DERIVATIONS
An introduction to non associative algebra (or, Playing havoc with the product rule)

Series 2—Part 6
Universal enveloping associative triple systems

Colloquium
Fullerton College

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History of these lectures

"Slides" for all series 1 and series 2 talks available at
http://www.math.uci.edu/INSERT a "$\sim$" HERE brusso/undergraduate.html

Series 1

- PART I FEBRUARY 8, 2011 **ALGEBRAS; DERIVATIONS**
- PART II JULY 21, 2011 **TRIPLE SYSTEMS; DERIVATIONS**
- PART III FEBRUARY 28, 2012 **MODULES; DERIVATIONS**
- PART IV JULY 26, 2012 **COHOMOLOGY (ASSOCIATIVE ALGEBRAS)**
- PART V OCTOBER 25, 2012 **THE SECOND COHOMOLOGY GROUP**
- PART VI MARCH 7, 2013 **COHOMOLOGY (LIE ALGEBRAS)**
- PART VII JULY 25, 2013 **COHOMOLOGY (JORDAN ALGEBRAS)**
- PART VIII SEPTEMBER 17, 2013 **VANISHING THEOREMS IN DIMENSIONS 1 AND 2 (ASSOCIATIVE ALGEBRAS)**
- PART IX FEBRUARY 18, 2014 **VANISHING THEOREMS IN DIMENSIONS 1 AND 2 (JORDAN ALGEBRAS)**

Series 2

- PART I JULY 24, 2014 **THE REMARKABLE CONNECTION BETWEEN JORDAN ALGEBRAS AND LIE ALGEBRAS (Two theorems relating different types of derivations)**
- PART II NOVEMBER 18, 2014 **THE REMARKABLE CONNECTION BETWEEN JORDAN ALGEBRAS AND LIE ALGEBRAS (Two theorems embedding triple systems in Lie algebras)**
- (digression) FEBRUARY 24, 2015 **GENETIC ALGEBRAS**
- PART III JULY 15, 2015 **LOCAL DERIVATIONS**
- (Fall 2015 missed due to the flu)
- PART IV FEBRUARY 23, 2016 **2-LOCAL DERIVATIONS**
- PART V JUNE 28, 2016 **LINKING ALGEBRA OF A TRIPLE SYSTEM**
- PART VI (today) OCTOBER 18, 2016 **UNIVERSAL ENVELOPING ASSOCIATIVE TRIPLE SYSTEMS**

Outline of today's talk

Series 2—Part 6

- Part 1: Square and Rectangular Matrices
- Part 2: Algebras and Triple Systems
- Part 3: Examples of Embeddings (Relations between different structures)
- Part 4: Universal Enveloping Associative Triple Systems

(Part 1 starts here) RECTANGULAR MATRICES

$$M_{p,q} = \text{ all } p \text{ by } q \text{ real matrices}$$

$$\mathbf{a} = [a_{ij}]_{p\times q} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \cdots & \cdots & \cdots & \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix} \quad (a_{ij} \in \mathbb{R})$$

Matrix Multiplication $M_{p,q} \times M_{q,r} \subset M_{p,r}$

$$\mathbf{ab} = [a_{ij}]_{p\times q}[b_{kl}]_{q\times r} = [c_{ij}]_{p\times r} \text{ where } c_{ij} = \sum_{k=1}^{q} a_{ik} b_{kj}$$

Example

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = [a_{ij}]_{3\times 2}[b_{ij}]_{2\times 1} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} \\ a_{31}b_{11} + a_{32}b_{21} \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} = [c_{ij}]_{3\times 1}$$

SQUARE MATRICES

$$M_p = M_{p,p} = \text{ all } p \text{ by } p \text{ real matrices}$$

$$\mathbf{a} = [a_{ij}]_{p\times p} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \cdots & \cdots & \cdots & \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{bmatrix} \quad (a_{ij} \in \mathbb{R})$$

Matrix Multiplication $M_p \times M_p \subset M_p$

$$\mathbf{ab} = [a_{ij}]_{p\times p}[b_{kl}]_{p\times p} = [c_{ij}]_{p\times p} \text{ where } c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

