DERIVATIONS ON BANACH ALGEBRAS

Introduction to continuous non-associative cohomology

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PART I—ALGEBRAS

PART II—TRIPLE SYSTEMS

ANALYSIS SEMINAR
OCTOBER 18, 2011

PART I—ALGEBRAS

PART II—TRIPLE SYSTEMS

I—DERIVATIONS ON FINITE DIMENSIONAL ALGEBRAS

Sophus Lie (1842–1899)

Marius Sophus Lie was a Norwegian mathematician. He largely created the theory of continuous symmetry, and applied it to the study of geometry and differential equations.

Pascual Jordan (1902–1980)

Pascual Jordan was a German theoretical and mathematical physicist who made significant contributions to quantum mechanics and quantum field theory.

LET $\mathcal{C}$ DENOTE THE ALGEBRA OF CONTINUOUS FUNCTIONS ON A LOCALLY COMPACT HAUSDORFF SPACE.

DEFINITION 1

A DERIVATION ON $\mathcal{C}$ IS A LINEAR MAPPING $\delta : \mathcal{C} \to \mathcal{C}$ SATISFYING THE "PRODUCT" RULE:

$$\delta(f+g) = \delta(f) + \delta(g)$$

$$\delta(cf) = c\delta(f)$$

$$\delta(fg) = \delta(f)g + f\delta(g)$$

THEOREM 1

There are no (non-zero) derivations on $\mathcal{C}$.

In other words,
Every derivation of $\mathcal{C}$ is identically zero

THEOREM 1A

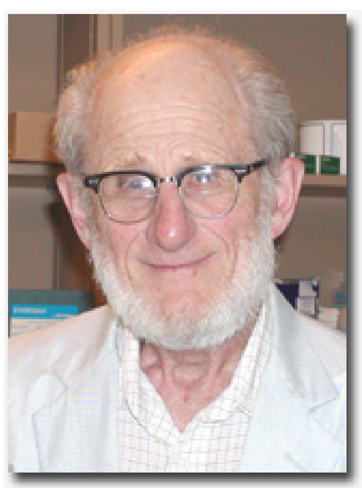
(1955-Singer and Wermer)

Every continuous derivation on $\mathcal{C}$ is zero.

Theorem 1B

(1960-Sakai)

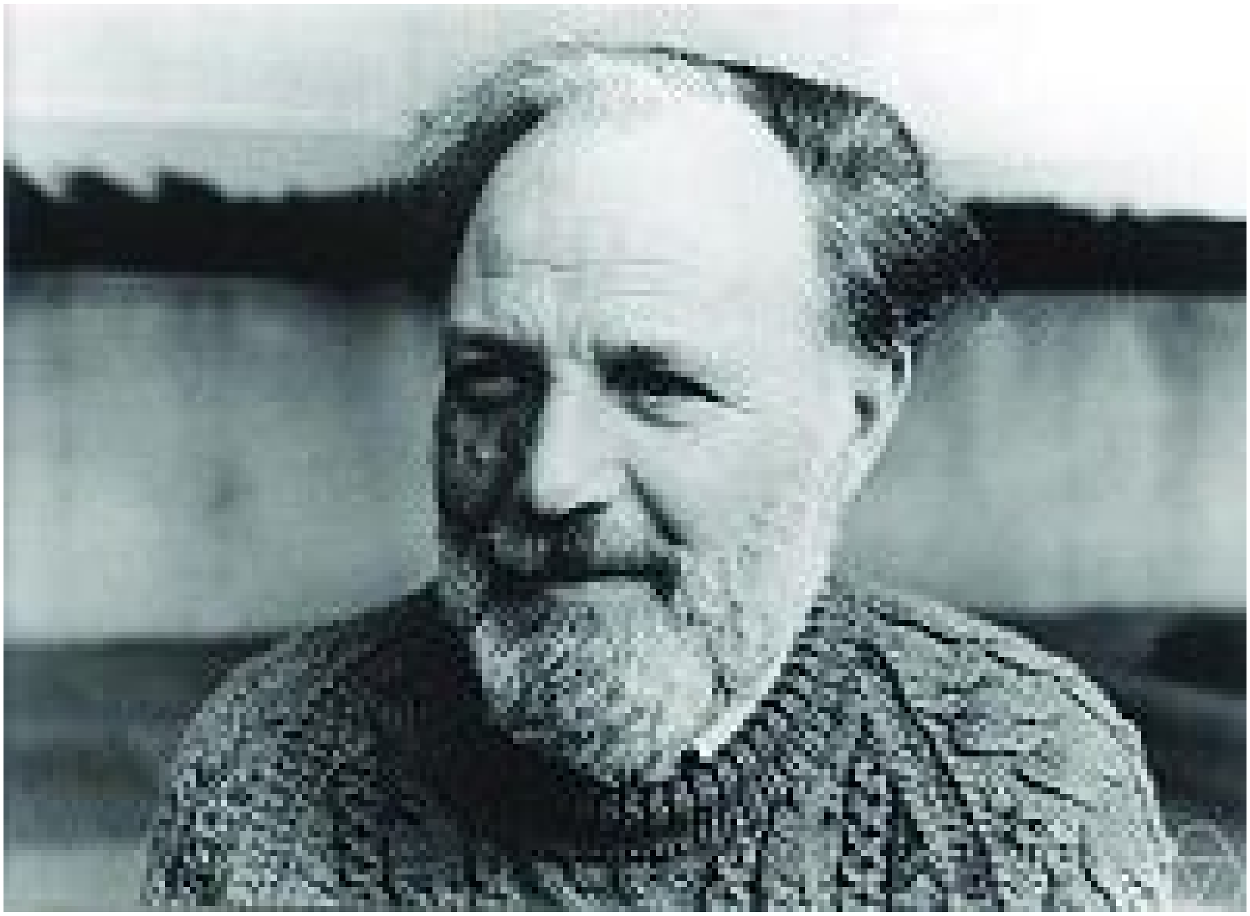
Every derivation on $\mathcal{C}$ is continuous.



John Wermer (b. 1925)

Soichiro Sakai (b. 1926)

Isadore Singer (b. 1924)



Isadore Manuel Singer is an Institute Professor in the Department of Mathematics at the Massachusetts Institute of Technology. He is noted for his work with Michael Atiyah in 1962, which paved the way for new interactions between pure mathematics and theoretical physics.

LET $M_n(\mathbf{C})$ DENOTE THE ALGEBRA OF ALL n by n COMPLEX MATRICES, OR MORE GENERALLY, ANY FINITE DIMENSIONAL SEMISIMPLE ASSOCIATIVE ALGEBRA. .

DEFINITION 2

A DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO MATRIX MULTIPLICATION IS A LINEAR MAPPING δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(AB) = \delta(A)B + A\delta(B)$$

.

PROPOSITION 2

FIX A MATRIX A in $M_n(\mathbf{C})$ AND DEFINE

$$\delta_A(X) = AX - XA.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION

THEOREM 2

(1942 Hochschild)

EVERY DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO MATRIX MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{C})$.

Gerhard Hochschild (1915–2010)

(Photo 1968)

Gerhard Paul Hochschild was an American mathematician who worked on Lie groups, algebraic groups, homological algebra and algebraic number theory.

Joseph Henry Maclagan Wedderburn (1882–1948)

Scottish mathematician, who taught at Princeton University for most of his career. A significant algebraist, he proved that a finite division algebra is a field, and part of the Artin–Wedderburn theorem on simple algebras. He also worked on group theory and matrix algebra.

Amalie Emmy Noether (1882–1935)

Amalie Emmy Noether was an influential German mathematician known for her groundbreaking contributions to abstract algebra and theoretical physics. Described as the most important woman in the history of mathematics, she revolutionized the theories of rings, fields, and algebras. In physics, Noether's theorem explains the fundamental connection between symmetry and conservation laws.

