $\{0\} \times A$ is an ideal in M_h
- $(\{0\} \times A)^2 = 0$
- $g : M_h \to A$ defined by $g(a, x) = a$ is a homomorphism
- M_h is an extension of $\{0\} \times A$ by A.

EQUIVALENCE OF EXTENSIONS

Extensions
$I \stackrel{\subsetneq}{\to} M \stackrel{g}{\to} A$
and
$I \stackrel{\subsetneq}{\to} M' \stackrel{g'}{\to} A$
are said to be equivalent if
there is an isomorphism $\psi : M \to M'$
such that

- $\psi(x) = x$ for all $x \in I$
- $g = g' \circ \psi$

(Is this an equivalence relation?)

EXAMPLE 2—continued

Let $h_1, h_2 \in \ker T_2$.
We then have two extensions of A by $\{0\} \times A$, namely
$\{0\} \times A \overset{\subsetneq}{\to} M_{h_1} \overset{g_1}{\to} A$
and
$\{0\} \times A \overset{\subsetneq}{\to} M_{h_2} \overset{g_2}{\to} A$
Now suppose that h_1 is equivalent[1] to h_2,
$h_1 - h_2 = T_1 f$ for some $f \in L(A)$

- The above two extensions are equivalent.
- We thus have a mapping from $H^2(A, A)$ into the set of equivalence classes of extensions of A by the ideal $\{0\} \times A$

[1] This is the same as saying that $[h_1] = [h_2]$ as elements of $H^2(A, A) = \ker T_2 / \mathrm{im}\, T_1$

GRADUS AD PARNASSUM (COHOMOLOGY)

1. Verify that there is a one to one correspondence between partitions of a set X and equivalence relations on that set.
   Precisely, show that
   - If $X = \cup X_i$ is a partition of X, then $R := \{(x, y) \in X \times X : x, y \in X_i \text{ for some } i\}$ is an equivalence relation whose equivalence classes are the subsets X_i.
   - If R is an equivalence relation on X with equivalence classes X_i, then $X = \cup X_i$ is a partition of X.

2. Verify that $T_{n+1} \circ T_n = 0$ for $n = 0, 1, 2$. Then prove it for all $n \geq 3$.

3. Show that if $f : G_1 \to G_2$ is a homomorphism of groups, then $G_1/\ker f$ is isomorphic to $f(G_1)$
   Hint: Show that the map $[x] \mapsto f(x)$ is an isomorphism of $G_1/\ker f$ onto $f(G_1)$

4. Show that if $h : A_1 \to A_2$ is a homomorphism of algebras, then $A_1/\ker h$ is isomorphic to $h(A_1)$
   Hint: Show that the map $[x] \mapsto h(x)$ is an isomorphism of $A_1/\ker h$ onto $h(A_1)$

5. Show that the algebra M_h in Example 2 is associative.
Hint: You use the fact that A is associative AND the fact that, since $h \in \ker T_2$, $h(a,b)c + h(ab.c) = ah(b,c) + h(a,bc)$

6. Show that equivalence of extensions is actually an equivalence relation.
Hint:
   - reflexive: $\psi : M \to M$ is the identity map
   - symmetric: replace $\psi : M \to M'$ by its inverse $\psi^{-1} : M' \to M$
   - transitive: given $\psi : M \to M'$ and $\psi' : M' \to M''$ let $\psi'' = \psi' \circ \psi : M \to M''$

7. Show that in example 2, if h_1 and h_2 are equivalent bilinear maps, that is, $h_1 - h_2 = T_1 f$ for some linear map f, then M_{h_1} and M_{h_2} are equivalent extensions of $\{0\} \times A$ by A. **Hint:** $\psi : M_{h_1} \to M_{h_2}$ is defined by

$$\psi(a,x) = (a, x + f(a))$$

Cohomology groups were defined in various contexts as follows

- associative algebras (1945)
- Lie algebras (1952)
- Lie triple systems (1961,2002)
- Jordan algebras (1971)
- associative triple systems (1976)
- Jordan triple systems (1982)

FASHIONABLE TRIPLE SYSTEMS

Table 4
TRIPLE SYSTEMS
associative triple systems
$(abc)de = ab(cde) = a(dcb)e$

Lie triple systems
$aab = 0$
$abc + bca + cab = 0$
$de(abc) = (dea)bc + a(deb)c + ab(dec)$

Jordan triple systems
$abc = cba$
$de(abc) = (dea)bc - a(edb)c + ab(dec)$

DERIVATIONS INTO A MODULE

CONTEXTS
(i) ASSOCIATIVE ALGEBRAS
(ii) LIE ALGEBRAS
(iiI) JORDAN ALGEBRAS
Could also consider:
(i') ASSOCIATIVE TRIPLE SYSTEMS
(ii') LIE TRIPLE SYSTEMS
(iii') JORDAN TRIPLE SYSTEMS