Examples $p = 1, 2$

- $M_1 = \{[a_{11}] : a_{11} \in \mathbb{R}\}$ (Behaves exactly as $\mathbb{R}$)
- $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = [a_{ij}]_{2\times 2}[b_{ij}]_{2\times 2} = \begin{bmatrix} a_{11}b_{11} + a_{11}b_{12} & a_{21}b_{11} + a_{22}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

Important special cases

$M_{p,q}$ is a linear space

$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$, $c[a_{ij}] = [ca_{ij}]$

$M_{1,p} = \mathbb{R}^p$ (row vectors)

$M_{p,1} = \mathbb{R}^p$ (column vectors)

M_p is an <u>algebra</u>
(matrix addition and matrix multiplication)

$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$, $c[a_{ij}] = [ca_{ij}]$, $[a_{ij}] \times [b_{ij}] = [\sum_{k=1}^{n} a_{ik} b_{kj}]$

$M_{p,q}$ is a <u>triple system</u>
(matrix addition and <u>triple matrix multiplication</u>)

What is triple matrix multiplication? You need the **transpose**.

Transpose (to the rescue)

If $\mathbf{a} = [a_{ij}] \in M_{p,q}$ then $\mathbf{a}^t = [a^t_{ij}] \in M_{q,p}$ where $a^t_{ij} = a_{ji}$

$$\mathbf{a} = [a_{ij}]_{p\times q} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \cdots & \cdots & \cdots & \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix}$$

$$\mathbf{a}^t = [a^t_{ij}]_{q\times p} = \begin{bmatrix} a^t_{11} & a^t_{12} & \cdots & a^t_{1p} \\ a^t_{21} & a^t_{22} & \cdots & a^t_{2p} \\ \cdots & \cdots & \cdots & \\ a^t_{q1} & a^t_{q2} & \cdots & a^t_{qp} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{p1} \\ a_{12} & a_{22} & \cdots & a_{p2} \\ \cdots & \cdots & \cdots & \\ a_{1q} & a_{2q} & \cdots & a_{pq} \end{bmatrix}$$

$$(ab)^t = b^t a^t$$

Proof: If $\mathbf{a} = [a_{ij}]$, $\mathbf{b} = [b_{ij}]$ and $\mathbf{c} = \mathbf{ab} = [c_{ij}]$, so $c_{ij} = \sum_{k=1}^q a_{ik} b_{kj}$,

$$\mathbf{b}^t\mathbf{a}^t = [b^t_{ij}][a^t_{ij}] = [\textstyle\sum_{k=1}^q b^t_{ik} a^t_{kj}] \qquad \text{and}$$

$$(\mathbf{ab})^t = [c^t_{ij}] = [\textstyle\sum_{k=1}^q a_{jk} b_{ki}] = [\sum_{k=1}^q a^t_{kj} b^t_{ik}] \quad Q.E.D.$$

Three (binary) multiplications on Matrices

Matrix Multiplication

$(M_p, +, ab)$ is an **associative algebra** , $ab =$ matrix multiplication

Bracket Multiplication

$(M_p, +, [a, b])$ is a **Lie algebra** , $[a, b] = ab - ba$

Circle Multiplication

$(M_p, +, a \circ b)$ is a **Jordan algebra** , $a \circ b = (ab + ba)/2$

Three triple multiplications on Matrices

Triple Matrix Multiplication

$ab^tc, \quad a,b,c \in M_{p,q}(\mathbb{C})$ Denote ab^tc by $\langle abc\rangle$. Then
$(M_{p,q}, +, \langle abc\rangle)$ is an **associative triple system**

Triple Bracket Multiplication

$[[a,b],c], \quad a,b,c \in M_p(\mathbb{C})$ Denote $[[a,b],c]$ by $[abc]$. Then
$(M_p, +, [abc])$ is a **Lie triple system**

Triple Circle Multiplication

$(ab^tc + cb^ta)/2, \quad a,b,c \in M_{p,q}(\mathbb{C})$ Denote $(ab^tc + cb^ta)/2$ by $\{abc\}$. Then
$(M_p, +, \{abc\})$ is a **Jordan triple system**