RECOMMENDED READING

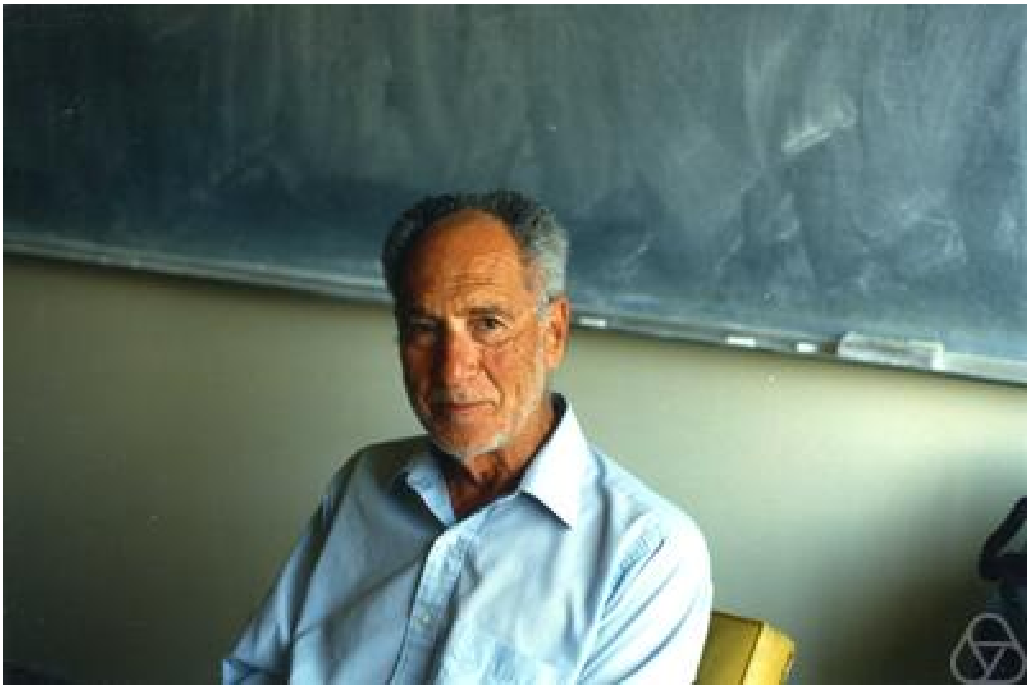
Gerhard Hochschild
A mathematician of the XXth Century

Walter Ferrer Santos

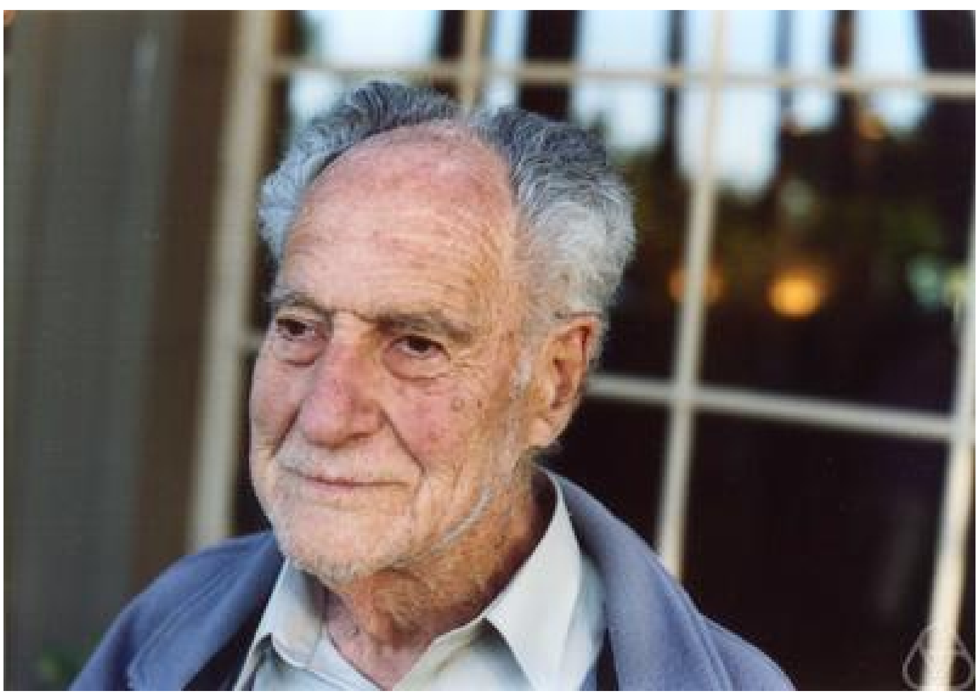
arXiv:1104.0335v1 [math.HO] 2 Apr 2011

(Photo 1976)

Gerhard Hochschild (1915–2010)



(Photo 1986)



(Photo 2003)

DEFINITION 3

A DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO BRACKET MULTIPLICATION

$$[X, Y] = XY - YX$$

IS A LINEAR MAPPING δ WHICH SATISFIES THE PRODUCT RULE

$$\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$$

.

PROPOSITION 3

FIX A MATRIX A in $M_n(\mathbf{C})$ AND DEFINE

$$\delta_A(X) = [A, X] = AX - XA.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION

THEOREM 3

(1942 Hochschild, Zassenhaus)
EVERY DERIVATION ON $M_n(\mathbf{C})^*$ WITH RESPECT TO BRACKET MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{C})$.

Hans Zassenhaus (1912–1991)

Hans Julius Zassenhaus was a German mathematician, known for work in many parts of abstract algebra, and as a pioneer of computer algebra.

*not a semisimple Lie algebra: trace$(X)\,I$ is a derivation which is not inner

DEFINITION 4

A DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO CIRCLE MULTIPLICATION

$$X \circ Y = (XY + YX)/2$$

IS A LINEAR MAPPING δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$

PROPOSITION 4

FIX A MATRIX A in $M_n(\mathbf{C})$ AND DEFINE

$$\delta_A(X) = AX - XA.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION

THEOREM 4

(1972-Sinclair)

EVERY DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{C})$.

REMARK

(1937-Jacobson)

THE ABOVE PROPOSITION AND THEOREM NEED TO BE MODIFIED FOR THE SUBALGEBRA (WITH RESPECT TO CIRCLE MULTIPLICATION) OF HERMITIAN MATRICES.

Alan M. Sinclair (retired)

Nathan Jacobson (1910–1999)

Nathan Jacobson was an American mathematician who was recognized as one of the leading algebraists of his generation, and he was also famous for writing more than a dozen standard monographs.

Table 1

$M_n(\mathbf{C})$ (SEMISIMPLE ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a, b] = ab - ba$	$a \circ b = ab + ba$
Th. 2	Th.3	Th.4
$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$

Table 2

ALGEBRAS

commutative algebras

$$ab = ba$$

associative algebras

$$a(bc) = (ab)c$$

Lie algebras

$$a^2 = 0$$

$$(ab)c + (bc)a + (ca)b = 0$$

Jordan algebras

$$ab = ba$$

$$a(a^2b) = a^2(ab)$$

DERIVATIONS ON C^*-ALGEBRAS

THE ALGEBRA $M_n(\mathbf{C})$, WITH MATRIX MULTIPLICATION, AS WELL AS THE ALGEBRA $\mathcal{C}$, WITH ORDINARY MULTIPLICATION, ARE EXAMPLES OF C^*-ALGEBRAS (FINITE DIMENSIONAL; resp. COMMUTATIVE).

THE FOLLOWING THEOREM THUS EXPLAINS THEOREMS 1 AND 2.

THEOREM (1966-Sakai, Kadison)
EVERY DERIVATION OF A C^*-ALGEBRA IS OF THE FORM $x \mapsto ax - xa$ FOR SOME a IN THE WEAK CLOSURE OF THE C^*-ALGEBRA

Irving Kaplansky (1917–2006)

Kaplansky made major contributions to group theory, ring theory, the theory of operator algebras and field theory.

Richard Kadison (b. 1925)

Richard V. Kadison is an American mathematician known for his contributions to the study of operator algebras.

II—DERIVATIONS ON OPERATOR ALGEBRAS

TWO BASIC QUESTIONS ON DERIVATIONS OF BANACH ALGEBRAS (AND TRIPLES)

$A \to A$ and $A \to M$ (MODULE)

- AUTOMATIC CONTINUITY?
- INNER?

CONTEXTS

(i) C*-ALGEBRAS
(associative Banach algebras)

(ii) JC*-ALGEBRAS
(Jordan Banach algebras)

(iii) JC*-TRIPLES
(Banach Jordan triples)

Could also consider:
(ii') <u>Banach Lie algebras</u>
(iii')<u>Banach Lie triple systems</u>
(i')<u>Banach associative triple systems</u>

(i) C*-ALGEBRAS

derivation: $D(ab) = a \cdot Db + Da \cdot b$

inner derivation: ad x$(a) = x \cdot a - a \cdot x$ $(x \in M)$

- AUTOMATIC CONTINUITY RESULTS

  KAPLANSKY 1949: $C(X)$

  SAKAI 1960

  RINGROSE 1972: (module)

- INNER DERIVATION RESULTS

  SAKAI, KADISON 1966

  CONNES 1976 (module)

  HAAGERUP 1983 (module)

THEOREM (Sakai 1960)
Every derivation from a C*-algebra into itself is continuous.

