DERIVATIONS AND PROJECTIONS ON JORDAN TRIPLES

Introduction to non-associative algebra, continuous cohomology, and quantum functional analysis

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PART I—DERIVATIONS

PART II—COHOMOLOGY

PART III—QUANTUM FUNCTIONAL ANALYSIS

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PART I—DERIVATIONS

PART II—COHOMOLOGY

PART III—QUANTUM FUNCTIONAL ANALYSIS

I—ALGEBRAS

Sophus Lie (1842–1899)

Marius Sophus Lie was a Norwegian mathematician. He largely created the theory of continuous symmetry, and applied it to the study of geometry and differential equations.

Pascual Jordan (1902–1980)

Pascual Jordan was a German theoretical and mathematical physicist who made significant contributions to quantum mechanics and quantum field theory.

LET $\mathcal{C}$ DENOTE THE ALGEBRA OF CONTINUOUS FUNCTIONS ON A LOCALLY COMPACT HAUSDORFF SPACE.

DEFINITION 1

A DERIVATION ON $\mathcal{C}$ IS A LINEAR MAPPING $\delta : \mathcal{C} \to \mathcal{C}$ SATISFYING THE "PRODUCT" RULE:

$$\delta(f+g) = \delta(f) + \delta(g)$$

$$\delta(cf) = c\delta(f)$$

$$\delta(fg) = \delta(f)g + f\delta(g)$$

THEOREM 1

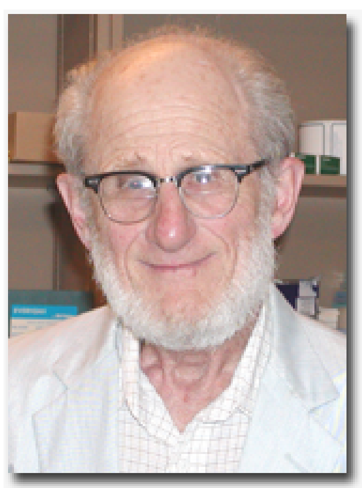
There are no (non-zero) derivations on $\mathcal{C}$.

In other words,

Every derivation of $\mathcal{C}$ is identically zero

THEOREM 1A
(1955-Singer and Wermer)
Every continuous derivation on $\mathcal{C}$ is zero.

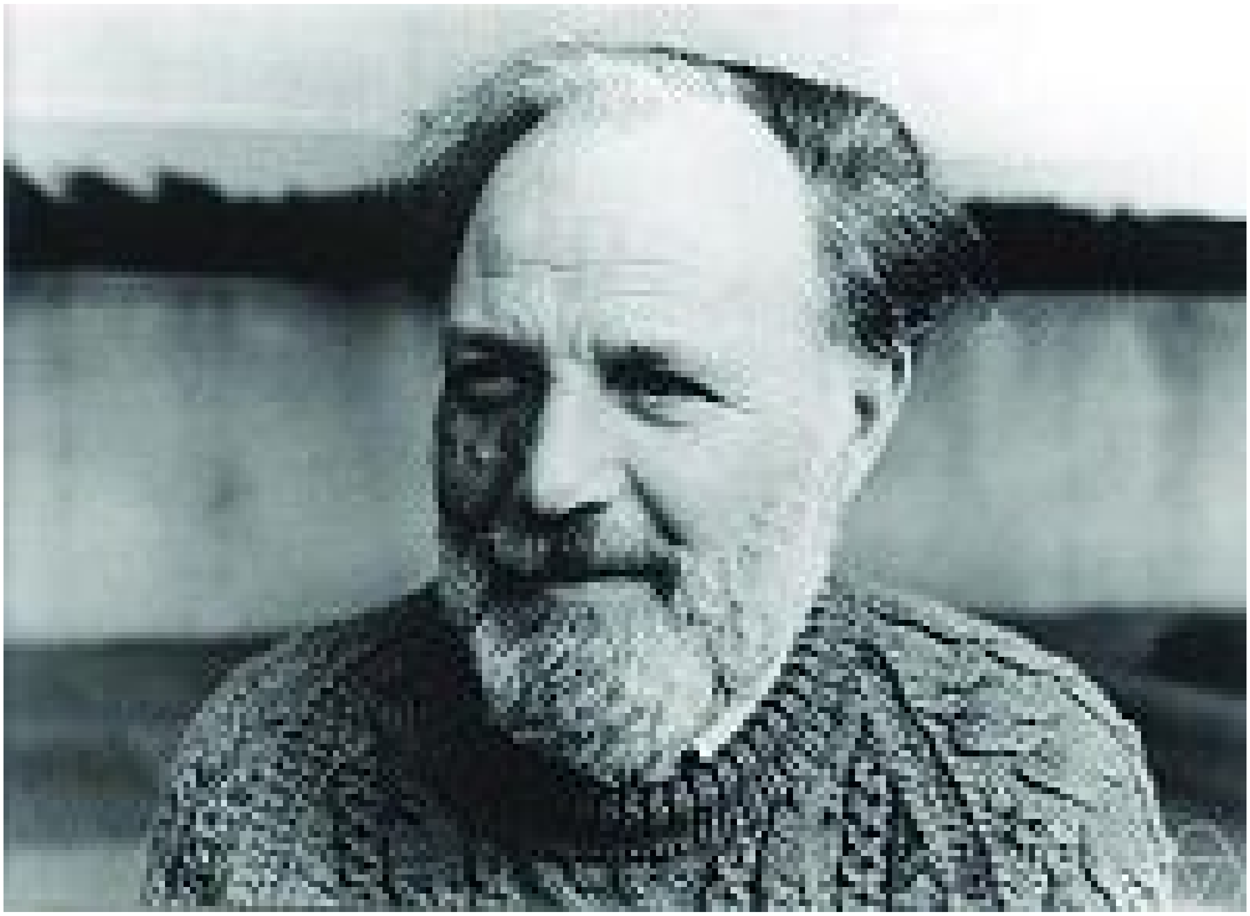
Theorem 1B
(1960-Sakai)
Every derivation on $\mathcal{C}$ is continuous.



John Wermer
(b. 1925)

Soichiro Sakai
(b. 1926)

Isadore Singer (b. 1924)



Isadore Manuel Singer is an Institute Professor in the Department of Mathematics at the Massachusetts Institute of Technology. He is noted for his work with Michael Atiyah in 1962, which paved the way for new interactions between pure mathematics and theoretical physics.

LET $M_n(\mathbf{C})$ DENOTE THE ALGEBRA OF ALL n by n COMPLEX MATRICES, OR MORE GENERALLY, ANY FINITE DIMENSIONAL SEMISIMPLE ASSOCIATIVE ALGEBRA. .

DEFINITION 2

A DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO MATRIX MULTIPLICATION IS A LINEAR MAPPING δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(AB) = \delta(A)B + A\delta(B)$$

.

PROPOSITION 2

FIX A MATRIX A in $M_n(\mathbf{C})$ AND DEFINE

$$\delta_A(X) = AX - XA.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION

THEOREM 2

(1942 Hochschild)

EVERY DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO MATRIX MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{C})$.

Gerhard Hochschild (1915–2010)

(Photo 1968)

Gerhard Paul Hochschild was an American mathematician who worked on Lie groups, algebraic groups, homological algebra and algebraic number theory.

Joseph Henry Maclagan Wedderburn (1882–1948)

Scottish mathematician, who taught at Princeton University for most of his career. A significant algebraist, he proved that a finite division algebra is a field, and part of the Artin–Wedderburn theorem on simple algebras. He also worked on group theory and matrix algebra.

Amalie Emmy Noether (1882–1935)

Amalie Emmy Noether was an influential German mathematician known for her groundbreaking contributions to abstract algebra and theoretical physics. Described as the most important woman in the history of mathematics, she revolutionized the theories of rings, fields, and algebras. In physics, Noether's theorem explains the fundamental connection between symmetry and conservation laws.

RECOMMENDED READING

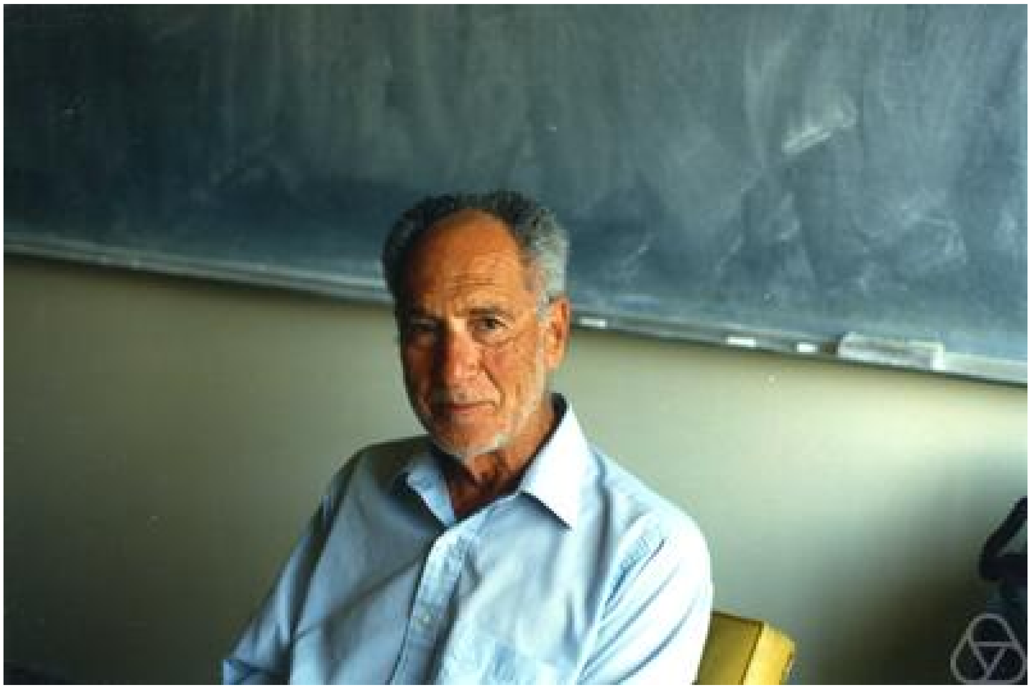
Gerhard Hochschild
A mathematician of the XXth Century