(i) ASSOCIATIVE ALGEBRAS

derivation: $D(ab) = a \cdot Db + Da \cdot b$
inner derivation: $(\text{ad x})(a) = x \cdot a - a \cdot x \ (x \in M)$
THEOREM (Noether,Wedderburn) (early 20th century)) [2]
EVERY DERIVATION OF <u>SEMISIMPLE</u> ASSOCIATIVE ALGEBRA IS INNER, THAT IS, OF THE FORM $x \mapsto ax - xa$ FOR SOME a IN THE ALGEBRA
THEOREM (Hochschild 1942)
EVERY DERIVATION OF <u>SEMISIMPLE</u> ASSOCIATIVE ALGEBRA INTO A MODULE IS INNER, THAT IS, OF THE FORM $x \mapsto ax - xa$ FOR SOME a IN THE MODULE

[2] The operational word here, and in all of these results is SEMISIMPLE—think primes, fundamental theorem of arithmetic

(iii) JORDAN ALGEBRAS

derivation: $D(a \circ b) = a \circ Db + Da \circ b$
inner derivation: $\sum_i [L(x_i)L(a_i) - L(a_i)L(x_i)]$
$(x_i \in M, a_i \in A)$
$b \mapsto \sum_i [x_i \circ (a_i \circ b) - a_i \circ (x_i \circ b)]$

THEOREM (1949-Jacobson)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN ALGEBRA INTO ITSELF IS INNER

THEOREM (1951-Jacobson)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN ALGEBRA INTO A (JORDAN) **MODULE** IS INNER
(Lie algebras, Lie triple systems)

(iii') JORDAN TRIPLE SYSTEMS

derivation: $D\{a, b, c\} = \{Da.b, c\} + \{a, Db, c\} + \{a, b, Dc\}$
$\{x, y, z\} = (xy^*z + zy^*x)/2$
inner derivation: $\sum_i [L(x_i, a_i) - L(a_i, x_i)]$
$(x_i \in M, a_i \in A)$
$b \mapsto \sum_i [\{x_i, a_i, b\} - \{a_i, x_i, b\}]$

THEOREM (1972 Meyberg)

EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN TRIPLE SYSTEM IS INNER

(Lie algebras, Lie triple systems)

THEOREM (1978 Kühn-Rosendahl)

EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN TRIPLE SYSTEM INTO A JORDAN TRIPLE MODULE IS INNER

(Lie algebras)

(i') ASSOCIATIVE TRIPLE SYSTEMS

derivation: $D(ab^tc) = ab^tDc + a(Db)^tc + (Da)b^tc$
inner derivation: see Table 3
The (non-module) result can be derived from the result for Jordan triple systems. (See an exercise)
THEOREM (1976 Carlsson)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE ASSOCIATIVE TRIPLE SYSTEM INTO A MODULE IS INNER
(reduces to associative ALGEBRAS)

(ii) LIE ALGEBRAS

THEOREM (Zassenhaus)
(early 20th century)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE LIE ALGEBRA INTO ITSELF IS INNER
THEOREM (Hochschild 1942)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE LIE ALGEBRA INTO A MODULE IS INNER

(ii') LIE TRIPLE SYSTEMS

THEOREM (Lister 1952)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE LIE TRIPLE SYSTEM INTO ITSELF IS INNER
THEOREM (Harris 1961)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE LIE TRIPLE SYSTEM INTO A MODULE IS INNER

Table 1 $M_n(\mathbb{R})$ (ALGEBRAS)

associative	Lie	Jordan
$ab = a \times b$	$[a,b] = ab - ba$	$a \circ b = ab + ba$
Noeth,Wedd 1920	Zassenhaus 1930	Jacobson 1949
Hochschild 1942	Hochschild 1942	Jacobson 1951

Table 3 $M_{m,n}(\mathbb{R})$ (TRIPLE SYSTEMS)

associative triple	Lie triple	Jordan triple
ab^tc	$[[a,b],c]$	$ab^tc + cb^ta$
	Lister 1952	Meyberg 1972
Carlsson 1976	Harris 1961	Kühn-Rosendahl 1978

Table 2 $M_n(\mathbb{R})$ (ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a, b] = ab - ba$	$a \circ b = ab + ba$
Th. 2	Th.3	Th.4
$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$

COHOMOLOGY OF NONASSOCIATIVE ALGEBRAS

$n = 0$

ASSOCIATIVE

$f : A \to A$ is a constant function, say $f(x) = b$ for all x
$T_0(f) : A \to A$ is a linear function
$T_0(f)(x_1) = x_1 b - b x_1$

LIE

$f : A \to A$ is a constant function, say $f(x) = b$ for all x
$T_0(f) : A \to A$ is a linear function
$T_0(f)(x_1) = [b, x_1]$

JORDAN

$f \in A \times A$ is an ordered pair, say $f = (a, b)$
$T_0(f) : A \to A$ is a linear function
$T_0(f)(x_1) = a \circ (b \circ x_1) - b \circ (a \circ x_1)$

$$n = 1$$

ASSOCIATIVE

$f : A \to A$ is a linear function
$T_1(f) : A \times A \to A$ is a bilinear function
$T_1(f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$

LIE

$f : A \to A$ is a linear function
$T_1(f) : A \times A \to A$ is a skew-symmetric bilinear function
$T_1(f)(x_1, x_2) = -[f(x_2), x_1] + [f(x_1), x_2] - f([x_1, x_2])$