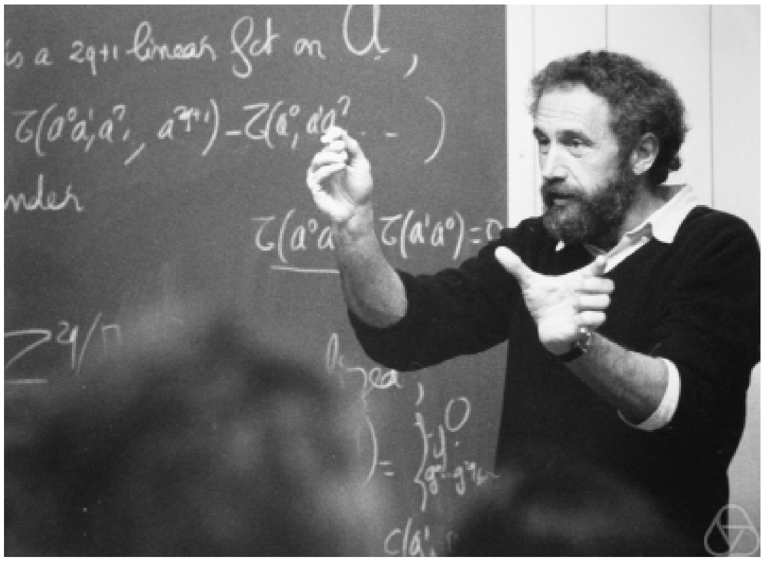
THEOREM (Ringrose 1972)
Every derivation from a C*-algebra into a Banach A-**bimodule** is continuous.

THEOREM (1966-Sakai, Kadison)
EVERY DERIVATION OF A C^*-ALGEBRA IS OF THE FORM $x \mapsto ax - xa$ FOR SOME a IN THE WEAK CLOSURE OF THE C^*-ALGEBRA

John Ringrose (b. 1932)

John Ringrose is a leading world expert on non-self-adjoint operators and operator algebras. He has written a number of influential texts including Compact non-self-adjoint operators (1971) and, with R V Kadison, Fundamentals of the theory of operator algebras in four volumes published in 1983, 1986, 1991 and 1992.

THEOREM (1976-Connes)
EVERY AMENABLE C^*-ALGEBRA IS NUCLEAR.



Alain Connes b. 1947

Alain Connes is the leading specialist on operator algebras.

In his early work on von Neumann algebras in the 1970s, he succeeded in obtaining the almost complete classification of injective factors.

Following this he made contributions in operator K-theory and index theory, which culminated in the Baum-Connes conjecture.

He also introduced cyclic cohomology in the early 1980s as a first step in the study of noncommutative differential geometry.

Connes has applied his work in areas of mathematics and theoretical physics, including number theory, differential geometry and particle physics.

THEOREM (1983-Haagerup)
EVERY NUCLEAR C^*-ALGEBRA IS AMENABLE.

THEOREM (1983-Haagerup)
EVERY C^*-ALGEBRA IS WEAKLY AMENABLE.

Uffe Haagerup b. 1950

Haagerup's research is in operator theory, and covers many subareas in the subject which are currently very active - random matrices, free probability, C*-algebras and applications to mathematical physics.

DIGRESSION #1
A BRIDGE TO JORDAN ALGEBRAS

A *Jordan derivation* from a Banach algebra A into a Banach A-module is a linear map D satisfying $D(a^2) = aD(a) + D(a)a$, $(a \in A)$, or equivalently,

$$D(ab + ba) = aD(b) + D(b)a + D(a)b + bD(a),$$
$$(a, b \in A).$$

Sinclair proved in 1970 that a bounded Jordan derivation from a semisimple Banach algebra to itself is a derivation, although this result fails for derivations of semisimple Banach algebras into a Banach bi-module.

Nevertheless, a celebrated result of B.E. Johnson in 1996 states that every bounded Jordan derivation from a C*-algebra A to a Banach A-bimodule is an associative derivation.

In view of the intense interest in automatic continuity problems in the past half century, it is therefore somewhat surprising that the following problem has remained open for fifteen years.

PROBLEM

Is every Jordan derivation from a C*-algebra A to a Banach A-bimodule automatically continuous (and hence a derivation, by Johnson's theorem)?

In 2004, J. Alaminos, M. Brešar and A.R. Villena gave a positive answer to the above problem for some classes of C*-algebras including the class of abelian C*-algebras

Combining a theorem of Cuntz from 1976 with the theorem just quoted yields

THEOREM

Every Jordan derivation from a C*-algebra A to a Banach A-module is continuous.

In the same way, using the solution in 1996 by Hejazian-Niknam in the commutative case we have

THEOREM

Every Jordan derivation from a C*-algebra A to a _Jordan_ Banach A-module is continuous.

(Jordan module will be defined below)

These two results will also be among the consequences of our results on automatic continuity of derivations into Jordan triple modules.

(END OF DIGRESSION)

(ii) JC*-ALGEBRA

derivation: $D(a \circ b) = a \circ Db + Da \circ b$

inner derivation: $\sum_i [L(x_i)L(a_i) - L(a_i)L(x_i)]$

$(x_i \in M, a_i \in A)$

$b \mapsto \sum_i [x_i \circ (a_i \circ b) - a_i \circ (x_i \circ b)]$

- AUTOMATIC CONTINUITY RESULTS

  UPMEIER 1980

  HEJAZIAN-NIKNAM 1996 (module)

  ALAMINOS-BRESAR-VILLENA 2004 (module)

- INNER DERIVATION RESULTS

  JACOBSON 1951 (module)

  UPMEIER 1980

THEOREM (1951-Jacobson)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN ALGEBRA INTO A (JORDAN) **MODULE** IS INNER
(Lie algebras, Lie triple systems)

THEOREM (1980-Upmeier)
EVERY DERIVATION OF A REVERSIBLE JC*-ALGEBRA EXTENDS TO A DERIVATION OF ITS ENVELOPING C*-ALGEBRA. (IMPLIES SINCLAIR)

THEOREM (1980-Upmeier)

1. Purely exceptional JBW-algebras have the inner derivation property
2. Reversible JBW-algebras have the inner derivation property
3. $\oplus L^\infty(S_j, U_j)$ has the inner derivation property if and only if $\sup_j \dim U_j < \infty$, U_j spin factors.

Nathan Jacobson (1910-1999)

Harald Upmeier (b. 1950)

Digression #2—LIE DERIVATIONS

Miers, C. Robert
Lie derivations of von Neumann algebras.
DukeMath. J. 40 (1973), 403–409.

If M is a von Neumann algebra, $[M, M]$ the Lie algebra linearly generated by $\{[X, Y] = XY - YX : X, Y \in M\}$ and $L : [M, M] \to M$ a Lie derivation, i.e., L is linear and $L[X, Y] = [LX, Y] + [X, LY]$, then the author shows that L has an extension $D : M \to M$ that is a derivation of the associative algebra.

The proof involves matrix-like computations.

A theorem of S. Sakai [Ann. of Math. (2) 83 (1966), 273–279] now states that $DX = [A, X]$ with $A \in M$ fixed.

Using this the author finally shows that if $L : M \to M$ is a Lie derivation, then $L = D + \lambda$, where D is an associative derivation and λ is a linear map into the center of M vanishing on $[M, M]$.

For primitive rings with nontrivial idempotent and characteristic $\neq 2$ a slightly weaker result is due to W. S. Martindale, III [Michigan Math. J. 11 (1964), 183187].

Reviewed by Gerhard Janssen

Miers, C. Robert
Lie triple derivations of von Neumann algebras.
Proc. Amer. Math. Soc. 71 (1978), no. 1, 57–61.

Authors summary: A Lie triple derivation of an associative algebra M is a linear map $L : M \to M$ such that

$$L[[X,Y],Z] = [[L(X),Y],Z] +$$

$$[[X,L(Y)],Z] + [[X,Y],L(Z)]$$

for all $X, Y, Z \in M$.

We show that if M is a von Neumann algebra with no central Abelian summands then there exists an operator $A \in M$ such that $L(X) = [A,X] + \lambda(X)$ where $\lambda : M \to Z_M$ is a linear map which annihilates brackets of operators in M.

Reviewed by Jozsef Szucs

THEOREM

(JOHNSON 1996)

EVERY <u>CONTINUOUS</u> LIE DERIVATION OF A SYMMERTICALLY AMENABLE BANACH ALGEBRA A INTO A BANACH BIMODULE X IS THE SUM OF AN ASSOCIATIVE DERIVATION AND A "TRIVIAL" DERIVATION

(TRIVIAL=ANY LINEAR MAP WHICH VANISHES ON COMMUTATORS AND MAPS INTO THE "CENTER" OF THE MODULE).

The continuity assumption can be dropped if $X = A$ and A is a C*-algebra or a semisimple symmetrically amenable Banach algebra

Mathieu, Martin; Villena, Armando R.
The structure of Lie derivations on C*-algebras.
J. Funct. Anal. 202 (2003), no. 2, 504–525.

Alaminos, J.; Mathieu, M.; Villena, A. R.
Symmetric amenability and Lie derivations.
Math. Proc. Cambridge Philos. Soc. 137 (2004), no. 2, 433–439.

“ It remains an open question whether an analogous result for Lie derivations from A into a Banach A-bimodule holds when A is an arbitrary C*-algebra and when A is an arbitrary symmetrically amenable Banach algebra.”

“It is also an interesting open question whether or not every Lie derivation on a semisimple Banach algebra to itself has this form.”