Walter Ferrer Santos

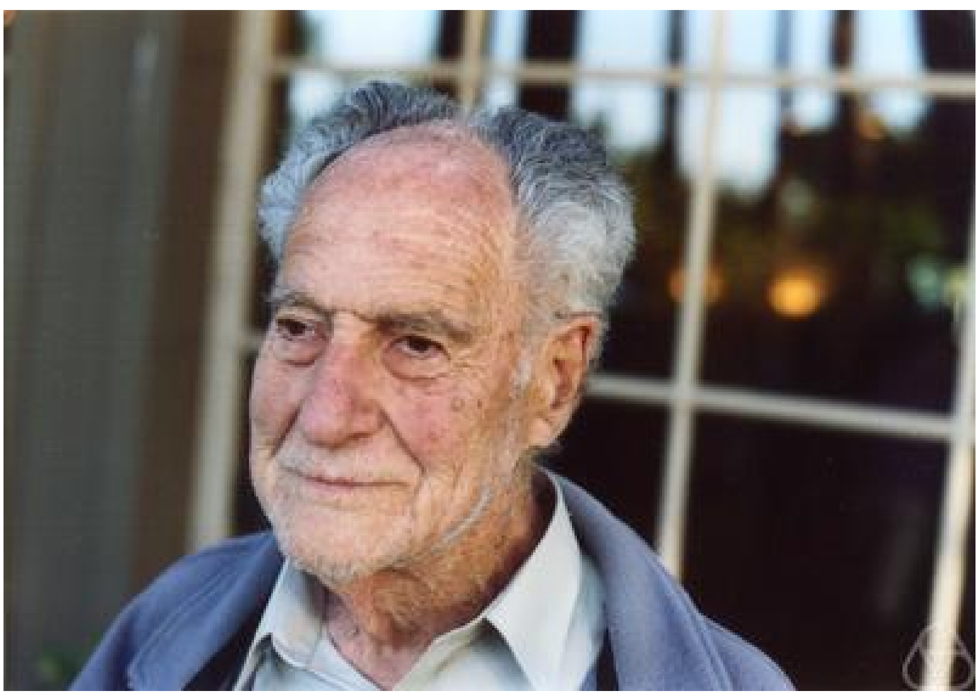
arXiv:1104.0335v1 [math.HO] 2 Apr 2011

(Photo 1976)

Gerhard Hochschild (1915–2010)



(Photo 1986)



(Photo 2003)

DEFINITION 3

A DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO BRACKET MULTIPLICATION

$$[X, Y] = XY - YX$$

IS A LINEAR MAPPING δ WHICH SATISFIES THE PRODUCT RULE

$$\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$$

.

PROPOSITION 3

FIX A MATRIX A in $M_n(\mathbf{C})$ AND DEFINE

$$\delta_A(X) = [A, X] = AX - XA.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION

THEOREM 3

(1942 Hochschild, Zassenhaus)
EVERY DERIVATION ON $M_n(\mathbf{C})^*$ WITH RESPECT TO BRACKET MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{C})$.

Hans Zassenhaus (1912–1991)

Hans Julius Zassenhaus was a German mathematician, known for work in many parts of abstract algebra, and as a pioneer of computer algebra.

*not a semisimple Lie algebra: trace$(X)\, I$ is a derivation which is not inner

DEFINITION 4

A DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO CIRCLE MULTIPLICATION

$$X \circ Y = (XY + YX)/2$$

IS A LINEAR MAPPING δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$

PROPOSITION 4

FIX A MATRIX A in $M_n(\mathbf{C})$ AND DEFINE

$$\delta_A(X) = AX - XA.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION

THEOREM 4

(1972-Sinclair)

EVERY DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{C})$.

REMARK

(1937-Jacobson)

THE ABOVE PROPOSITION AND THEOREM NEED TO BE MODIFIED FOR THE SUBALGEBRA (WITH RESPECT TO CIRCLE MULTIPLICATION) OF HERMITIAN MATRICES.

Alan M. Sinclair (retired)

Nathan Jacobson (1910–1999)

Nathan Jacobson was an American mathematician who was recognized as one of the leading algebraists of his generation, and he was also famous for writing more than a dozen standard monographs.

Table 1

$M_n(\mathbf{C})$ (SEMISIMPLE ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a,b] = ab - ba$	$a \circ b = ab + ba$
Th. 2	Th.3	Th.4
$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$

Table 2

ALGEBRAS

commutative algebras

$$ab = ba$$

associative algebras

$$a(bc) = (ab)c$$

Lie algebras

$$a^2 = 0$$

$$(ab)c + (bc)a + (ca)b = 0$$

Jordan algebras

$$ab = ba$$

$$a(a^2b) = a^2(ab)$$

DERIVATIONS ON C^*-ALGEBRAS

THE ALGEBRA $M_n(\mathbf{C})$, WITH MATRIX MULTIPLICATION, AS WELL AS THE ALGEBRA $\mathcal{C}$, WITH ORDINARY MULTIPLICATION, ARE EXAMPLES OF C^*-ALGEBRAS (FINITE DIMENSIONAL; resp. COMMUTATIVE).

THE FOLLOWING THEOREM THUS EXPLAINS THEOREMS 1 AND 2.

THEOREM (1966-Sakai, Kadison)
EVERY DERIVATION OF A C^*-ALGEBRA IS OF THE FORM $x \mapsto ax - xa$ FOR SOME a IN THE WEAK CLOSURE OF THE C^*-ALGEBRA

Irving Kaplansky (1917–2006)

Kaplansky made major contributions to group theory, ring theory, the theory of operator algebras and field theory.

Richard Kadison (b. 1925)

Richard V. Kadison is an American mathematician known for his contributions to the study of operator algebras.

GRADUS AD PARNASSUM FOR SECTION I—ALGEBRAS

1. Prove Proposition 2: FIX A MATRIX A in $M_n(\mathbf{C})$ AND DEFINE
$$\delta_A(X) = AX - XA.$$
THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION
2. Prove Proposition 3: FIX A MATRIX A in $M_n(\mathbf{C})$ AND DEFINE
$$\delta_A(X) = [A, X] = AX - XA.$$
THEN δ_A IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION
3. Prove Proposition 4: FIX A MATRIX A in $M_n(\mathbf{C})$ AND DEFINE
$$\delta_A(X) = AX - XA.$$
THEN δ_A IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION

4. Let A, B are two fixed matrices in $M_n(\mathbf{C})$. Show that the linear mapping

$$\delta_{A,B}(X) = A \circ (B \circ X) - B \circ (A \circ X)$$

is a derivation of $M_n(\mathbf{C})$ with respect to circle multiplication.
(cf. Remark following Theorem 4)

5. Show that $M_n(\mathbf{C})$ is a Lie algebra with respect to bracket multiplication.
6. Show that $M_n(\mathbf{C})$ is a Jordan algebra with respect to circle multiplication.
7. Let us write $\delta_{a,b}$ for the linear mapping $\delta_{a,b}(x) = a(bx) - b(ax)$ in a Jordan algebra. Show that $\delta_{a,b}$ is a derivation of the Jordan algebra by following the outline below. (cf. problem 4 above.)

(a) In the Jordan algebra axiom

$$u(u^2 v) = u^2(uv),$$

replace u by $u + w$ to obtain the equation

$$2u((uw)v) + w(u^2 v) = 2(uw)(uv) + u^2(wv) \tag{1}$$

(b) In (1), interchange v and w and subtract the resulting equation from (1) to obtain the equation

$$2u(\delta_{v,w}(u)) = \delta_{v,w}(u^2). \qquad (2)$$

(c) In (2), replace u by $x+y$ to obtain the equation

$$\delta_{v,w}(xy) = y\delta_{v,w}(x) + x\delta_{v,w}(y),$$

which is the desired result.

END OF SECTION I

II—AUTOMATIC CONTINUITY

The automatic continuity of various algebraic mappings plays important roles in the general theory of Banach algebras and in particular in operator algebra theory

H. G. Dales, Banach algebras and automatic continuity. London Mathematical Society Monographs. New Series, 24. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 2000.

Digression #1—SURVEY OF

UNIQUENESS OF NORM; AUTOMATIC CONTINUITY OF HOMOMORPHISMS

A Banach algebra $(A, \|\cdot\|)$ has a unique norm if each norm with respect to which A is a normed algebra is equivalent to the given norm.

In this case, the topological and algebraic structures of A are intimately linked.

PROPOSTION A commutative semisimple Banach algebra has a unique norm.

Follows from

THEOREM (Silov 1947) Any homomorphism of a Banach algebra A to a commutative semisimple Banach algebra is continuous.

TWO MILESTONE RESULTS

Johnson 1967 Each semisimple Banach algebra has a unique norm.

Eidelheit 1940 $B(E)$ has a unique norm for any Banach space

(This last result was proved before Banach algebras were invented!)

AN OBSERVATION

PROPOSITION If $(A, \|\cdot\|)$ is a Banach algebra with the property that for any algebra norm $\|\cdot\|_1$ there is a constant C such that $\|a\| \leq C\|a\|_1$, then A has a unique norm if and only if every homomorphism from A to any Banach algebra is continuous.

MORE EXAMPLES

Jewell and Sinclair 1976

$L^1(\mathbf{R}^+, \omega)$ has a unique norm

All finite dimensional Banach algebras have a unique norm

Let A be any linear space with two inequivalent norms, and make it an algebra via $ab = 0$

Let A be a commutative Banach algebra, E a Banach A-module. Then $A \oplus E$ is a commutative algebra via $(a, x)(b, y) = (ab, a \cdot y + b \cdot x)$. If $D : A \to E$ is a derivation, define two norms $\|(a, x)\|_1 = \|a\| + \|x\|$ and $\|(a, x)\|_2 = \|a\| + \|Da - x\|$. These norms are equivalent if and only if D is continuous.

SOME BASIC UNSOLVED PROBLEMS

Johnson's 1967 milestone is a consequence of the following theorem, which is called "the seed from which automatic continuity theory has grown"

THEOREM (Johnson 1967) A homomorphism of a Banach algebra A **ONTO** a semisimple Banach algebra B is continuous.

QUESTION

In Johnson's theorem, can you replace **onto** by **dense range**.

SUB-QUESTION

What if A is a C^*-algebra?

SUB-SUB-QUESTION

What if A and B are both C^*-algebras?