JORDAN

$f : A \to A$ is a linear function
$T_1(f) : A \times A \to A$ is a symmetric bilinear function
$T_1(f)(x_1, x_2) = x_1 \circ f(x_2) - f(x_1 \circ x_2) + f(x_1) \circ x_2$

$n = 2$

ASSOCIATIVE

$f : A \times A \to A$ is a bilinear function
$T_2(f) : A \times A \times A \to A$ is a trilinear function
$T_2(f)(x_1, x_2, x_3) = x_1 f(x_2, x_3) - f(x_1 x_2, x_3) - f(x_1, x_2 x_3) + f(x_1, x_2) x_3$

LIE

$f : A \times A \to A$ is a skew-symmetric bilinear function
$T_2(f) : A \times A \times A \to A$ is a skew-symmetric trilinear function

$$\begin{aligned} T_2(f)(x_1, x_2, x_3) &= [f(x_2, x_3), x_1] - [f(x_1, x_3), x_2] + [f(x_1, x_2), x_3] \\ &- f(x_3, [x_1, x_2]) + f(x_2, [x_1, x_3]) - f(x_1, [x_2, x_3]) \end{aligned}$$

JORDAN

JUST AHEAD

OBJECTIVES

INTERPRETATION OF COHOMOLOGY GROUPS

FIRST COHOMOLOGY GROUP
DERIVATIONS (AND INNER DERIVATIONS)

SECOND COHOMOLOGY GROUP
EXTENSIONS (AND SPLIT EXTENSIONS)

VANISHING THEOREMS

FOR EACH CLASS OF ALGEBRAS (ASSOCIATIVE, LIE, JORDAN), UNDER WHAT CONDITIONS IS $H^n(A) = 0$, ESPECIALLY FOR $n = 1, 2$

Unified approach to second cohomology group

(Jacobson book on Jordan algebras 1968)

Basic setting

Let M, E and A be algebras satisfying the same set of axioms (associative, Lie, Jordan).
Let α and β be algebra homomorphisms

$$M \overset{\alpha}{\to} E \overset{\beta}{\to} A$$

such that

$\ker \alpha = \{0\}$ (i.e., α is one-to-one)
$\operatorname{Im} \alpha = \ker \beta$
$\operatorname{Im} \beta = A$ (ı.e., β is onto)

There is a linear transformation $\delta : A \to E$ such that $\beta(\delta(a)) = a$ for every $a \in A$.
Define the bilinear transformation $h : A \times A \to M, \quad h(a,b) = \delta(ab) - \delta(a)\delta(b)$

THEOREM (Properties of h)

ASSOCIATIVE ALGEBRAS $h(a,b)c + h(ab,c) = ah(b,c) + h(a,bc)$
(Hochschild 2-cocycle)
Hochschild 2-coboundary: $h(a,b) = af(b) - f(ab) + f(a)b$

LIE ALGEBRAS $h(a,a) = 0$ **and**
$h(a,b)c + h(ab,c) + h(b,c)a + h(bc,a) + h(c,a)b + h(ca,b) = 0$
(Lie 2-cocycle)
Lie 2-coboundary: $h(a,b) = -[f(b),a] + [f(a),b] - f([a,b])$
$= -f(b)a + af(b) + f(a)b - bf(a) - f(ab) + f(ba)$

JORDAN ALGEBRAS $h(a,b) = h(b,a)$ **and**
$(h(a,a)b)c + h(a^2,b)a + h(a^2b,a) = a^2h(b,a) + h(a,a)(ba) + h(a^2,ba)$
(Jordan 2-cocycle)
Jordan 2-coboundary: $h(a,b) = a \circ f(b) - f(a \circ b) + f(a) \circ b$
$= ag(b) + g(b)a - g(ab) - g(ba) + g(a)b + bg(a)$

Beginning of the proof

Algebras defined by identities

If A is an algebra then a function $f : A \times \cdots \times A \to A$ is said to be an identity for A if $f(a_1, \cdots, a_n) = 0$ for every set of n elements of A
Let I denote the set of identites defining a class of algebras. For example
$I = \{f\}$ where $f(x_1, x_2, x_3) = (x_1x_2)x_3 - x_1(x_2x_3)$ defines the class of associative algebras
$I = \{f, g\}$ where $f(x_1) = x_1^2$ and $g(x_1, x_2) = (x_1x_2)x_3 + (x_2x_3)x_1 + (x_3x_1)x_2$ defines the class of Lie algebras
$I = \{f, g\}$ where $f(x_1, x_2) = x_1x_2 - x_2x_1$ and $g(x_1, x_2) = (x_1^2x_2)x_1 - x_1^2(x_2x_1)$ defines the class of Jordan algebras

AND SO FORTH ...

Whitehead Type Theorems

ASSOCIATIVE ALGEBRAS
$H^1 = H^2 = 0$

LIE ALGEBRAS
$H^1 = H^2 = 0$

JORDAN ALGEBRAS
$H^1 = H^2 = 0$