(H.G. Dales)

END OF DIGRESSION

III—COHOMOLOGY OF FINITE DIMENSIONAL ALGEBRAS

ASSOCIATIVE ALGEBRAS

HOCHSCHILD

ANNALS OF MATHEMATICS 1945

Let M be an associative algebra and X a two-sided M-module. For $n \geq 1$, let
$L^n(M, X) =$ all n-linear maps
$(L^0(M, X) = X)$

Coboundary operator
$\partial : L^n \to L^{n+1}$ (for $n \geq 1$)

$$\partial\phi(a_1, \cdots, a_{n+1}) = a_1\phi(a_2, \cdots, a_{n+1})$$
$$+ \sum(-1)^j \phi(a_1, \cdots, a_{j-1}, a_j a_{j+1}, \cdots, a_{n+1})$$
$$+(-1)^{n+1}\phi(a_1, \cdots, a_n)a_{n+1}$$

For $n = 0$,
$$\partial : X \to L(M, X) \qquad \partial x(a) = ax - xa$$

Since $\partial \circ \partial = 0$,
$$\text{Im}(\partial : L^{n-1} \to L^n) \subset \ker(\partial : L^n \to L^{n+1})$$

$H^n(M, X) = \ker\partial/\text{Im}\partial$ is a vector space.

For $n = 1$, $\ker\partial =$
$\{\phi : M \to X : a_1\phi(a_2) - \phi(a_1a_2) + \phi(a_1)a_2 = 0\}$
$=$ the space of derivations from M to X

$$\partial : X \to L(M, X) \qquad \partial x(a) = ax - xa$$

$\mathrm{Im}\partial =$ the space of inner derivations

Thus $H^1(M, X)$ measures how close derivations are to inner derivations.

An associative algebra B is an extension of associative algebra A if there is a homomorphism σ of B onto A. The extension splits if $B = \ker\sigma \oplus A^*$ where A^* is an algebra isomorphic to A, and is singular if $(\ker\sigma)^2 = 0$.

PROPOSITION

There is a one to one correspondence between isomorphism classes of singular extensions of A and $H^2(A, A)$

LIE ALGEBRAS

JACOBSON

LIE ALGEBRAS 1962

If L is a Lie algebra, then an L-module is a vector space M and a mapping of $M \times L$ into M, $(m, x) \mapsto mx$, satisfying

$$(m_1 + m_2)x = m_1 x + m_2 x$$

$$\alpha(mx) = (\alpha m)x = m(\alpha x)$$

$$m[x_1, x_2] = (mx_1)x_2 - (mx_2)x_1.$$

Let L be a Lie algebra, M an L-module. If $i \geq 1$, an *i-dimensional M-cochain* for L is a skew symmetric i-linear mapping f of $L \times L \times \cdots \times L$ into M. Skew symmetric means that if two arguments in $f(x_1, \cdots, x_i)$ are interchanged, the value of f changes sign. A 0-dimensional cochain is a constant function from L to M.

The coboundary operator δ (for $i \geq 1$) is:

$$\delta(f)(x_1, \cdots, x_{i+1})$$
$$= \textstyle\sum_{q=1}^{i+1} (-1)^{i+1} f(x_1, \cdots, \widehat{x}_q, \cdots, x_{i+1}) x_q$$
$$+ \sum_{q<r=1}^{i+1} (-1)^{r+q} f(x_1, \cdots, \widehat{x}_q, \cdots, \widehat{x}_r, \cdots, x_{i+1}, [x_q, x_r]).$$

and for $i = 0$, $\delta(f)(x) = ux$ (module action), if f is the constant $u \in M$.

One verifies that $\delta^2 = 0$ giving rise to cohomology groups

$$H^i(L, M) = Z^i(L, M)/B^i(L, M)$$

If $i = 0$ we take $B^i = 0$ and $H^0(L, M) = Z^0(L, M) = \{u \in M : ux = 0, \forall x \in L\}$.

THEOREM (WHITEHEAD'S LEMMAS)

If L is a finite dimensional semisimple Lie algebra over a field of characteristic 0, then

$$H^1(L, M) = H^2(L, M) = 0$$

for every finite dimensional module M of L.

THEOREM (WHITEHEAD)

If L is a finite dimensional semisimple Lie algebra over a field of characteristic 0, then

$$H^i(L, M) = 0 \ (\forall i \geq 0)$$

for every finite dimensional <u>irreducible</u> module M of L such that $ML \neq 0$.

JORDAN ALGEBRAS

GERSTENHABER
PROCEEDING OF THE NATIONAL ACADEMY OF SCIENCES 1964

GLASSMAN/JACOBSON
JOURNAL OF ALGEBRA 1970

Let A be an algebra defined by a set of identities and let M be an A-module. A singular extension of length 2 is, by definition, a null extension of A by M. So we need to know what a null extension is.

It is simply a short exact sequence

$$0 \to M \xrightarrow{\alpha} E \xrightarrow{\beta} A \to 0$$

where, provisionally, M is an algebra (rather than an A-module) with $M^2 = 0$.

If $n > 2$, a singular extension of length n is an exact sequence of bimodules

$$0 \to M \to M_{n-1} \to \cdots \to M_2 \to E \to A \to 0$$

Morphisms, equivalences, addition, and scalar multiplication of equivalence classes of singular extensions can be defined.

Then for $n \geq 2$, $H^n(A, M) :=$ equivalence classes of singular extensions of length n

These definitions are equivalent to the classical ones in the associative and Lie cases.

IV. COHOMOLOGY OF BANACH ALGEBRAS

Let M be a Banach algebra and X a Banach M-module.

For $n \geq 1$, let

$L^n(M,X) =$ all continuous n-linear maps

$(L^0(M,X) = X)$

Coboundary operator

$\partial : L^n \to L^{n+1}$ (for $n \geq 1$)

$$\partial\phi(a_1, \cdots, a_{n+1}) = a_1\phi(a_2, \cdots, a_{n+1})$$
$$+ \sum(-1)^j \phi(a_1, \cdots, a_{j-1}, a_j a_{j+1}, \cdots, a_{n+1})$$
$$+(-1)^{n+1}\phi(a_1, \cdots, a_n)a_{n+1}$$

For $n = 0$,

$$\partial : X \to L(M,X) \qquad \partial x(a) = ax - xa$$

so

Im$\partial =$ the space of inner derivations

Since $\partial \circ \partial = 0$,
$\mathrm{Im}(\partial : L^{n-1} \to L^n) \subset \ker(\partial : L^n \to L^{n+1})$

$H^n(M, X) = \ker\partial / \mathrm{Im}\partial$ is a vector space.

For $n = 1$, $\ker\partial =$
$\{\phi : M \to X : a_1\phi(a_2) - \phi(a_1 a_2) + \phi(a_1)a_2 = 0\}$
$=$ the space of continuous derivations from M to X

Thus,

$$H^1(M, X) = \frac{\text{derivations from } M \text{ to } X}{\text{inner derivations from } M \text{ to } X}$$

measures how close continuous derivations are to inner derivations.

(What do $H^2(M, X)$, $H^3(M, X)$,... measure?)

Sneak Peak at Banach algebra cohomology

- $H^1(C(\Omega), E) = H^2(C(\Omega), E) = 0$
  (Kamowitz 1962 A PIONEER!)

  (Question: $H^3(C(\Omega), E) =?$)

- $H^1(A, B(H)) = 0??$ $(A \subset B(H))$
  "The major open question in the theory of derivations on C*-algebras"

- A derivation from A into $B(H)$ is inner if and only if it is completely bounded.
  (Christensen 1982)

- Barry Johnson, "**Cohomology of Banach algebras**", Memoirs of the American Mathematical Society 1972

Barry Johnson (1942–2002)

1,2 and 3 DIMENSIONAL COHOMOLOGY FOR BANACH JORDAN ALGEBRAS; PERTURBATION THEORY

Dosi-RMJM 2009 pp516-520

DEFINITION:

A BANACH ALGEBRA IS **STABLE** IF ANY TWO SUFFICIENTLY CLOSE BANACH ALGEBRA MULTIPLICATIONS ARE TOPOLOGICALLY ALGEBRAICALLY ISOMORPHIC

MORE PRECISELY

If m is a Banach algebra multiplication on A, then $\|m(x,y)\| \le \|m\|\|x\|\|y\|$.

THEOREM

If $H^2(A,A) = H^3(A,A) = 0$, then there exists $\epsilon > 0$ such that if $\|m_1 - m_2\| < \epsilon$ then (A, m_1) and (A, m_2) are topologically algebraically isomorphic.

- Johnson, Proc. Lon. Math. Soc. 1977
- Raeburn and Taylor, Jour. Funct. Anal. 1977

The origin of perturbation theory is deformation theory.

Let c_{ij}^k be the structure constants of a finite dimensional Lie algebra L.