Barry Johnson (1942–2002)

SOME MISCELLANEOUS INTERESTING RESULTS

RODRIGUEZ-PALACIOS 1985
Uniqueness of norm for semisimple nonassociative Banach algebras (Jordan algebras)

ESTERLE 1980 Any homomorphism from $C(\Omega)$ onto a Banach algebra is continuous. (Obvious question: replace $C(\Omega)$ by any C^*-algebra)

SINCLAIR 1974 If a homomorphism from a C^*-algebra to a Banach algebra is continuous when restricted to all singly generated (by self-adjoint elements) subalgebras, then it is continuous.

It follows from elementary spectral theory that *-homomorphisms of C*-algebras are continuous.

Proof:
If $\Phi : A \to B$ satisfies $\Phi(a^*) = \Phi(a)^*$, $\Phi(ab) = \Phi(a)\Phi(b)$, and $\Phi(1) = 1$, then $\sigma(\Phi(a)) \subset \sigma(a)$ and

$$\|\Phi(a)\|^2 = \|\Phi(a)^*\Phi(a)\| = \|\Phi(a^*a)\|$$

$$= \|\Phi(a^*a)\|_{\mathrm{sp}} \leq \|a^*a\|_{\mathrm{sp}} = \|a^*a\| = \|a\|^2$$

END OF SURVEY

BACK TO DERIVATIONS!

AUTOMATIC CONTINUITY OF DERIVATIONS FROM BANACH ALGEBRAS INTO BANACH MODULES

THE AUTOMATIC CONTINUITY PROBLEM FOR DERIVATIONS

Under what conditions on a Banach algebra A are all derivations from A into some or all Banach A-modules automatically continuous?

OBSERVATION

If all homomorphisms from a Banach algebra A into any Banach algebra are continuous, then all derivations from A into any Banach A-module E are continuous.
(The converse is false: $C(\Omega)$, CH, Ringrose)

WHY IS THIS IMPORTANT?

COHOMOLOGY OF BANACH ALGEBRAS

Let M be a Banach algebra and X a Banach M-module.

For $n \geq 1$, let

$L^n(M, X) =$ all continuous n-linear maps

$(L^0(M, X) = X)$

Coboundary operator

$\partial : L^n \to L^{n+1}$ (for $n \geq 1$)

$$\partial\phi(a_1, \cdots, a_{n+1}) = a_1\phi(a_2, \cdots, a_{n+1})$$
$$+\sum(-1)^j\phi(a_1, \cdots, a_{j-1}, a_j a_{j+1}, \cdots, a_{n+1})$$
$$+(-1)^{n+1}\phi(a_1, \cdots, a_n)a_{n+1}$$

For $n = 0$,

$$\partial : X \to L(M, X) \qquad \partial x(a) = ax - xa$$

so

$\text{Im}\partial =$ the space of inner derivations

Since $\partial \circ \partial = 0$,
$\text{Im}(\partial : L^{n-1} \to L^n) \subset \ker(\partial : L^n \to L^{n+1})$

$H^n(M, X) = \ker\partial/\text{Im}\partial$ is a vector space.

For $n = 1$, $\ker\partial =$
$\{\phi : M \to X : a_1\phi(a_2) - \phi(a_1 a_2) + \phi(a_1)a_2 = 0\}$
$=$ the space of continuous derivations from M to X

Thus,

$$H^1(M, X) = \frac{\text{derivations from } M \text{ to } X}{\text{inner derivations from } M \text{ to } X}$$

measures how close continuous derivations are to inner derivations.

(What do $H^2(M, X)$, $H^3(M, X)$,... measure?)

Sneak Preview of section VI

- $H^1(C(\Omega), E) = H^2(C(\Omega), E) = 0$
(Kamowitz 1962 A PIONEER!)
(Question: $H^3(C(\Omega), E) = ?$)

- $H^1(A, B(H)) = 0??$ $(A \subset B(H))$
"The major open question in the theory of derivations on C*-algebras"

- A derivation from A into $B(H)$ is inner if and only if it is completely bounded.
(Christensen 1982)

- Barry Johnson, "**Cohomology of Banach algebras**", Memoirs of the American Mathematical Society 1972

Digression #2—LIE DERIVATIONS

Miers, C. Robert
Lie derivations of von Neumann algebras.
DukeMath. J. 40 (1973), 403–409.

If M is a von Neumann algebra, $[M,M]$ the Lie algebra linearly generated by $\{[X,Y]=XY-YX : X,Y \in M\}$ and $L : [M,M] \to M$ a Lie derivation, i.e., L is linear and $L[X,Y]=[LX,Y]+[X,LY]$, then the author shows that L has an extension $D : M \to M$ that is a derivation of the associative algebra.

The proof involves matrix-like computations.

A theorem of S. Sakai [Ann. of Math. (2) 83 (1966), 273–279] now states that $DX = [A, X]$ with $A \in M$ fixed.

Using this the author finally shows that if $L : M \to M$ is a Lie derivation, then $L = D + \lambda$, where D is an associative derivation and λ is a linear map into the center of M vanishing on $[M, M]$.

For primitive rings with nontrivial idempotent and characteristic $\neq 2$ a slightly weaker result is due to W. S. Martindale, III [Michigan Math. J. 11 (1964), 183187].

Reviewed by Gerhard Janssen

Miers, C. Robert
Lie triple derivations of von Neumann algebras.
Proc. Amer. Math. Soc. 71 (1978), no. 1, 57–61.

Authors summary: A Lie triple derivation of an associative algebra M is a linear map $L : M \to M$ such that

$$L[[X, Y], Z] = [[L(X), Y], Z] + [[X, L(Y)], Z] + [[X, Y], L(Z)]$$

for all $X, Y, Z \in M$.

We show that if M is a von Neumann algebra with no central Abelian summands then there exists an operator $A \in M$ such that $L(X) = [A, X] + \lambda(X)$ where $\lambda : M \to Z_M$ is a linear map which annihilates brackets of operators in M.

Reviewed by Jozsef Szucs

THEOREM

(JOHNSON 1996)

EVERY CONTINUOUS LIE DERIVATION OF A SYMMERTICALLY AMENABLE BANACH ALGEBRA A INTO A BANACH BIMODULE X IS THE SUM OF AN ASSOCIATIVE DERIVATION AND A "TRIVIAL" DERIVATION

(TRIVIAL=ANY LINEAR MAP WHICH VANISHES ON COMMUTATORS AND MAPS INTO THE "CENTER" OF THE MODULE).

The continuity assumption can be dropped if $X = A$ and A is a C*-algebra or a semisimple symmetrically amenable Banach algebra

Mathieu, Martin; Villena, Armando R.
The structure of Lie derivations on C*-algebras.
J. Funct. Anal. 202 (2003), no. 2, 504–525.

Alaminos, J.; Mathieu, M.; Villena, A. R.
Symmetric amenability and Lie derivations.
Math. Proc. Cambridge Philos. Soc. 137 (2004), no. 2, 433–439.

“ It remains an open question whether an analogous result for Lie derivations from A into a Banach A-bimodule holds when A is an arbitrary C*-algebra and when A is an arbitrary symmetrically amenable Banach algebra.”

“It is also an interesting open question whether or not every Lie derivation on a semisimple Banach algebra to itself has this form.”

(H.G. Dales)

END OF DIGRESSION

END OF SECTION II

III—TRIPLE SYSTEMS

DEFINITION 5

A DERIVATION ON $M_{m,n}(\mathbf{C})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION

IS A LINEAR MAPPING δ WHICH SATISFIES THE (TRIPLE) PRODUCT RULE

$$\delta(AB^*C) =$$
$$\delta(A)B^*C + A\delta(B)^*C + AB^*\delta(C)$$

PROPOSITION 5

FOR TWO MATRICES $A \in M_m(\mathbf{C}), B \in \mathrm{M}_n(\mathbf{C})$, WITH $A^* = -A, B^* = -B$,

DEFINE $\delta_{A,B}(X) =$

$$AX + XB$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION

THEOREM 5

EVERY DERIVATION ON $M_{m,n}(\mathbf{C})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

REMARK

THESE RESULTS HOLD TRUE AND ARE OF INTEREST FOR THE CASE $m = n$.

TRIPLE BRACKET MULTIPLICATION

LET'S GO BACK FOR A MOMENT TO SQUARE MATRICES AND THE BRACKET MULTIPLICATION.

MOTIVATED BY THE LAST REMARK, WE DEFINE THE TRIPLE BRACKET MULTIPLICATION TO BE $[[X,Y],Z]$

DEFINITION 6

A DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION

IS A LINEAR MAPPING δ WHICH SATISFIES THE TRIPLE PRODUCT RULE

$$\delta([[A,B],C]) = [[\delta(A),B],C] + [[A,\delta(B)],C] + [[A,B],\delta(C)]$$

PROPOSITION 6

FIX TWO MATRICES A, B IN $M_n(\mathbf{C})$ AND DEFINE $\delta_{A,B}(X) = [[A,B],X]$
THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION.

THEOREM 6

EVERY DERIVATION OF $M_n(\mathbf{C})^\dagger$ WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION IS A SUM OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

†not a semisimple Lie triple system, as in Theorem 3

TRIPLE CIRCLE MULTIPLICATION

LET'S RETURN TO RECTANGULAR MATRICES AND FORM THE TRIPLE CIRCLE MULTIPLICATION

$$(AB^*C + CB^*A)/2$$

For sanity's sake, let us write this as

$$\{A, B, C\} = (AB^*C + CB^*A)/2$$

DEFINITION 7

A <u>DERIVATION</u> ON $M_{m,n}(\mathbf{C})$ WITH <u>RESPECT TO TRIPLE CIRCLE MULTIPLICATION</u>

IS A LINEAR MAPPING δ WHICH SATISFIES THE TRIPLE PRODUCT RULE

$$\delta(\{A,B,C\}) = \{\delta(A), B, C\} + \{A, \delta(B), C\} + \{B, A, \delta(C)\}$$

PROPOSITION 7

FIX TWO MATRICES A, B IN $M_{m,n}(\mathbf{C})$ AND DEFINE

$$\delta_{A,B}(X) = \{A, B, X\} - \{A, B, X\}$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION.