(Part 2 starts here) Review of Algebras—Axiomatic approach

AN ALGEBRA IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE OVER A FIELD) WITH TWO BINARY OPERATIONS, CALLED ADDITION AND MULTIPLICATION—we are downplaying multiplication by scalars (=numbers=field elements)

ADDITION IS DENOTED BY $a+b$ AND IS REQUIRED TO BE COMMUTATIVE $\quad a+b=b+a$
AND ASSOCIATIVE $\quad (a+b)+c=a+(b+c)$

MULTIPLICATION IS DENOTED BY ab AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION
$(a+b)c=ac+bc, \quad a(b+c)=ab+ac$

AN ALGEBRA IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)
(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

Table 1 (FASHIONABLE) ALGEBRAS

commutative algebras $ab = ba$
(Real numbers, Complex numbers, Continuous functions)

****
* **associative algebras** $a(bc) = (ab)c$ *
* (Matrix multiplication) *
****

Lie algebras $a^2 = 0$, $(ab)c + (bc)a + (ca)b = 0$
(Bracket multiplication on associative algebras: $[x, y] = xy - yx$)

Jordan algebras $ab = ba$, $a(a^2b) = a^2(ab)$
(Circle multiplication on associative algebras: $x \circ y = (xy + yx)/2$)

AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

AN TRIPLE SYSTEM IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH ONE BINARY OPERATION, CALLED ADDITION AND ONE TERNARY OPERATION CALLED TRIPLE MULTIPLICATION
ADDITION IS DENOTED BY
$a + b$
AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE
$a + b = b + a, \quad (a + b) + c = a + (b + c)$
TRIPLE MULTIPLICATION IS DENOTED
$\langle abc \rangle$
AND IS REQUIRED TO BE LINEAR IN EACH VARIABLE

$\langle (a + b)cd \rangle = \langle acd \rangle + \langle bcd \rangle$
$\langle a(b + c)d \rangle = \langle abd \rangle + \langle acd \rangle$
$\langle ab(c + d) \rangle = \langle abc \rangle + \langle abd \rangle$

THE AXIOM WHICH CHARACTERIZES TRIPLE MATRIX MULTIPLICATION

$$\langle\langle abc\rangle de\rangle = \langle ab\langle cde\rangle\rangle = \langle a\langle dcb\rangle e\rangle$$

Why not $\langle a\langle bcd\rangle e\rangle$?

THESE ARE CALLED **ASSOCIATIVE TRIPLE SYSTEMS**
(MAGNUS HESTENES, OTTMAR LOOS)

Example: Triple matrix multiplication $abc := ab^tc$ satisfies

$$(ab^tc)d^te = ab^t(cd^te) = a(dc^tb)^te$$

which is just ab^tcd^te

THE AXIOMS WHICH CHARACTERIZE TRIPLE BRACKET MULTIPLICATION

$$aab = 0$$

$$abc + bca + cab = 0$$

$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

THESE ARE CALLED **LIE TRIPLE SYSTEMS**

(NATHAN JACOBSON, MAX KOECHER)

Example: Triple bracket multiplication $abc := [[a, b], c]$ satisfies

$$[[a, a], b] = 0$$

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$$

$$[[d, e], [[a, b], c]] = [[[[d, e], a], b], c] + [[a, [[d, e], b]], c] + [[a, b], [[d, e], c]]$$

THE AXIOMS WHICH CHARACTERIZE TRIPLE CIRCLE MULTIPLICATION ARE

$$abc = cba$$

$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$

THESE ARE CALLED **JORDAN TRIPLE SYSTEMS**
Example: Triple circle multiplication $abc := \{abc\} : (abTc + cb^t a)/2$ satisfies

$$\{abc\} = \{cba\}$$

$$\{de\{abc\}\} = \{\{dea\}bc\} - \{a\{edb\}c\} + \{ab\{dec\}\}$$

SUMMARY

Table 1 (FASHIONABLE) ALGEBRAS

commutative algebras $ab = ba$
associative algebras $a(bc) = (ab)c$
Lie algebras $a^2 = 0$, $(ab)c + (bc)a + (ca)b = 0$
Jordan algebras $ab = ba$, $a(a^2 b) = a^2(ab)$

Table 2 TRIPLE SYSTEMS

associative triple systems
$(abc)de = ab(cde) = a(dcb)e$
Lie triple systems
$aab = 0$
$abc + bca + cab = 0$
$de(abc) = (dea)bc + a(deb)c + ab(dec)$
Jordan triple systems
$abc = cba$
$de(abc) = (dea)bc - a(edb)c + ab(dec)$

(What is a **commutative** triple system?)