Let $c_{ij}^k(\epsilon) \to c_{ij}^k$

Stability means $(L, c_{ij}^k(\epsilon))$ is isomorphic to (L,c$_{ij}^k$) if ϵ is sufficiently small.

THEOREM

(Gerstenhaber, Ann. of Math. 1964)
Finite dimensional semisimple Lie algebras are stable.

ANAR DOSI (ALSO USES DOSIEV)
(Middle East Technical University, TURKEY)

THEOREM

IF L IS A BANACH LIE ALGEBRA AND $H^2(L,L) = H^3(L,L) = 0$, THEN L IS A STABLE BANACH LIE ALGEBRA

THEOREM

SIMILAR FOR BANACH JORDAN ALGEBRAS (WITH APPROPRIATE DEFINITIONS OF LOW DIMENSIONAL COHOMOLOGY GROUPS)

Survey of operator algebra cohomology 1971-2009

Ringrose, presidential address
Bull. LMS 1996

Sinclair and Smith: Survey
Contemporary Mathematics 2004

Hochschild cohomology involves an associative algebra A and A-bimodules X and gives rise to

- n-cochains $L^n(A, X)$,
- coboundary operators Δ_n,
- n-coboundaries B^n,
- n-cocycles Z^n and
- cohomology groups $H^n(A, X)$.

If A is a Banach algebra and X is a Banach A-bimodule (=Banach space with module actions jointly continuous) we have the continuous versions of the above concepts

$$L_c^n \ , \ B_c^n \ , \ Z_c^n \ , \ H_c^n(A, X).$$

Warning: B_c^n is not always closed, so H_c^n is still only a vector space.

Let A be a C*-algebra of operators acting on a Hilbert space H and let X be a dual normal A-module (X is a dual space and the module actions are separately ultra weakly-weak*-continuous). We now have

- normal n-cochains $L^n_w(A, X)$= bounded and separately weakly continuous n-cochains
- coboundary operators Δ_n,
- normal n-coboundaries B^n_w,
- normal n-cocycles Z^n_w and
- normal cohomology groups $H^n_w(A, X)$.

For a C*-algebra acting on a Hilbert space we thus have three possible cohomology theories:

- the purely algebraic Hochschild theory H^n
- the bounded theory H_c^n
- the normal theory H_w^n

THEOREM 1C (1971)

$$H_w^n(A, X) \sim H_w^n(R, X)$$

(R =ultraweak closure of A)

THEOREM 2C (1972)

$$H_w^n(A, X) \sim H_w^c(A, X)$$

By Theorems 1C and 2C, due to **Johnson-Kadison-Ringrose**, all four cohomology groups

$$H_w^n(A, X)\ ,\ H_w^n(R, X)\ ,\ H_c^n(R, X)\ ,\ H_w^n(R, X)$$

are isomorphic.

THEOREM 3C (1971)

(Johnson-Kadison-Ringrose)

$H_c^n(R, X) = 0 \; \forall n \geq 1$

($R =$hyperfinite von Neumann algebra)

THEOREM 4C (1978)

(Connes)

If R is a von Neumann algebra with a separable predual, and $H_c^1(R, X) = 0$ for every dual normal R-bimodule X, then R is hyperfinite.

At this point, there were two outstanding problems of special interest;

Problem A

$$H_c^n(R,R) = 0 \ \forall n \geq 1?$$

for every von Neumann algebra R

Problem B

$$H_c^n(R,B(H)) = 0 \ \forall n \geq 1?$$

for every von Neumann algebra R acting on a Hilbert space H

(**Problem C** will come later)

ENTER COMPLETE BOUNDEDNESS

FAST FORWARD ONE DECADE

"The main obstacle to advance was a paucity of information about the general bounded linear (or multilinear) mapping between operator algebras. The major breakthrough, leading to most of the recent advances, came through the development of a rather detailed theory of completely bounded mappings."
(Ringrose)

Let A be a C*-algebra and let S be a von Neumann algebra, both acting on the same Hilbert space H with $A \subset S$. We can view S as a dual normal A-module with A acting on S by left and right multiplication. We now have

- completely bounded n-cochains $L^n_{cb}(A, S)$
- coboundary operators Δ_n,
- completely bounded n-coboundaries B^n_{cb},
- completely bounded n-cocycles Z^n_{bc}
- completely bounded cohomology groups $H^n_{cb}(A, S)$.

Let A be a C*-algebra and let S be a von Neumann algebra, both acting on the same Hilbert space H with $A \subset S$. We can view S as a dual normal A-module with A acting on S by left and right multiplication. We now have

- completely bounded n-cochains $L^n_{cb}(A, S)$
- coboundary operators Δ_n,
- completely bounded n-coboundaries B^n_{cb},
- completely bounded n-cocycles Z^n_{bc}
- completely bounded cohomology groups $H^n_{cb}(A, S)$.

For a C*-algebra A and a von Neumann algebra S with $A \subset S \subset B(H)$ we thus have two new cohomology theories:

- the completely bounded theory H^n_{cb}
- the completely bounded normal theory H^n_{cbw}

By straightforward analogues of Theorems 1C and 2C, all four cohomology groups

$$H^n_{cb}(A,S)\ ,\ H^n_{cbw}(A,S)\ ,\ H^n_{cb}(R,S)\ ,\ H^n_{cbw}(R,S)$$

are isomorphic, where R is the ultraweak closure of A.

THEOREM 5C (1987)

(Christensen-Effros-Sinclair)

$$H^n_{cb}(R, B(H)) = 0 \ \forall n \geq 1$$

($R =$any von Neumann algebra acting on H)

THEOREM 6C (1987)[†]

(Christensen-Sinclair)

$$H^n_{cb}(R, R) = 0 \ \forall n \geq 1$$

($R =$any von Neumann algebra)

[†]unpublished as of 2004

"Cohomology and complete boundedness have enjoyed a symbiotic relationship where advances in one have triggered progress in the other" (Sinclair-Smith)

Theorems 7C and 8C are due to Christensen-Effros-Sinclair.

THEOREM 7C (1987)

$$H_c^n(R, R) = 0 \ \forall n \geq 1$$

($R =$ von Neumann algebra of type I, II_∞, III, or of type II_1 and stable under tensoring with the hyperfinite factor)

THEOREM 8C (1987)

$$H_c^n(R, B(H)) = 0 \ \forall n \geq 1$$

($R =$ von Neumann algebra of type I, II_∞, III, or of type II_1 and stable under tensoring with the hyperfinite factor, acting on a Hilbert space H)

THEOREM 9C (1998)

(Sinclair-Smith based on earlier work of Christensen,Pop,Sinclair,Smith)

$$H_c^n(R,R) = 0 \ \forall n \geq 1$$

($R =$ von Neumann algebra of type II_1 with a Cartan subalgebra and a separable‡ predual)

THEOREM 10C (2003)

(Christensen-Pop-Sinclair-Smith $n \geq 3$)

$$H_c^n(R,R) = H_c^n(R,B(H)) = 0 \ \forall n \geq 1$$

($R =$ von Neumann algebra factor of type II_1 with property Γ, acting on a Hilbert space H)

($n = 1$:Kadison-Sakai '66 and Christensen '86
$n = 2$: Christensen-Sinclair '87, '01)

‡The separability assumption was removed in 2009—Jan Cameron

We can now add a third problem (C) to our previous two (A,B)

Problem A

$$H_c^n(R, R) = 0 \ \forall n \geq 1?$$

for every von Neumann algebra R

Problem B

$$H_c^n(R, B(H)) = 0 \ \forall n \geq 1?$$

for every von Neumann algebra R acting on a Hilbert space H

Problem C

$$H_c^n(R, R)) = 0 \ \forall n \geq 2?$$

(R is a von Neumann algebra of type II_1)

A candidate is the factor arising from the free group on 2 generators.

V—DERIVATIONS ON FINITE DIMENSIONAL TRIPLE SYSTEMS

DEFINITION 5

A DERIVATION ON $M_{m,n}(\mathbf{C})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION

IS A LINEAR MAPPING δ WHICH SATISFIES THE (TRIPLE) PRODUCT RULE

$$\delta(AB^*C) =$$
$$\delta(A)B^*C + A\delta(B)^*C + AB^*\delta(C)$$

PROPOSITION 5

FOR TWO MATRICES $A \in M_m(\mathbf{C}), B \in \mathrm{M}_n(\mathbf{C})$, WITH $A^* = -A, B^* = -B$,

DEFINE $\delta_{A,B}(X) =$

$$AX + XB$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION

THEOREM 5

EVERY DERIVATION ON $M_{m,n}(\mathbf{C})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

REMARK

THESE RESULTS HOLD TRUE AND ARE OF INTEREST FOR THE CASE $m = n$.

TRIPLE BRACKET MULTIPLICATION

LET'S GO BACK FOR A MOMENT TO SQUARE MATRICES AND THE BRACKET MULTIPLICATION.