THEOREM 7

EVERY DERIVATION OF $M_{m,n}(\mathbf{C})$ WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

Table 3[‡]

$M_{m,n}(\mathbf{C})$ (SS TRIPLE SYSTEMS)

triple matrix	triple bracket	triple circle
ab^*c	$[[a,b],c]$	$ab^*c + cb^*a$
Th. 5	Th.6	Th.7
$\delta_{a,b}(x)$ $=$ ab^*x $+xb^*a$ $-ba^*x$ $-xa^*b$	$\delta_{a,b}(x)$ $=$ abx $+xba$ $-bax$ $-xab$	$\delta_{a,b}(x)$ $=$ ab^*x $+xb^*a$ $-ba^*x$ $-xa^*b$
(sums)	(sums) $(m = n)$	(sums)

[‡]Note: for triple matrix and triple circle multiplication,

$$(ab^* - ba^*)^* = -(ab^* - ba^*)$$

and

$$(b^*a - a^*b)^* = -(b^*a - a^*b)$$

Table 1

$M_n(\mathrm{C})$ (SS ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a,b] = ab - ba$	$a \circ b = ab + ba$
Th. 2	Th.3	Th.4
$\delta_a(x)$ $=$ $ax - xa$	$\delta_a(x)$ $=$ $ax - xa$	$\delta_a(x)$ $=$ $ax - xa$

Table 3

$M_{m,n}(\mathrm{C})$ (SS TRIPLE SYSTEMS)

triple matrix	triple bracket	triple circle
ab^*c	$[[a,b],c]$	$ab^*c + cb^*a$
Th. 5	Th.6	Th.7
$\delta_{a,b}(x)$ $=$ ab^*x $+xb^*a$ $-ba^*x$ $-xa^*b$	$\delta_{a,b}(x)$ $=$ abx $+xba$ $-bax$ $-xab$	$\delta_{a,b}(x)$ $=$ ab^*x $+xb^*a$ $-ba^*x$ $-xa^*b$
(sums)	(sums) $(m = n)$	(sums)

AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

AN TRIPLE SYSTEM IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH ONE BINARY OPERATION, CALLED ADDITION AND ONE TERNARY OPERATION CALLED TRIPLE MULTIPLICATION

ADDITION IS DENOTED BY

$$a + b$$

AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE

$$a + b = b + a, \quad (a + b) + c = a + (b + c)$$

TRIPLE MULTIPLICATION IS DENOTED

$$abc$$

AND IS REQUIRED TO BE LINEAR IN EACH VARIABLE

$$(a + b)cd = acd + bcd$$
$$a(b + c)d = abd + acd$$
$$ab(c + d) = abc + abd$$

IMPORTANT BUT SIMPLE EXAMPLES OF TRIPLE SYSTEMS CAN BE FORMED FROM ANY ALGEBRA

IF ab DENOTES THE ALGEBRA PRODUCT, JUST DEFINE A TRIPLE MULTIPLICATION TO BE $(ab)c$

LET'S SEE HOW THIS WORKS IN THE ALGEBRAS WE INTRODUCED IN SECTION I

$\mathcal{C}$; $fgh = (fg)h$, OR $fgh = (f\overline{g})h$

$(M_n(\mathbf{C}), \times)$; $abc = abc$ OR $abc = ab^*c$

$(M_n(\mathbf{C}), [,])$; $abc = [[a, b], c]$

$(M_n(\mathbf{C}), \circ)$; $abc = (a \circ b) \circ c$

A TRIPLE SYSTEM IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)

(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

IN THE TRIPLE CONTEXT THIS MEANS THE FOLLOWING

ASSOCIATIVE

$$ab(cde) = (abc)de = a(bcd)e$$

OR $ab(cde) = (abc)de = a(dcb)e$

COMMUTATIVE: $abc = cba$

AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

THE AXIOM WHICH CHARACTERIZES TRIPLE MATRIX MULTIPLICATION IS

$$(abc)de = ab(cde) = a(dcb)e$$

THESE ARE CALLED

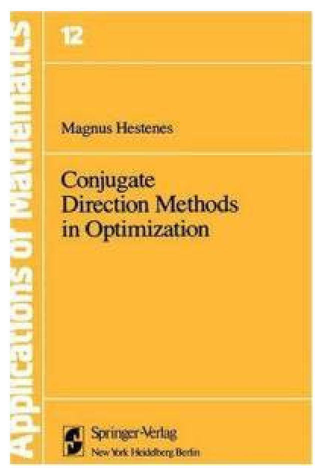
ASSOCIATIVE TRIPLE SYSTEMS

or

HESTENES ALGEBRAS

Magnus Hestenes (1906–1991)

Magnus Rudolph Hestenes was an American mathematician. Together with Cornelius Lanczos and Eduard Stiefel, he invented the conjugate gradient method.



THE AXIOMS WHICH CHARACTERIZE TRIPLE BRACKET MULTIPLICATION ARE

$$aab = 0$$

$$abc + bca + cab = 0$$

$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

THESE ARE CALLED
LIE TRIPLE SYSTEMS

(NATHAN JACOBSON, MAX KOECHER)

Max Koecher (1924–1990)

Max Koecher was a German mathematician. His main research area was the theory of Jordan algebras, where he introduced the KantorKoecherTits construction.

Nathan Jacobson (1910–1999)

THE AXIOMS WHICH CHARACTERIZE TRIPLE CIRCLE MULTIPLICATION ARE

$$abc = cba$$

$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$

THESE ARE CALLED
JORDAN TRIPLE SYSTEMS

Kurt Meyberg

Ottmar Loos + Erhard Neher

Table 4

TRIPLE SYSTEMS

associative triple systems

$$(abc)de = ab(cde) = a(dcb)e$$

Lie triple systems

$$aab = 0$$

$$abc + bca + cab = 0$$

$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

Jordan triple systems

$$abc = cba$$

$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$

THE PHYSICAL UNIVERSE SEEMS TO BE ASSOCIATIVE.

RIGHT?

WRONG!

THEOREM

(1985 FRIEDMAN-RUSSO)

THE RANGE OF A CONTRACTIVE PROJECTION ON A C^*-ALGEBRA (ASSOCIATIVE) IS A JB^*-TRIPLE (NON-ASSOCIATIVE).

Digression #3—PROJECTIVE STABILITY

A well-known and useful result in the structure theory of operator triple systems is the "contractive projection principle," that is, the fact that the range of a contractive projection on a JB^*-triple is linearly isometric in a natural way to another JB^*-triple (Kaup, Stacho, Friedman-Russo).

Thus, the category of JB^*-triples and contractions is stable under contractive projections.

To put this result in proper prospective, let $\mathcal{B}$ be the category of Banach spaces and contractions.

We say that a sub-category $\mathcal{S}$ of $\mathcal{B}$ is **projectively stable** if it has the property that whenever A is an object of $\mathcal{S}$ and X is the range of a morphism of $\mathcal{S}$ on A which is a projection, then X is isometric (that is, isomorphic in $\mathcal{S}$) to an object in $\mathcal{S}$.

Examples of projectively stable categories

- L_1, contractions (Grothendieck 1955)
- L^p, $1 \leq p < \infty$, contractions (Douglas 1965, Ando 1966, Bernau-Lacey 1974, Tzafriri 1969)
- C^*-algebras, completely positive unital maps (Choi-Effros 1977)
- ℓ_p, $1 \leq p < \infty$, contractions (Lindenstrauss-Tzafriri 1978)
- JC^*-algebras, positive unital maps (Effros-Stormer 1979)
- $TROs$ (ternary rings of operators), complete contractions (Youngson 1983)
- JB^*-triples, contractions (Kaup, Stacho 1984, Friedman-Russo 1985)
- ℓ^p-direct sums of $C_p(H)$, $1 \leq p < \infty$, H Hilbert space, contractions (Arazy-Friedman, 1978,2000)
- ℓ^p-direct sums of $L^p(\Omega, H)$, $1 \leq p < \infty$, H Hilbert space, contractions (Raynaud 2004)
- ℓ^p-direct sums of $C_p(H)$, $1 \leq p \neq 2 < \infty$, H Hilbert space, complete contractions (LeMerdy-Ricard-Roydor 2009)

It follows immediately that if $\mathcal{S}$ is projectively stable, then so is the category $\mathcal{S}_*$ of spaces whose dual spaces belong to $\mathcal{S}$.

It should be noted that $TROs$, C^*-algebras and JC^*-algebras are not stable under contractive projections and JB^*-triples are not stable under bounded projections.

More about JB*-triples

JB*-triples are generalizations of JB*-algebras and C*-algebras. The axioms can be said to come geometry in view of Kaup's Riemann mapping theorem.

Kaup showed in 1983 that JB*-triples are exactly those Banach spaces whose open unit ball is a bounded symmetric domain.

Kaup's holomorphic characterization of JB*-triples directly led to the proof of the projective stability of JB*-triples mentioned above.

Many authors have studied the interplay between JB*-triples and infinite dimensional holomorphy.

Contractive projections have proved to be a valuable tool for the study of problems on JB*-triples (Gelfand-Naimark theorem, structure of inner ideals, operator space characterization of TROs, to name a few)

They are justified both as a natural generalization of operator algebras as well as because of their connections with complex geometry.

. . .

Preduals of JBW*-triples have been called pre-symmetric spaces and have been proposed as mathematical models of physical systems.

In this model the operations on the physical system are represented by contractive projections on the pre-symmetric space.

JB*-triples first arose in Koecher's proof of the classification of bounded symmetric domains in $\mathbf{C}^n$.

The original proof of this fact, done in the 1930's by Cartan, used Lie algebras and Lie groups, techniques which do not extend to infinite dimensions.

On the other hand, to a large extent, the Jordan algebra techniques do so extend, as shown by Kaup and Upmeier.[§]

[§]The opposite is true concerning cohomology. Lie algebra cohomology is well developed, Jordan algebra cohomology is not

APPLICATION

GELFAND NAIMARK THEOREM (FRIEDMAN-RUSSO 1986)

Every JB*-triple is isometically isomorphic to a subtriple of a direct sum of Cartan factors.

The theorem was not unexpected. However, the proof required new techniques because of the lack of an order structure on a JB*-triple.

Step 1: February 1983 Friedman-Russo

Let $P : A \to A$ be a linear projection of norm 1 on a JC*-triple A. Then $P(A)$ is a JB*-triple under $\{xyz\}_{P(A)} = P(\{xyz\})$ for $x, y, z \in P(A)$.

Step 2: April 1983 Friedman-Russo

Same hypotheses. Then P is a conditional expectation in the sense that

$$P\{PaPbPc\} = P\{Pa, b, Pc\}$$

and

$$P\{PaPbPc\} = P\{aPbPc\}.$$

Step 3: May 1983 Kaup

Let $P : U \to U$ be a linear projection of norm 1 on a JB*-triple U. Then $P(U)$ is a JB*-triple under $\{xyz\}_{P(U)} = P(\{xyz\}_U)$ for $x, y, z \in P(U)$.