(Part 3 starts here)

Recall that any associative algebra A with product ab can be made into a Lie algebra, denoted by A^-, by defining $[a, b] = ab - ba$ and into a Jordan algebra, denoted by A^+, by defining $a \circ b = (ab + ba)/2$

Ado's Theorem 1947

Every finite dimensional Lie algebra over an algebraically closed field is isomorphic to a subalgebra of A^- for some associative algebra A

Exceptional Jordan algebras

There exist finite dimensional Jordan algebras which cannot be isomorphic to a subalgebra of A^+ for any associative algebra A

Theorem

Every finite dimensional Lie triple system F is isomorphic to a Lie subtriple system of a Lie algebra.

Proof:

Let $(\mathrm{Inder}\, F)$ be the set of all sums of mappings on F of the form $L(a,b)x = [[a,b],x]$, where a, b are fixed elements of F. Let $\mathcal{L}$ be the Lie algebra $(\mathrm{Inder}\, F) \oplus F$ with product

$$[(H_1, x_1), (H_2, x_2)] = ([H_1, H_2] + L(x_1, x_2), H_1 x_2 - H_2 x_1).$$

Exercise

$\mathcal{L}$ is a Lie algebra.

Exercise

Let $\alpha : F \to \mathcal{L}$ be defined by $\alpha(x) = (0, x)$. Then α is a Lie triple isomorphism of F onto a Lie triple subsystem of the Lie algebra $\mathcal{L}$, that is, α is linear and $\alpha([[a,b],c]) = [[\alpha(a), \alpha(b)], \alpha(c)]$.

The TKK construction (Tits-Kantor-Koecher)

Let V be a Jordan triple and let $\mathcal{L}(V)$ be its TKK Lie algebra .
$\mathcal{L}(V) = V \oplus V_0 \oplus V$ and the Lie product is given by

$$[(x,h,y),(u,k,v)] = (hu - kx,\, [h,k] + x \square v - u \square y,\, k^\natural y - h^\natural v).$$

Here, $a \square b$ is the left multiplication operator $x \mapsto \{abx\}$ (also called the box operator), $V_0 = \mathrm{span}\{V \square V\}$ is a Lie subalgebra of $\mathcal{L}(V)$ and for $h = \sum_i a_i \square b_i \in V_0$, the map $h^\natural : V \to V$ is defined by

$$h^\natural = \sum_i b_i \square a_i.$$

Exercise

$\mathfrak{L}(V)$ is a Lie algebra

Example

Let $V = M_2$ be the 2 by 2 matrix algebra. Then $\mathcal{L}(V)$ is isomorphic to the Lie algebra of all 4 by 4 matrices of trace zero.

Application

Every derivation of a finite dimensional semisimple Jordan triple system is inner.

Application of the earlier theorem

Every derivation of a finite dimensional semisimple Lie triple system is inner.

Linking algebra of an associative triple system

Let X denote $M_{p,q}$ and let $x, y, z, \ldots$ denote elements of X. We define

$$XX^t = \{x_1 y_1^t + x_2 y_2^t + \cdots + x_n y_n^t : x_i, y_i \in X, n = 1, 2 \ldots\}$$

and

$$X^t X = \{x_1^t y_1 + x_2^t y_2 + \cdots + x_n^t y_n : x_i, y_i \in X, n = 1, 2 \ldots\}$$