MOTIVATED BY THE LAST REMARK, WE DEFINE THE TRIPLE BRACKET MULTIPLICATION TO BE $[[X, Y], Z]$

DEFINITION 6

A <u>DERIVATION</u> ON $M_n(\mathbf{C})$ WITH <u>RESPECT TO</u> <u>TRIPLE BRACKET MULTIPLICATION</u>

IS A LINEAR MAPPING δ WHICH SATISFIES THE TRIPLE PRODUCT RULE

$$\delta([[A, B], C]) = [[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$$

PROPOSITION 6

FIX TWO MATRICES A, B IN $M_n(\mathbf{C})$ AND DEFINE $\delta_{A,B}(X) = [[A, B], X]$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION.

THEOREM 6

EVERY DERIVATION OF $M_n(\mathbf{C})^{\S}$ WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION IS A SUM OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

§not a semisimple Lie triple system, as in Theorem 3

TRIPLE CIRCLE MULTIPLICATION

LET'S RETURN TO RECTANGULAR MATRICES AND FORM THE TRIPLE CIRCLE MULTIPLICATION

$$(AB^*C + CB^*A)/2$$

For sanity's sake, let us write this as

$$\{A, B, C\} = (AB^*C + CB^*A)/2$$

DEFINITION 7

A <u>DERIVATION</u> ON $M_{m,n}(\mathbf{C})$ WITH <u>RESPECT TO</u> <u>TRIPLE CIRCLE MULTIPLICATION</u>

IS A LINEAR MAPPING δ WHICH SATISFIES THE TRIPLE PRODUCT RULE

$$\delta(\{A,B,C\}) =$$
$$\{\delta(A), B, C\} + \{A, \delta(B), C\} + \{B, A, \delta(C)\}$$

PROPOSITION 7

FIX TWO MATRICES A, B IN $M_{m,n}(\mathbf{C})$ AND DEFINE

$$\delta_{A,B}(X) = \{A, B, X\} - \{A, B, X\}$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION.

THEOREM 7

EVERY DERIVATION OF $M_{m,n}(\mathbf{C})$ WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

Table 3[¶]

$M_{m,n}(\mathbf{C})$ (SS TRIPLE SYSTEMS)

triple matrix	triple bracket	triple circle
ab^*c	$[[a,b],c]$	$ab^*c + cb^*a$
Th. 5	Th.6	Th.7
$\delta_{a,b}(x)$ $=$ ab^*x $+xb^*a$ $-ba^*x$ $-xa^*b$	$\delta_{a,b}(x)$ $=$ abx $+xba$ $-bax$ $-xab$	$\delta_{a,b}(x)$ $=$ ab^*x $+xb^*a$ $-ba^*x$ $-xa^*b$
(sums)	(sums) $(m = n)$	(sums)

[¶]Note: for triple matrix and triple circle multiplication,

$$(ab^* - ba^*)^* = -(ab^* - ba^*)$$

and

$$(b^*a - a^*b)^* = -(b^*a - a^*b)$$

Table 1

$M_n(\mathrm{C})$ (SS ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a,b] = ab - ba$	$a \circ b = ab + ba$
Th. 2	Th.3	Th.4
$\delta_a(x)$ $=$ $ax - xa$	$\delta_a(x)$ $=$ $ax - xa$	$\delta_a(x)$ $=$ $ax - xa$

Table 3

$M_{m,n}(\mathrm{C})$ (SS TRIPLE SYSTEMS)

triple matrix	triple bracket	triple circle
ab^*c	$[[a,b],c]$	$ab^*c + cb^*a$
Th. 5	Th.6	Th.7
$\delta_{a,b}(x)$ $=$ ab^*x $+xb^*a$ $-ba^*x$ $-xa^*b$	$\delta_{a,b}(x)$ $=$ abx $+xba$ $-bax$ $-xab$	$\delta_{a,b}(x)$ $=$ ab^*x $+xb^*a$ $-ba^*x$ $-xa^*b$
(sums)	(sums) $(m = n)$	(sums)

AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

AN <u>TRIPLE SYSTEM</u> IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH ONE BINARY OPERATION, CALLED <u>ADDITION</u> AND ONE TERNARY OPERATION CALLED <u>TRIPLE MULTIPLICATION</u>

ADDITION IS DENOTED BY

$$a+b$$

AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE

$$a+b=b+a, \quad (a+b)+c=a+(b+c)$$

TRIPLE MULTIPLICATION IS DENOTED

$$abc$$

AND IS REQUIRED TO BE LINEAR IN EACH VARIABLE

$$(a+b)cd = acd + bcd$$
$$a(b+c)d = abd + acd$$
$$ab(c+d) = abc + abd$$

IMPORTANT BUT SIMPLE EXAMPLES OF TRIPLE SYSTEMS CAN BE FORMED FROM ANY ALGEBRA

IF ab DENOTES THE ALGEBRA PRODUCT, JUST DEFINE A TRIPLE MULTIPLICATION TO BE $(ab)c$

LET'S SEE HOW THIS WORKS IN THE ALGEBRAS WE INTRODUCED IN SECTION I

$\mathcal{C}$; $fgh = (fg)h$, OR $fgh = (f\overline{g})h$

$(M_n(\mathbf{C}), \times)$; $abc = abc$ OR $abc = ab^*c$

$(M_n(\mathbf{C}), [,])$; $abc = [[a,b],c]$

$(M_n(\mathbf{C}), \circ)$; $abc = (a \circ b) \circ c$

A TRIPLE SYSTEM IS SAID TO BE <u>ASSOCIATIVE</u> (RESP. <u>COMMUTATIVE</u>) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)

(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

IN THE TRIPLE CONTEXT THIS MEANS THE FOLLOWING

ASSOCIATIVE

$$ab(cde) = (abc)de = a(bcd)e$$

OR $ab(cde) = (abc)de = a(dcb)e$

COMMUTATIVE: $abc = cba$

AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

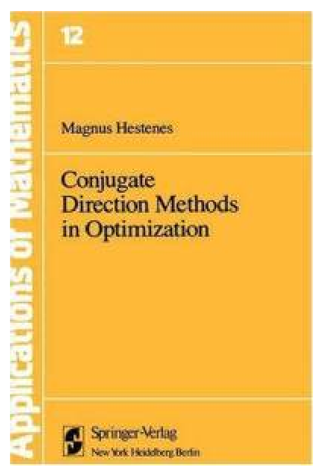
THE AXIOM WHICH CHARACTERIZES TRIPLE MATRIX MULTIPLICATION IS

$$(abc)de = ab(cde) = a(dcb)e$$

THESE ARE CALLED
ASSOCIATIVE TRIPLE SYSTEMS
or
HESTENES ALGEBRAS

Magnus Hestenes (1906–1991)

Magnus Rudolph Hestenes was an American mathematician. Together with Cornelius Lanczos and Eduard Stiefel, he invented the conjugate gradient method.



THE AXIOMS WHICH CHARACTERIZE TRIPLE BRACKET MULTIPLICATION ARE

$$aab = 0$$

$$abc + bca + cab = 0$$

$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

THESE ARE CALLED

LIE TRIPLE SYSTEMS

(NATHAN JACOBSON, MAX KOECHER)

Max Koecher (1924–1990)

Max Koecher was a German mathematician. His main research area was the theory of Jordan algebras, where he introduced the KantorKoecherTits construction.

Nathan Jacobson (1910–1999)

THE AXIOMS WHICH CHARACTERIZE TRIPLE CIRCLE MULTIPLICATION ARE

$$abc = cba$$

$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$

THESE ARE CALLED

JORDAN TRIPLE SYSTEMS

Kurt Meyberg

Ottmar Loos + Erhard Neher

Table 4

TRIPLE SYSTEMS

associative triple systems

$$(abc)de = ab(cde) = a(dcb)e$$

Lie triple systems

$$aab = 0$$

$$abc + bca + cab = 0$$

$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

Jordan triple systems

$$abc = cba$$

$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$

VI—DERIVATIONS ON BANACH TRIPLES

(iii) JC*-TRIPLE

derivation:

$$D\{a,b,c\} = \{Da.b,c\} + \{a,Db,c\} + \{a,b,Dc\}$$

$$\{x,y,z\} = (xy^*z + zy^*x)/2$$

inner derivation: $\sum_i [L(x_i,a_i) - L(a_i,x_i)]$

$$(x_i \in M, a_i \in A)$$

$$b \mapsto \sum_i [\{x_i,a_i,b\} - \{a_i,x_i,b\}]$$

- AUTOMATIC CONTINUITY RESULTS

  BARTON-FRIEDMAN 1990

  (**NEW**) PERALTA-RUSSO 2010 (module)