Also, $P\{PaPbPc\} = P\{Pa, b, Pc\}$ for $a, b, c \in U$, which extends one of the formulas in the previous step.

Step 4: February 1984 Friedman-Russo

Every JBW*-triple splits into atomic and purely non-atomic ideals.

Step 5: August 1984 Dineen

The bidual of a JB*-triple is a JB*-triple.

Step 6: October 1984 Barton-Timoney

The bidual of a JB*-triple is a JBW*-triple, that is, the triple product is separately weak*-continuous.

Step 7: December 1984 Horn

Every JBW*-triple factor of type I is isomorphic to a Cartan factor.

More generally, every JBW*-triple of type I is isomorphic to a direct sum of L^∞ spaces with values in a Cartan factor.

Step 8: March 1985 Friedman-Russo

Putting it all together

$$\pi : U \to U^{**} = A \oplus N = (\oplus_\alpha C_\alpha) \oplus N = \sigma(U^{**}) \oplus N$$

implies that $\sigma \circ \pi : U \to A = \oplus_\alpha C_\alpha$ is an isometric isomorphism.

Consequences of the Gelfand-Naimark theorem

- Every JB*-triple is isomorphic to a subtriple of a JB*-algebra.
- In every JB*-triple, $\|\{xyz\}\| \leq \|x\|\|y\|\|z\|$
- Every JB*-triple contains a unique norm-closed ideal J such that U/J is isomorphic to a JC*-triple and J is purely exceptional, that is, every homomorphism of J into a C*-algebra is zero.

Digression #4—Structural Projections; Structure of inner ideals

Edwards,Ruttiman,Hugli

AN OBSERVATION ABOUT PROJECTIVE STABILITY FOR JBW*-TRIPLES

Preservation of type (Chu-Neal-Russo)

Simultaneously: Bunce-Peralta

Theorem 1:
Let P be a normal contractive projection on a JBW^*-triple Z of type I. Then $P(Z)$ is of type I.

Theorem 2:
Let P be a normal contractive projection on a semifinite JBW^*-triple Z. Then $P(Z)$ is a semifinite JW^*-triple.

BEING MATHEMATICIANS, WE NATURALLY WONDERED ABOUT A CONVERSE:

THEOREM
(2008 NEAL-RUSSO)

A JB*-SUBTRIPLE OF A C*-ALGEBRA IS THE RANGE OF A CONTRACTIVE PROJECTION ON THE C*-ALGEBRA.

Matthew Neal (b. 1972)

Digression #5—PROJECTIVE RIGIDITY

By considering the converse of projective stability, one is lead to the following definition.

A sub-category $\mathcal{S}$ of $\mathcal{B}$ is **projectively rigid** if it has the property that whenever A is an object of $\mathcal{S}$ and X is a subspace of A which is isometric to an object in $\mathcal{S}$, then X is the range of a morphism of $\mathcal{S}$ on A which is a projection.

Examples of projectively rigid categories,

- ℓ_p, $1 < p < \infty$, contractions (Pelczynski 1960)
- L^p, $1 \leq p < \infty$, contractions (Douglas 1965, Ando 1966, Bernau-Lacey 1974)
- $C_p, 1 \leq p < \infty$, contractions (Arazy-Friedman 1977)
- Preduals of von Neumann algebras, contractions (Kirchberg 1993)
- Preduals of $TROs$, complete contractions (Ng-Ozawa 2002)
- Preduals of JBW*-triples, contractions* (Neal-Russo 2008/2011)
- ℓ^p-direct sums of $C_p(H)$, $1 \leq p \neq 2 < \infty$, H Hilbert space, complete contractions (LeMerdy-Ricard-Roydor 2009)

(*with one exception—see the next page)

THEOREM
(Neal-Russo)

The category of preduals of JBW^*-triples with no summands of the form $L^1(\Omega, H)$ where H is a Hilbert space of dimension at least two, is projectively rigid.

The result of Ng and Ozawa fails in the category of operator spaces with complete contractions.

GRADUS AD PARNASSUM
FOR SECTION III—TRIPLE SYSTEMS

1. Prove Proposition 5: FOR TWO MATRICES $A \in M_m(\mathbf{C}), B \in$M$_n(\mathbf{C})$, WITH $A^* = -A, B^* = -B$, DEFINE $\delta_{A,B}(X) = AX + XB$. THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION. (Use the notation $\langle abc\rangle$ for ab^*c)
2. Prove Proposition 6: FIX TWO MATRICES A, B IN $M_n(\mathbf{C})$ AND DEFINE $\delta_{A,B}(X) = [[A, B], X]$. THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION. (Use the notation $[abc]$ for $[[a, b], c]$)
3. Prove Proposition 7: FIX TWO MATRICES A, B IN $M_{m,n}(\mathbf{C})$ AND DEFINE $\delta_{A,B}(X) = \{A, B, X\} - \{A, B, X\}$. THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION. (Use the notation $\{abc\}$ for $ab^*c + cb^*a$)

4. Show that $M_n(\mathbf{R})$ is a Lie triple system with respect to triple bracket multiplication. In other words, show that the three axioms for Lie triple systems in Table 4 are satisfied if abc denotes $[[a,b],c] = (ab-ba)c-c(ab-ba)$ (a, b and c denote matrices). (Use the notation $[abc]$ for $[[a,b],c]$)
5. Show that $M_{m,n}(\mathbf{R})$ is a Jordan triple system with respect to triple circle multiplication. In other words, show that the two axioms for Jordan triple systems in Table 4 are satisfied if abc denotes $ab^*c + cb^*a$ (a, b and c denote matrices). (Use the notation $\{abc\}$ for $ab^*c + cb^*a$)
6. Let us write $\delta_{a,b}$ for the linear process

$$\delta_{a,b}(x) = abx$$

in a Lie triple system. Show that $\delta_{a,b}$ is a derivation of the Lie triple system by using the axioms for Lie triple systems in Table 4. (Use the notation $[abc]$ for the triple product in any Lie triple system, so that, for example, $\delta_{a,b}(x)$ is denoted by $[abx]$)

7. Let us write $\delta_{a,b}$ for the linear process

$$\delta_{a,b}(x) = abx - bax$$

in a Jordan triple system. Show that $\delta_{a,b}$ is a derivation of the Jordan triple system by using the axioms for Jordan triple systems in Table 4. (Use the notation $\{abc\}$ for the triple product in any Jordan triple system, so that, for example, $\delta_{a,b}(x) = \{abx\} - \{bax\}$)

8. On the Jordan algebra $M_n(\mathbf{R})$ with the circle product $a \circ b = ab + ba$, define a triple product

$$\{abc\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b.$$

Show that $M_n(\mathbf{R})$ is a Jordan triple system with this triple product.
Hint: show that $\{abc\} = 2abc + 2cba$

9. On the vector space $M_n(\mathbf{R})$, define a triple product $\langle abc \rangle = abc$ (matrix multiplication without the adjoint in the middle). Formulate the definition of a derivation of the resulting triple system, and state and prove a result corresponding to Proposition 5. Is this triple system associative?
10. In an associative algebra, define a triple product $\langle abc \rangle$ to be abc. Show that the algebra becomes an associative triple system with this triple product.
11. In an associative triple system with triple product denoted $\langle abc \rangle$, define a binary product ab to be $\langle aub \rangle$, where u is a fixed element. Show that the triple system becomes an associative algebra with this product. Suppose further that $\langle auu \rangle = a$ for all a. Show that we get a unital involutive algebra with involution $a^\sharp = \langle uau \rangle$.

12. In a Lie algebra with product denoted by $[a, b]$, define a triple product $[abc]$ to be $[[a, b], c]$. Show that the Lie algebra becomes a Lie triple system with this triple product.
(Meyberg Lectures, chapter 6, example 1, page 43)
13. Let A be an algebra (associative, Lie, or Jordan; it doesn't matter). Show that the set $\mathcal{D} := \text{Der}\,(A)$ of all derivations of A is a Lie subalgebra of $\text{End}\,(A)$. That is, $\mathcal{D}$ is a linear subspace of the vector space of linear transformations on A, and if $D_1, D_2 \in \mathcal{D}$, then $D_1D_2 - D_2D_1 \in \mathcal{D}$.
14. Let A be a triple system (associative, Lie, or Jordan; it doesn't matter). Show that the set $\mathcal{D} := \text{Der}\,(A)$ of derivations of A is a Lie subalgebra of $\text{End}\,(A)$. That is, $\mathcal{D}$ is a linear subspace of the vector space of linear transformations on A, and if $D_1, D_2 \in \mathcal{D}$, then $D_1D_2 - D_2D_1 \in \mathcal{D}$.

GRADUS AD PARNASSUM ADDITIONAL PROBLEMS FOR SECTIONS I AND III (ALGEBRAS AND TRIPLE SYSTEMS)

1. In an arbitrary Jordan triple system, with triple product denoted by $\{abc\}$, define a triple product by

$$[abc] = \{abc\} - \{bac\}.$$

Show that the Jordan triple system becomes a Lie triple system with this new triple product.
(Meyberg Lectures, chapter 11, Theorem 1, page 108)

2. In an arbitrary associative triple system, with triple product denoted by $\langle abc \rangle$, define a triple product by

$$[xyz] = \langle xyz \rangle - \langle yxz \rangle - \langle zxy \rangle + \langle zyx \rangle.$$

Show that the associative triple system becomes a Lie triple system with this new triple product.