$$XX^t = M_{p,q} M_{q,p} \text{ is a subalgebra of } M_p$$

and

$$X^t X = M_{q,p} M_{p,q} \text{ is a subalgebra of } M_q$$

$$\text{Let } A = \begin{bmatrix} XX^t & X \\ X^t & X^t X \end{bmatrix} \qquad \left(\text{Note: } A \subset \left[\begin{smallmatrix} M_p & M_{p,q} \\ M_{q,p} & M_q \end{smallmatrix}\right] \subset M_{p+q}\right)$$

$$= \left\{ \left[\begin{smallmatrix} \sum_{i=1}^n x_i y_i^t & x \\ y^t & \sum_{j=1}^m z_j^t w_j \end{smallmatrix}\right] : x, y, x_i, y_i, z_j, w_j \in X, n = 1, 2, \ldots, m = 1, 2, \ldots \right\}$$

Exercise

A is an algebra: $a = \begin{pmatrix} \alpha & x \\ y^t & \beta \end{pmatrix}$, $b = \begin{pmatrix} \alpha_1 & x_1 \\ y_1^t & \beta_1 \end{pmatrix}$, $ab = \begin{pmatrix} \alpha\alpha_1 + xy_1^t & \alpha x_1 + x\beta_1 \\ y^t\alpha_1 + \beta y_1^t & y^t x_1 + \beta\beta_1 \end{pmatrix}$

Exercise

$X = M_{1,2}, X^t = M_{2,1}, XX^t = \mathbb{R}, X^tX = M_2$

Example

If $X = M_{1,2}$,

$$A = \begin{pmatrix} M_1 & M_{1,2} \\ M_{2,1} & M_2 \end{pmatrix} = M_3$$

Application

Let $X = M_{p,q}$ and let $D : X \to X$ be a triple matrix derivation of X. If $A = \left(\begin{smallmatrix} XX^t & X \\ X^t & X^tX \end{smallmatrix}\right) \subset \left(\begin{smallmatrix} M_p & M_{p,q} \\ M_{q,p} & M_q \end{smallmatrix}\right) \subset M_{p+q}$, then A is an algebra and the map $\delta : A \to A$ given, for $x, y, x_i, y_i, z_j, w_j \in X$, by

$$\begin{bmatrix} \sum_i x_i y_i^t & x \\ y^t & \sum_j z_j^t w_j \end{bmatrix} \mapsto \begin{bmatrix} \sum_i (x_i (Dy_i)^t + (Dx_i) y_i^t) & Dx \\ (Dy)^t & \sum_j (z_j^t (Dw_j) + (Dz_j)^t w_j) \end{bmatrix}$$

is well defined and a derivation of A, which extends D (when X is embedded in A via $\varphi(x) = \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)$). If $a = \left(\begin{smallmatrix} \alpha & x \\ y^t & \beta \end{smallmatrix}\right) \in A$ then $\delta(a^t) = \delta(a)^t$ where $a^t = \left(\begin{smallmatrix} \alpha^t & y \\ x^t & \beta^t \end{smallmatrix}\right)$.

- δ is well-defined: $\sum_i x_i y_i^t = 0 \Rightarrow \sum_i (Dx_i) y_i^t + x_i (Dy_i)^t) = 0$
- δ is linear: $\delta(a + b) = \delta(a) + \delta(b)$; $\delta(ca) = c\delta(a)$
- $\delta(a^t) = \delta(a)^t$
- $\delta(ab) = a\delta(b) + \delta(a)b$
- $\delta \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & Dx \\ 0 & 0 \end{smallmatrix}\right)$
- $\varphi(xy^t z) = \varphi(x)\varphi(y)^t \varphi(z)$

(Part 4 starts here)

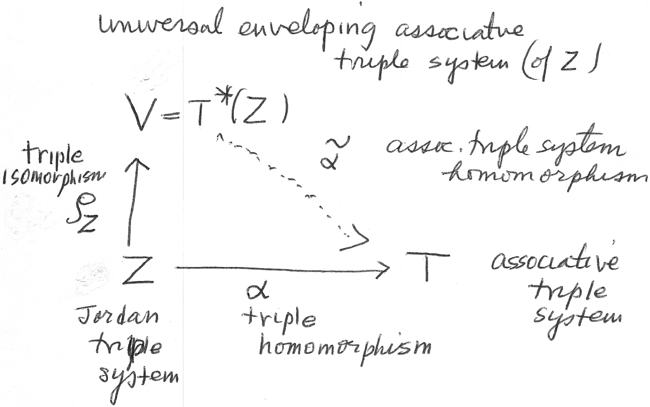
Theorem (Bunce-Feely-Timoney,Bohle-Werner 2012