- INNER DERIVATION RESULTS

  HO-MARTINEZ-PERALTA-RUSSO 2002

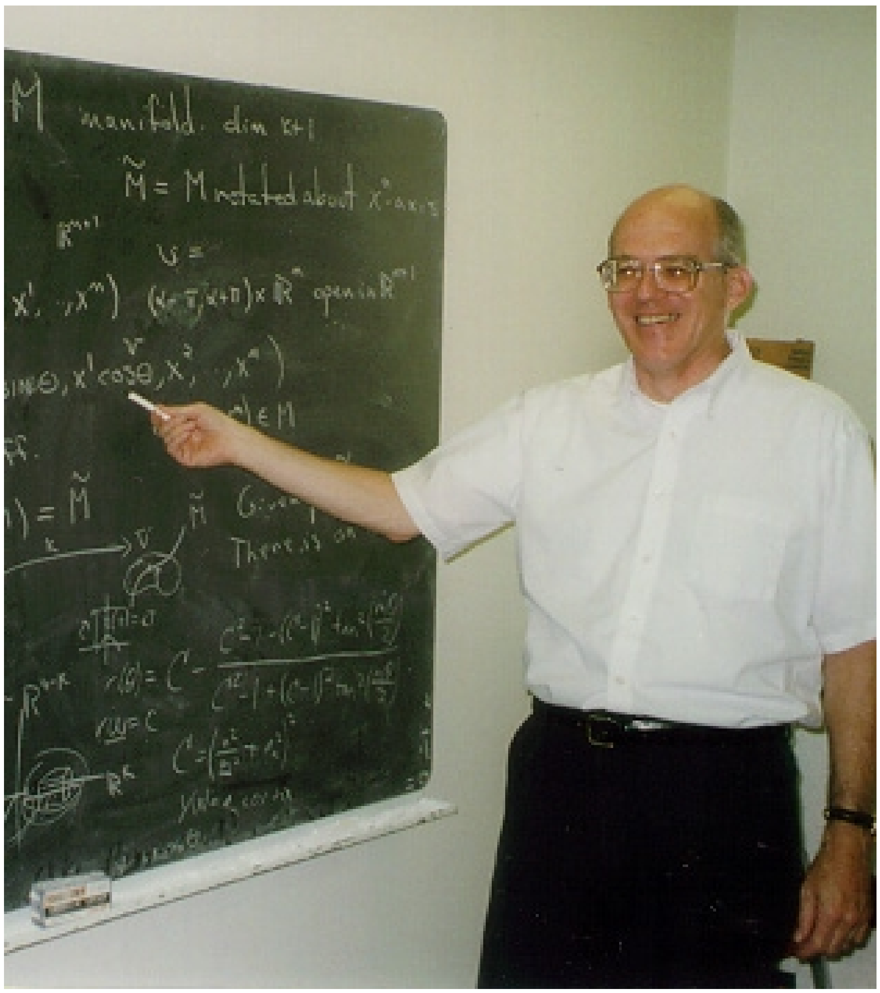
  MEYBERG 1972

  KÜHN-ROSENDAHL 1978 (module)

  (**NEW**) HO-PERALTA-RUSSO 2011
  (module, weak amenability)

KUDOS TO:

Lawrence A. Harris (PhD 1969)



1974 (infinite dimensional holomorphy)

1981 (spectral and ideal theory)

AUTOMATIC CONTINUITY RESULTS

THEOREM (1990 Barton-Friedman)
EVERY DERIVATION OF A JB*-TRIPLE IS CONTINUOUS

THEOREM (2010 Peralta-Russo)
NECESSARY AND SUFFICIENT CONDITIONS UNDER WHICH A DERIVATION OF A JB*-TRIPLE INTO A JORDAN TRIPLE **MODULE** IS CONTINUOUS

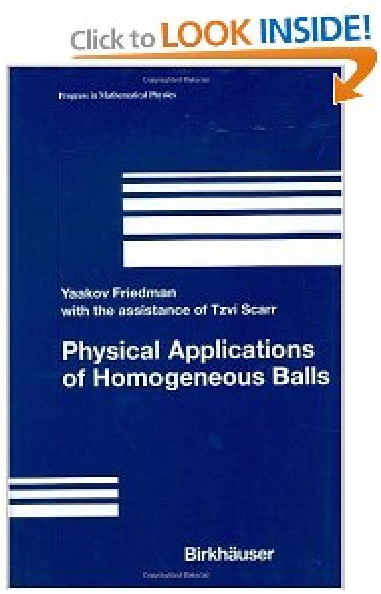
(JB*-triple and Jordan triple module are defined below)

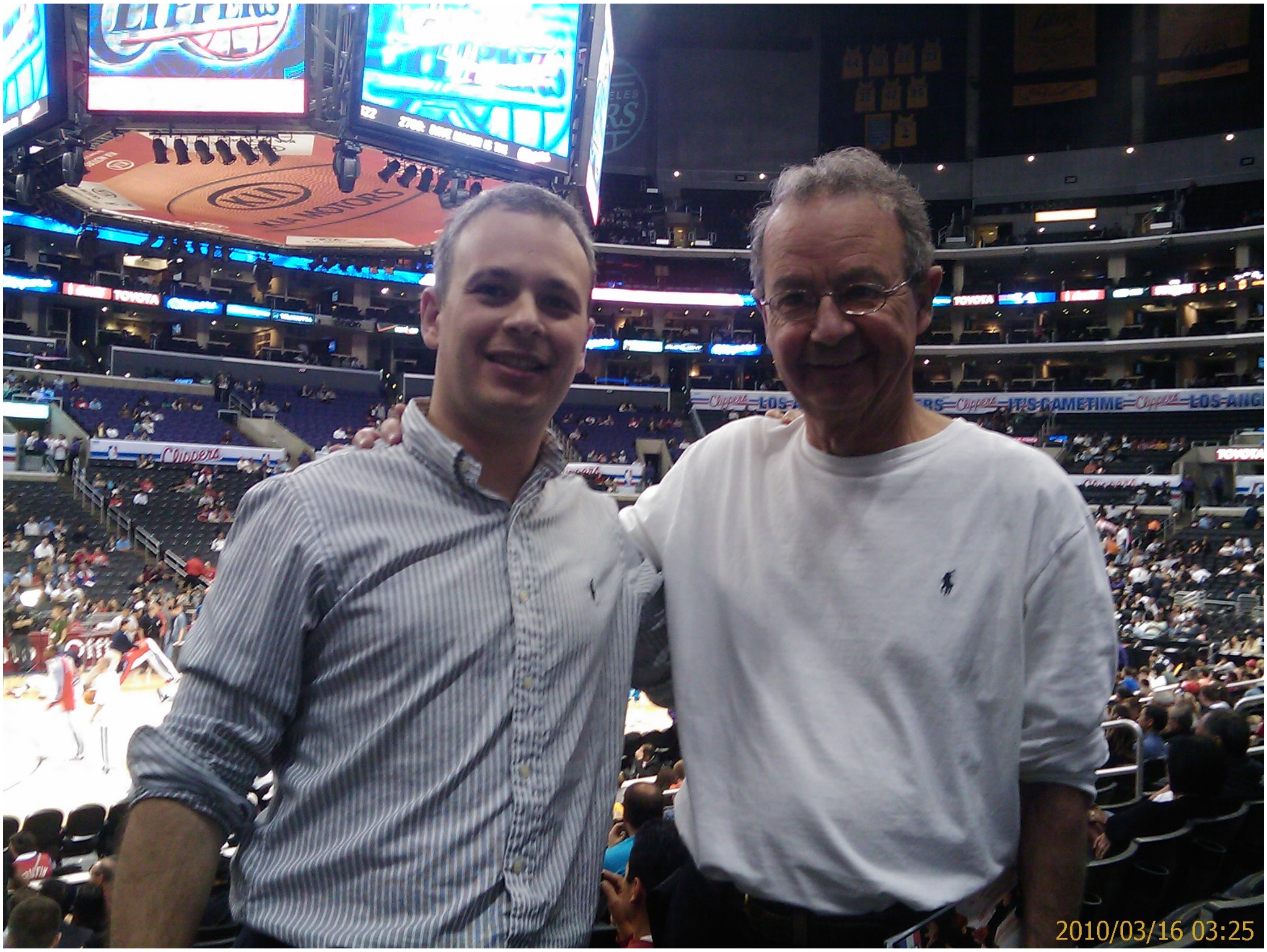
Tom Barton (b. 1955)

Tom Barton is Senior Director for Architecture, Integration and CISO at the University of Chicago. He had similar assignments at the University of Memphis, where he was a member of the mathematics faculty before turning to administration.

Yaakov Friedman (b. 1948)

Yaakov Friedman is director of research at Jerusalem College of Technology.





Antonio Peralta (b. 1974)
Bernard Russo (b. 1939)

GO LAKERS!