(Meyberg Lectures, chapter 6, example 3, page 43)

3. In an arbitrary Jordan algebra, with product denoted by xy, define a triple product by $[xyz] = x(yz) - y(xz)$. Show that the Jordan algebra becomes a Lie triple system with this new triple product.
   (Meyberg Lectures, chapter 6, example 4, page 43)
4. In an arbitrary Jordan triple system, with triple product denoted by $\{abc\}$, fix an element y and define a binary product by

$$ab = \{ayb\}.$$

   Show that the Jordan triple system becomes a Jordan algebra with this (binary) product.
   (Meyberg Lectures, chapter 10, Theorem 1, page 94—using different language; see also, Harald Upmeier, Symmetric Banach Manifolds and Jordan C*-algebras, 1985, Proposition 19.7, page 317))

5. In an arbitrary Jordan algebra with multiplication denoted by ab, define a triple product

$$\{abc\} = (ab)c + (cb)a - (ac)b.$$

Show that the Jordan algebra becomes a Jordan triple system with this triple product. (cf. Problem 15)
(Meyberg Lectures, chapter 10, page 93—using different language; see also, Harald Upmeier, Symmetric Banach Manifolds and Jordan C*-algebras, 1985, Corollary 19.10, page 320)
6. Show that every Lie triple system, with triple product denoted $[abc]$ is a subspace of some Lie algebra, with product denoted $[a, b]$, such that $[abc] = [[a, b], c]$.
(Meyberg Lectures, chapter 6, Theorem 1, page 45)
7. Find out what a semisimple associative algebra is and prove that every derivation of a finite dimensional semisimple associative algebra is inner, that is, of the form $x \mapsto ax - xa$ for some fixed a in the algebra.

(G. Hochschild, Semisimple algebras and generalized derivations, Amer. J. Math. 64, 1942, 677–694, Theorem 2.2)

8. Find out what a semisimple Lie algebra is and prove that every derivation of a finite dimensional semisimple Lie algebra is inner, that is, of the form $x \mapsto [a, x]$ for some fixed a in the algebra.

   (Meyberg Lectures, chapter 5, Theorem 2, page 42; see also G. Hochschild, Semisimple algebras and generalized derivations, Amer. J. Math. 64, 1942, 677–694, Theorem 2.1)

9. Find out what a semisimple Jordan algebra is and prove that every derivation of a finite dimensional semisimple Jordan algebra is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}(a_i(b_i x) - b_i(a_i x))$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the algebra.

   (N. Jacobson, Structure of Jordan algebras, 1968, around page 320 and Braun-Koecher, Jordan Algebren, around page 270)

10. In an associative triple system with triple product $\langle xyz \rangle$, show that you get a Jordan triple system with the triple product $\{xyz\} = \langle xyz \rangle + \langle zyx \rangle$. Then use Theorem 7 to prove Theorem 5.

    THEOREM 7
    EVERY DERIVATION OF $M_{m,n}(\mathbf{C})$ WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

    THEOREM 5
    EVERY DERIVATION ON $M_{m,n}(\mathbf{C})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

11. Find out what a semisimple associative triple system is and prove that every derivation of a finite dimensional semisimple associative triple system is inner (also find out what inner means in this context).
    (R. Carlsson, Cohomology for associative triple systems, P.A.M.S. 1976, 1–7)

12. Find out what a semisimple Lie triple system is and prove that every derivation of a finite dimensional semisimple Lie triple system is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}[a_i b_i x]$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the Lie triple system.
(Meyberg Lectures, chapter 6, Theorem 10, page 57)
13. Find out what a semisimple Jordan triple system is and prove that every derivation of a finite dimensional semisimple Jordan triple system is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}(\{a_i b_i x\} - \{b_i a_i x\})$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the Jordan triple system.
(Meyberg Lectures, chapter 11, Theorem 8, page 123 and Corollary 2, page 124)

END OF SECTION III

IV—WEAK AMENABILITY

TWO BASIC QUESTIONS ON DERIVATIONS OF BANACH ALGEBRAS AND TRIPLES

$A \to A$ and $A \to M$ (MODULE)

- AUTOMATIC CONTINUITY?
- INNER?

CONTEXTS

(i) C*-ALGEBRAS
(associative Banach algebras)

(ii) JC*-ALGEBRAS
(Jordan Banach algebras)

(iii) JC*-TRIPLES
(Banach Jordan triples)

Could also consider:
(ii') Banach Lie algebras
(iii')Banach Lie triple systems
(i')Banach associative triple systems

(i) C*-ALGEBRAS

derivation: $D(ab) = a \cdot Db + Da \cdot b$

inner derivation: ad x$(a) = x \cdot a - a \cdot x$ $(x \in M)$

- AUTOMATIC CONTINUITY RESULTS

  KAPLANSKY 1949: $C(X)$

  SAKAI 1960

  RINGROSE 1972: (module)

- INNER DERIVATION RESULTS

  SAKAI, KADISON 1966

  CONNES 1976 (module)

  HAAGERUP 1983 (module)

THEOREM (Sakai 1960)
Every derivation from a C*-algebra into itself is continuous.

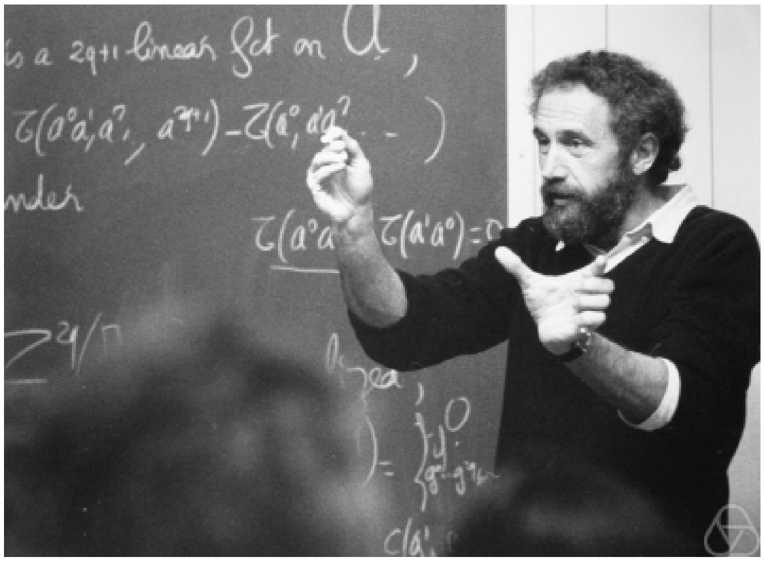
THEOREM (Ringrose 1972)
Every derivation from a C*-algebra into a Banach A-**bimodule** is continuous.

THEOREM (1966-Sakai, Kadison)
EVERY DERIVATION OF A C^*-ALGEBRA IS OF THE FORM $x \mapsto ax - xa$ FOR SOME a IN THE WEAK CLOSURE OF THE C^*-ALGEBRA

John Ringrose (b. 1932)

John Ringrose is a leading world expert on non-self-adjoint operators and operator algebras. He has written a number of influential texts including Compact non-self-adjoint operators (1971) and, with R V Kadison, Fundamentals of the theory of operator algebras in four volumes published in 1983, 1986, 1991 and 1992.

THEOREM (1976-Connes)
EVERY AMENABLE C^*-ALGEBRA IS NUCLEAR.



Alain Connes b. 1947

Alain Connes is the leading specialist on operator algebras.

In his early work on von Neumann algebras in the 1970s, he succeeded in obtaining the almost complete classification of injective factors.

Following this he made contributions in operator K-theory and index theory, which culminated in the Baum-Connes conjecture.

He also introduced cyclic cohomology in the early 1980s as a first step in the study of noncommutative differential geometry.

Connes has applied his work in areas of mathematics and theoretical physics, including number theory, differential geometry and particle physics.

THEOREM (1983-Haagerup)
EVERY NUCLEAR C^*-ALGEBRA IS AMENABLE.

THEOREM (1983-Haagerup)
EVERY C^*-ALGEBRA IS WEAKLY AMENABLE.

Uffe Haagerup b. 1950

Haagerup's research is in operator theory, and covers many subareas in the subject which are currently very active - random matrices, free probability, C*-algebras and applications to mathematical physics.

DIGRESSION #6
A BRIDGE TO JORDAN ALGEBRAS

A *Jordan derivation* from a Banach algebra A into a Banach A-module is a linear map D satisfying $D(a^2) = aD(a) + D(a)a$, ($a \in A$), or equivalently,

$$D(ab + ba) = aD(b) + D(b)a + D(a)b + bD(a),$$
$$(a, b \in A).$$

Sinclair proved in 1970 that a bounded Jordan derivation from a semisimple Banach algebra to itself is a derivation, although this result fails for derivations of semisimple Banach algebras into a Banach bi-module.

Nevertheless, a celebrated result of B.E. Johnson in 1996 states that every bounded Jordan derivation from a C*-algebra A to a Banach A-bimodule is an associative derivation.

In view of the intense interest in automatic continuity problems in the past half century, it is therefore somewhat surprising that the following problem has remained open for fifteen years.

PROBLEM

Is every Jordan derivation from a C*-algebra A to a Banach A-bimodule automatically continuous (and hence a derivation, by Johnson's theorem)?

In 2004, J. Alaminos, M. Brešar and A.R. Villena gave a positive answer to the above problem for some classes of C*-algebras including the class of abelian C*-algebras

Combining a theorem of Cuntz from 1976 with the theorem just quoted yields

THEOREM

Every Jordan derivation from a C*-algebra A to a Banach A-module is continuous.

In the same way, using the solution in 1996 by Hejazian-Niknam in the commutative case we have

THEOREM

Every Jordan derivation from a C*-algebra A to a *Jordan* Banach A-module is continuous.

(Jordan module will be defined below)

These two results will also be among the consequences of our results on automatic continuity of derivations into Jordan triple modules.

(END OF DIGRESSION)

(ii) JC*-ALGEBRA

derivation: $D(a \circ b) = a \circ Db + Da \circ b$

inner derivation: $\sum_i [L(x_i)L(a_i) - L(a_i)L(x_i)]$

$$(x_i \in M, a_i \in A)$$

$$b \mapsto \sum_i [x_i \circ (a_i \circ b) - a_i \circ (x_i \circ b)]$$

- AUTOMATIC CONTINUITY RESULTS

  UPMEIER 1980

  HEJAZIAN-NIKNAM 1996 (module)

  ALAMINOS-BRESAR-VILLENA 2004 (module)

- INNER DERIVATION RESULTS

  JACOBSON 1951 (module)

  UPMEIER 1980

THEOREM (1951-Jacobson)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN ALGEBRA INTO A (JORDAN) **MODULE** IS INNER
(Lie algebras, Lie triple systems)

THEOREM (1980-Upmeier)
EVERY DERIVATION OF A REVERSIBLE JC*-ALGEBRA EXTENDS TO A DERIVATION OF ITS ENVELOPING C*-ALGEBRA. (IMPLIES SINCLAIR)

THEOREM (1980-Upmeier)

1. Purely exceptional JBW-algebras have the inner derivation property
2. Reversible JBW-algebras have the inner derivation property
3. $\oplus L^\infty(S_j, U_j)$ has the inner derivation property if and only if $\sup_j \dim U_j < \infty$, U_j spin factors.