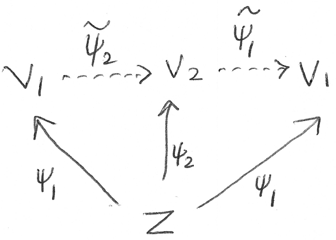
Let Z be a Jordan triple system. Then there exists an associative triple system $V = T^*(Z)$ and a triple homomorphism $\rho_Z : Z \to V$ (V is considered as a Jordan triple system under the triple product $\{abc\} = (ab^*c + cb^*a)/2$) such that if T is any associative triple system (also considered as a Jordan triple system under the triple product $\{abc\} = (ab^*c + cb^*a)/2$) and $\alpha : Z \to T$ is a triple homomorphism, then there is an associative triple system isomorphism $\tilde{\alpha} : V \to T$ such that $\tilde{\alpha} \circ \rho_Z = \alpha$.

And, V is the smallest associative triple system containing $\rho_Z(Z)$.

$\tilde{\alpha}$ is unique

V is unique (up to associative triple isomorphism)


universal enveloping associative triple system (of Z)
V = T*(Z)
triple isomorphism
ρ_Z
α̃
assoc. triple system homomorphism
Z
T
α
triple homomorphism
Jordan triple system
associative triple system



$\varphi := \tilde{\Psi}_1 \circ \tilde{\Psi}_2 : V_1 \to V_1$ is an assoc. triple system homomorphism

<u>claim</u> φ is the identity on V_1

so V_1 is isomorphic to V_2 as associative triple systems

Examples: Cartan Factors

$Z = M_{n,m}(\mathbb{C}),\ m,n \geq 2 \quad , \quad T^*(Z) = M_{n,m}(\mathbb{C}) \oplus M_{m,n}(\mathbb{C})$

$Z = M_{n,1}(\mathbb{C})$ or $M_{1,n}(\mathbb{C}) \quad , \quad T^*(Z) = \oplus_{k=1}^{n} M_{p_k,q_k}(\mathbb{C}),\ p_k = \binom{n}{k}, q_k = \binom{n}{k-1}$

$Z = A_n \subset M_n(\mathbb{C}),\ x^t = -x \quad , \quad T^*(Z) = M_n(\mathbb{C})$

$Z = S_n \subset M_n(\mathbb{C}),\ x^t = x \quad , \quad T^*(Z) = M_n(\mathbb{C})$

Z=spin factor, dimension $2n \quad , \quad T^*(Z) = M_{2^{n-1}}(\mathbb{C}) \oplus M_{2^{n-1}}(\mathbb{C})$
Z=spin factor, dimension $2n+1 \quad , \quad T^*(Z) = M_{2^n}(\mathbb{C})$

spin system: $S = \{I, s_1, \ldots, s_n\} \subset M_m(\mathbb{C}),\ n \geq 2,\ s_i^* = s_i,\ s_i s_j + s_j s_i = 2\delta_{ij}$
spin factor $Z \subset M_m(\mathbb{C})$ is the linear span of S

Applications

1. **K-theory of finite dimensional Jordan triple systems**
• Bohle, Dennis; Werner, Wend—The universal enveloping ternary ring of operators of a JB*-triple system. Proc. Edinb. Math. Soc. (2) 57 (2014), no. 2, 347–366.

There is something called "K-theory" for associative algebras that can be used to classify certain classes of operator algebras (C*-algebras).

Using the linking algebra of an associative triple system, one can obtain a K-theory for associative triple systems.

Using the universal enveloping associative triple system of a Jordan triple system, one can obtain a K-theory for Jordan triple systems, and hence a classification of a certain class of Jordan triple systems.

2. **Structure of infinite dimensional associative triple systems**
• Russo, Bernard—Universal enveloping TROs and Structure of W*-TROs, (preprint 2016)