PREVIOUS INNER DERIVATION RESULTS

THEOREM (1972 Meyberg)

EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN TRIPLE SYSTEM IS INNER

(Lie algebras, Lie triple systems)

THEOREM (1978 Kühn-Rosendahl)

EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN TRIPLE SYSTEM INTO A JORDAN TRIPLE MODULE IS INNER

(Lie algebras)

THEOREM 2002
(Ho-Martinez-Peralta-Russo)
CARTAN FACTORS OF TYPE $I_{n,n}$, II (even or ∞), and III HAVE THE INNER DERIVATION PROPERTY

THEOREM 2002
(Ho-Martinez-Peralta-Russo)
INFINITE DIMENSIONAL CARTAN FACTORS OF TYPE $I_{m,n}, m \neq n$, and IV DO NOT HAVE THE INNER DERIVATION PROPERTY.

HO-PERALTA-RUSSO WORK ON TERNARY WEAK AMENABILITY FOR C*-ALGEBRAS AND JB*-TRIPLES

1. COMMUTATIVE C*-ALGEBRAS ARE TERNARY WEAKLY AMENABLE (TWA)

2. COMMUTATIVE JB*-TRIPLES ARE APPROXIMATELY WEAKLY AMENABLE

3. $B(H), K(H)$ ARE TWA IF AND ONLY IF FINITE DIMENSIONAL

4. CARTAN FACTORS OF TYPE $I_{m,n}$ OF FINITE RANK WITH $m \neq n$, AND OF TYPE IV ARE TWA IF AND ONLY IF FINITE DIMENSIONAL

SAMPLE LEMMA

The C*-algebra $A = K(H)$ of all compact operators on an infinite dimensional Hilbert space H is not Jordan weakly amenable.

We shall identify A^* with the trace-class operators on H.

Supposing that A were Jordan weakly amenable, let $\psi \in A^*$ be arbitrary. Then D_ψ ($= \mathrm{ad}\,\psi$) is an associative derivation and hence a Jordan derivation, so by assumption would be an inner Jordan derivation. Thus there would exist $\varphi_j \in A^*$ and $b_j \in A$ such that

$$D_\psi(x) = \sum_{j=1}^{n} [\varphi_j \circ (b_j \circ x) - b_j \circ (\varphi_j \circ x)]$$

for all $x \in A$.

For $x, y \in A$, a direct calculation yields

$$\psi(xy - yx) = -\frac{1}{4}\left(\sum_{j=1}^{n} b_j\varphi_j - \varphi_j b_j\right)(xy - yx).$$

It is known (Pearcy-Topping 1971) that every compact operator on a separable (which we may assume WLOG) infinite dimensional Hilbert space is a finite sum of commutators of compact operators.

By the just quoted theorem of Pearcy and Topping, every element of $K(H)$ can be written as a finite sum of commutators $[x, y] = xy - yx$ of elements x, y in $K(H)$.

Thus, it follows that the trace-class operator

$$\psi = -\frac{1}{4}\left(\sum_{j=1}^{n} b_j\varphi_j - \varphi_j b_j\right)$$

is a finite sum of commutators of compact and trace-class operators, and hence has trace zero. This is a contradiction, since ψ was arbitrary.

PROPOSITION

The JB*-triple $A = M_n(\mathrm{C})$ is ternary weakly amenable.

By a Proposition which is a step in the proof that commutative C*-algebras are ternary weakly amenable,

$$\mathcal{D}_t(A, A^*) = \mathcal{I}nn_b^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*),$$

so it suffices to prove that

$$\mathcal{I}nn_b^*(A, A^*) \circ * \subset \mathcal{I}nn_t(A, A^*).$$

As in the proof of the Lemma, if $D \in \mathcal{I}nn_b^*(A, A^*)$ so that $Dx = \psi x - x\psi$ for some $\psi \in A^*$, then

$$\psi = [\varphi_1, b_1] - [\varphi_2, b_2] + \frac{\mathrm{Tr}\,(\psi)}{n} I,$$

where b_1, b_2 are self adjoint elements of A and φ_1 and φ_2 are self adjoint elements of A^*.

It is easy to see that, for each $x \in A$, we have

$$D(x^*) = \{\varphi_1, 2b_1, x\} - \{2b_1, \varphi_1, x\} - \{\varphi_2, 2b_2, x\} + \{2b_2, \varphi_2, x\},$$

so that

$$D \circ * \in \mathcal{I}nn_t(A, A^*).$$

VII—COHOMOLOGY OF FINITE DIMENSIONAL TRIPLE SYSTEMS

1. Cohomology of Lie triple systems and lie algebras with involution, B. Harris, TAMS 1961
2. Cohomology of associative triple systems, Renate Carlsson, PAMS 1976
3. On the representation theory of Lie triple systems, T.L.Hodge and B.J. Parshall, Trans. A.M.S. 2002

WEDDERBURN DECOMPOSITION

4. Der Wedderburnsche Hauptsatz für alternative Tripelsysteme und Paare, Renate Carlsson, Math. Ann 1977
5. Wedderburnzerlegung für Jordan-Paare, Oda Kühn und Adelheid Rosendahl, Manus. Math 1978

1
Cohomology of Lie triple systems and lie algebras with involution
B. Harris, TAMS 1961

MATHEMATICAL REVIEWS

A Lie triple system T is a subspace of a Lie algebra L closed under the ternary operation $[xyz] = [x, [y, z]]$ or, equivalently, it is the subspace of L consisting of those elements x such that $\sigma(x) = -x$, where σ is an involution of L.

A T-module M is a vector space such that the vector-space direct sum $T \oplus M$ is itself a Lie triple system in such a way that

1. T is a subsystem
2. $[xyz] \in M$ if any of x, y, z is in M
3. $[xyz] = 0$ if two of x, y, z are in M.

A universal Lie algebra $L_u(T)$ and an $L_u(T)$-module M_s can be constructed in such a way that both are operated on by an involution σ and so that T and M consist of those elements of $L_u(T)$ and M_s which are mapped into their negatives by σ.

Now suppose L is a Lie algebra with involution σ and N is an L-σ module. Then σ operates on $H^n(L,N)$ so that

$$H^n(L,N) = H^n_+(L,N) \oplus H^n_-(L,N)$$

with both summands invariant under σ.

The cohomology of the Lie triple system is defined by $H^n(T,M) = H^n_+(L_u(T), M_s)$.

The author investigates these groups for $n = 0, 1, 2$.

- $H^0(T, M) = 0$ for all T and M
- $H^1(T, M) =$ derivations of T into M modulo inner derivations
- $H^2(T, M) =$ factor sets of T into M modulo trivial factor sets.

Turning to the case of finite-dimensional simple T and ground field of characteristic 0, one has the Whitehead lemmas

$$H^1(T, M) = 0 = H^2(T, M)$$

Weyl's theorem: Every finite-dimensional module is semi-simple.

The paper ends by showing that if in addition, the ground field Φ is algebraically closed, then $H^3(T, \Phi)$ is 0 or not 0, according as $L_u(T)$ is simple or not.

2

On the representation theory of Lie triple systems,

Hodge, Terrell L., Parshall, Brian,

Trans. Amer. Math. Soc. 354 (2002), no. 11, 4359–4391

The authors of the paper under review study representations of Lie triple systems, both ordinary and restricted.

The theory is based on the connection between Lie algebras and Lie triple systems.

In addition, the authors begin the study of the cohomology theory for Lie triple systems and their restricted versions.

They also sketch some future applications and developments of the theory.

Reviewed by Plamen Koshlukov

3
Cohomology of associative triple systems, Renate Carlsson, PAMS 1976

MATHEMATICAL REVIEWS

A cohomology for associative triple systems is defined, with the main purpose to get quickly the cohomological triviality of finite-dimensional separable objects over fields of characteristic $\neq 2$, i.e., in particular the Whitehead lemmas and the Wedderburn principal theorem.

This is achieved by embedding an associative triple system A in an associative algebra $U(A)$ and associating with every trimodule M for A a bimodule M_u for $U(A)$ such that the cohomology groups $H^n(A, M)$ are subgroups of the classical cohomology groups $H^n(U(A), M_u)$.

Since $U(A)$ is chosen sufficiently close to A, in order to inherit separability, the cohomological triviality of separable A is an immediate consequence of the associative algebra theory.

The paper does not deal with functorialities, not even with the existence of a long exact cohomology sequence.

VIII—COHOMOLOGY OF BANACH TRIPLE SYSTEMS (PROSPECTUS)

- Lie derivations into a module; automatic continuity and weak amenability (Harris,Miers,Mathieu,Villena)
- Cohomology of commutative JB*-triples (Kamowitz,Carlsson)
- Cohomology of TROs (Zalar,Carlsson)
- Wedderburn decompositions for JB*-triples (Kühn-Rosendahl)
- Low dimensional cohomology for JBW*-triples and algebras-perturbation (Dosi,McCrimmon)
- Structure group of JB*-triple (McCrimmon—derivations)
- Alternative Banach triples (Carlsson,Braun)
- Completely bounded triple cohomology (Timoney et.al.,Christensen et.al)
- Local derivations on JB*algebras and triples (Kadison,Johnson,Ajupov,. . .)
- Chu's work on Koecher-Kantor-Tits construction