Nathan Jacobson (1910-1999)

Harald Upmeier (b. 1950)

(iii) JC*-TRIPLE

derivation:

$$D\{a,b,c\} = \{Da.b,c\} + \{a,Db,c\} + \{a,b,Dc\}$$

$$\{x,y,z\} = (xy^*z + zy^*x)/2$$

inner derivation: $\sum_i [L(x_i,a_i) - L(a_i,x_i)]$

$$(x_i \in M, a_i \in A)$$

$$b \mapsto \sum_i [\{x_i,a_i,b\} - \{a_i,x_i,b\}]$$

- AUTOMATIC CONTINUITY RESULTS

  BARTON-FRIEDMAN 1990

  (**NEW**) PERALTA-RUSSO 2010 (module)

- INNER DERIVATION RESULTS

  HO-MARTINEZ-PERALTA-RUSSO 2002

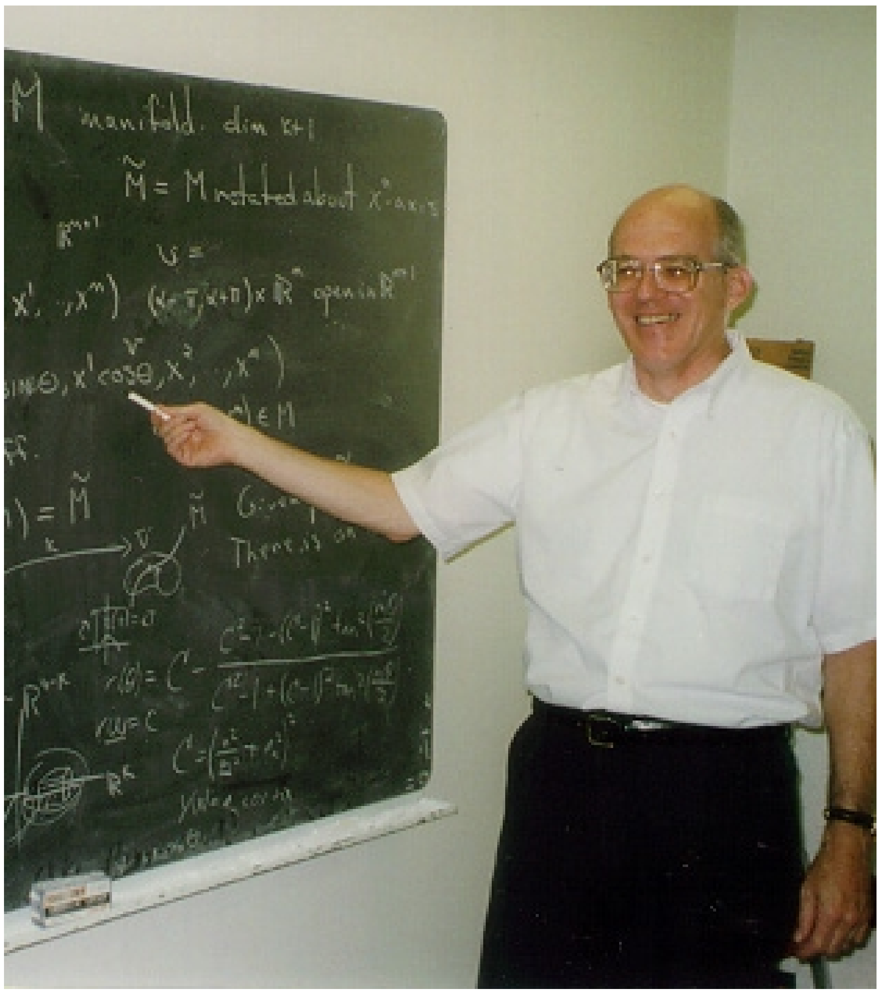
  MEYBERG 1972

  KÜHN-ROSENDAHL 1978 (module)

  (**NEW**) HO-PERALTA-RUSSO 2011 (module, weak amenability)

KUDOS TO:

Lawrence A. Harris (PhD 1969)



1974 (infinite dimensional holomorphy)

1981 (spectral and ideal theory)

AUTOMATIC CONTINUITY RESULTS

THEOREM (1990 Barton-Friedman)
EVERY DERIVATION OF A JB*-TRIPLE IS CONTINUOUS

THEOREM (2010 Peralta-Russo)
NECESSARY AND SUFFICIENT CONDITIONS UNDER WHICH A DERIVATION OF A JB*-TRIPLE INTO A JORDAN TRIPLE **MODULE** IS CONTINUOUS

(JB*-triple and Jordan triple module are defined below)

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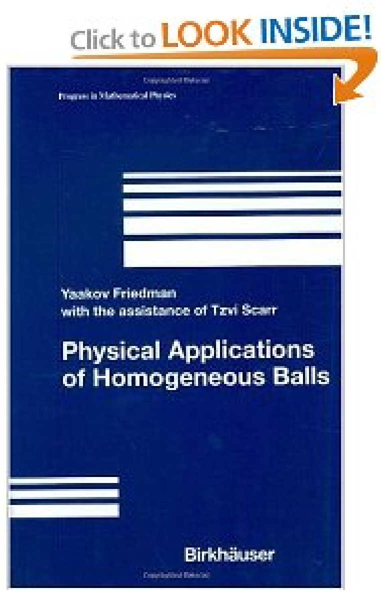

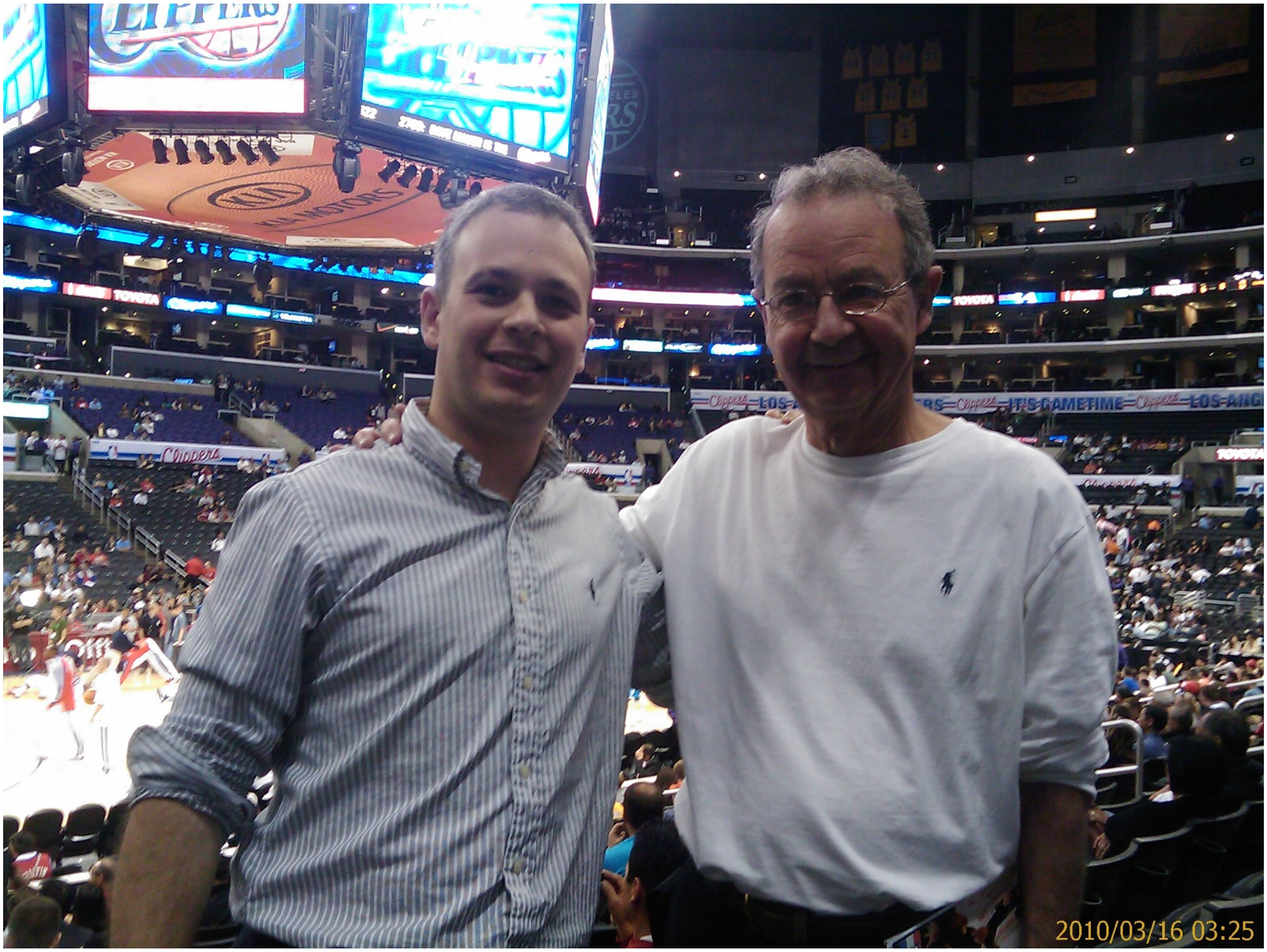

Antonio Peralta (b. 1974)
Bernard Russo (b. 1939)

GO LAKERS!

PREVIOUS INNER DERIVATION RESULTS

THEOREM (1972 Meyberg)

EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN TRIPLE SYSTEM IS INNER

(Lie algebras, Lie triple systems)

THEOREM (1978 Kühn-Rosendahl)

EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN TRIPLE SYSTEM INTO A JORDAN TRIPLE MODULE IS INNER

(Lie algebras)

THEOREM 2002
(Ho-Martinez-Peralta-Russo)
CARTAN FACTORS OF TYPE $I_{n,n}$,
II (even or ∞), and III HAVE THE INNER DERIVATION PROPERTY

THEOREM 2002
(Ho-Martinez-Peralta-Russo)
INFINITE DIMENSIONAL CARTAN FACTORS OF TYPE $I_{m,n}, m \neq n$, and IV DO NOT HAVE THE INNER DERIVATION PROPERTY.

SOME CONSEQUENCES FOR JB*-TRIPLES OF PERALTA-RUSSO WORK ON AUTOMATIC CONTINUITY

1. AUTOMATIC CONTINUITY OF DERIVATION ON JB*-TRIPLE (BARTON-FRIEDMAN)

2. AUTOMATIC CONTINUITY OF DERIVATION OF JB*-TRIPLE INTO DUAL (SUGGESTS WEAK AMENABILITY)

3. AUTOMATIC CONTINUITY OF DERIVATION OF JB*-ALGEBRA INTO A JORDAN MODULE (HEJAZIAN-NIKNAM)

SOME CONSEQUENCES FOR C*-ALGEBRAS OF PERALTA-RUSSO WORK ON AUTOMATIC CONTINUITY

1. AUTOMATIC CONTINUITY OF DERIVATION OF C*-ALGEBRA INTO A MODULE
(RINGROSE)

2. AUTOMATIC CONTINUITY OF JORDAN DERIVATION OF C*-ALGEBRA INTO A MODULE
(JOHNSON)

3. AUTOMATIC CONTINUITY OF JORDAN DERIVATION OF C*-ALGEBRA INTO A JORDAN MODULE
(HEJAZIAN-NIKNAM)

HO-PERALTA-RUSSO WORK ON TERNARY WEAK AMENABILITY FOR C*-ALGEBRAS AND JB*-TRIPLES

1. COMMUTATIVE C*-ALGEBRAS ARE TERNARY WEAKLY AMENABLE (TWA)

2. COMMUTATIVE JB*-TRIPLES ARE APPROXIMATELY WEAKLY AMENABLE

3. $B(H), K(H)$ ARE TWA IF AND ONLY IF FINITE DIMENSIONAL

4. CARTAN FACTORS OF TYPE $I_{m,n}$ OF FINITE RANK WITH $m \neq n$, AND OF TYPE IV ARE TWA IF AND ONLY IF FINITE DIMENSIONAL

SAMPLE LEMMA

The C*-algebra $A = K(H)$ of all compact operators on an infinite dimensional Hilbert space H is not Jordan weakly amenable.

We shall identify A^* with the trace-class operators on H.

Supposing that A were Jordan weakly amenable, let $\psi \in A^*$ be arbitrary. Then D_ψ ($= \mathrm{ad}\,\psi$) is an associative derivation and hence a Jordan derivation, so by assumption would be an inner Jordan derivation. Thus there would exist $\varphi_j \in A^*$ and $b_j \in A$ such that

$$D_\psi(x) = \sum_{j=1}^{n} [\varphi_j \circ (b_j \circ x) - b_j \circ (\varphi_j \circ x)]$$

for all $x \in A$.

For $x, y \in A$, a direct calculation yields

$$\psi(xy - yx) = -\frac{1}{4}\left(\sum_{j=1}^{n} b_j\varphi_j - \varphi_j b_j\right)(xy - yx).$$

It is known (Pearcy-Topping 1971) that every compact operator on a separable (which we may assume WLOG) infinite dimensional Hilbert space is a finite sum of commutators of compact operators.

By the just quoted theorem of Pearcy and Topping, every element of $K(H)$ can be written as a finite sum of commutators $[x, y] = xy - yx$ of elements x, y in $K(H)$.

Thus, it follows that the trace-class operator

$$\psi = -\frac{1}{4}\left(\sum_{j=1}^{n} b_j\varphi_j - \varphi_j b_j\right)$$

is a finite sum of commutators of compact and trace-class operators, and hence has trace zero. This is a contradiction, since ψ was arbitrary.

PROPOSITION

The JB*-triple $A = M_n(\mathrm{C})$ is ternary weakly amenable.

By a Proposition which is a step in the proof that commutative C*-algebras are ternary weakly amenable,

$$\mathcal{D}_t(A, A^*) = \mathcal{I}nn_b^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*),$$

so it suffices to prove that
$\mathcal{I}nn_b^*(A, A^*) \circ * \subset \mathcal{I}nn_t(A, A^*)$.

As in the proof of the Lemma, if $D \in \mathcal{I}nn_b^*(A, A^*)$ so that $Dx = \psi x - x\psi$ for some $\psi \in A^*$, then

$$\psi = [\varphi_1, b_1] - [\varphi_2, b_2] + \frac{\mathrm{Tr}\,(\psi)}{n} I,$$

where b_1, b_2 are self adjoint elements of A and φ_1 and φ_2 are self adjoint elements of A^*.

It is easy to see that, for each $x \in A$, we have

$$D(x^*) = \{\varphi_1, 2b_1, x\} - \{2b_1, \varphi_1, x\} - \{\varphi_2, 2b_2, x\} + \{2b_2, \varphi_2, x\},$$

so that

$$D \circ * \in \mathcal{I}nn_t(A, A^*).$$

Digression #7—LOCAL DERIVATIONS

THE QUESTIONS ARE: WHEN ARE THEY AUTOMATICALLY CONTINUOUS, AND WHEN ARE THEY GLOBAL DERIVATIONS. THIS MAKES SENSE IN ALL CONTEXTS.

APPENDIX
MAIN AUTOMATIC CONTINUITY RESULT
(Jordan triples, Jordan triple modules, Quadratic annihilator, Separating spaces)

Jordan triples

A complex (resp., real) *Jordan triple* is a complex (resp., real) vector space E equipped with a non-trivial triple product

$$E \times E \times E \to E$$

$$(x, y, z) \mapsto \{xyz\}$$

which is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one satisfying the so-called "*Jordan Identity*":

$$L(a, b)L(x, y) - L(x, y)L(a, b) =$$

$$L(L(a, b)x, y) - L(x, L(b, a)y),$$

for all a, b, x, y in E, where $L(x, y)z := \{xyz\}$.

A JB*-algebra is a complex Jordan Banach algebra A equipped with an algebra involution * satisfying $\|\{a,a^*,a\}\| = \|a\|^3$, $a \in A$. (Recall that $\{a,a^*,a\} = 2(a \circ a^*) \circ a - a^2 \circ a^*$).

A *(complex) JB*-triple* is a complex Jordan Banach triple E satisfying the following axioms:

(a) For each a in E the map $L(a,a)$ is an hermitian operator on E with non negative spectrum.

(b) $\|\{a,a,a\}\| = \|a\|^3$ for all a in A.

Every C*-algebra (resp., every JB*-algebra) is a JB*-triple with respect to the product $\{abc\} = \frac{1}{2}\,(ab^*c + cb^*a)$ (resp., $\{abc\} := (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$).

Jordan triple modules

If A is an associative algebra, an A-*bimodule* is a vector space X, equipped with two bilinear products $(a,x) \mapsto ax$ and $(a,x) \mapsto xa$ from $A \times X$ to X satisfying the following axioms:

$$a(bx) = (ab)x, \;\; a(xb) = (ax)b, \text{ and, } (xa)b = x(ab),$$

for every $a, b \in A$ and $x \in X$.

If J is a Jordan algebra, a *Jordan* J*-module* is a vector space X, equipped with two bilinear products $(a,x) \mapsto a \circ x$ and $(x,a) \mapsto x \circ a$ from $J \times X$ to X, satisfying:

$$a \circ x = x \circ a, \quad a^2 \circ (x \circ a) = (a^2 \circ x) \circ a, \text{ and,}$$

$$2((x \circ a) \circ b) \circ a + x \circ (a^2 \circ b) = 2(x \circ a) \circ (a \circ b) + (x \circ b) \circ a^2,$$

for every $a, b \in J$ and $x \in X$

If E is a complex Jordan triple, a *Jordan triple E-module* (also called *triple E-module*) is a vector space X equipped with three mappings

$$\{.,.,.\}_1 : X \times E \times E \to X$$
$$\{.,.,.\}_2 : E \times X \times E \to X$$
$$\{.,.,.\}_3 : E \times E \times X \to X$$

satisfying:

1. $\{x,a,b\}_1$ is linear in a and x and conjugate linear in b, $\{abx\}_3$ is linear in b and x and conjugate linear in a and $\{a,x,b\}_2$ is conjugate linear in a,b,x

2. $\{x,b,a\}_1 = \{a,b,x\}_3$, and $\{a,x,b\}_2 = \{b,x,a\}_2$ for every $a,b \in E$ and $x \in X$.

3. Denoting by $\{.,.,.\}$ any of the products $\{.,.,.\}_1$, $\{.,.,.\}_2$ and $\{.,.,.\}_3$, the identity $\{a,b,\{c,d,e\}\} = \{\{a,b,c\},d,e\} - \{c,\{b,a,d\},e\} + \{c,d,\{a,b,e\}\}$, holds whenever one of the elements a,b,c,d,e is in X and the rest are in E.

It is a little bit laborious to check that the dual space, E^*, of a complex (resp., real) Jordan Banach triple E is a complex (resp., real) triple E-module with respect to the products:

$$\{a,b,\varphi\}(x) = \{\varphi,b,a\}(x) := \varphi\{b,a,x\} \quad (3)$$

and

$$\{a,\varphi,b\}(x) := \overline{\varphi\{a,x,b\}}, \forall \varphi \in E^*, a,b,x \in E. \quad (4)$$

For each submodule S of a triple E-module X, we define its *quadratic annihilator*, $\text{Ann}_E(S)$, as the set

$$\{a \in E : Q(a)(S) = \{a,S,a\} = 0\}.$$

Separating spaces

Separating spaces have been revealed as a useful tool in results of automatic continuity.

Let $T : X \to Y$ be a linear mapping between two normed spaces. The *separating space*, $\sigma_Y(T)$, of T in Y is defined as the set of all z in Y for which there exists a sequence $(x_n) \subseteq X$ with $x_n \to 0$ and $T(x_n) \to z$.

A straightforward application of the closed graph theorem shows that a linear mapping T between two Banach spaces X and Y is continuous if and only if $\sigma_Y(T) = \{0\}$

Main Result

THEOREM Let E be a complex JB*-triple, X a Banach triple E-module, and let $\delta : E \to X$ be a triple derivation. Then δ is continuous if and only if $\text{Ann}_E(\sigma_X(\delta))$ is a (norm-closed) linear subspace of E and

$$\{\text{Ann}_E(\sigma_X(\delta)), \text{Ann}_E(\sigma_X(\delta)), \sigma_X(\delta)\} = 0.$$

COROLLARY Let E be a real or complex JB*-triple. Then

(a) Every derivation $\delta : E \to E$ is continuous.

(b) Every derivation $\delta : E \to E^*$ is continuous.

END OF SECTION IV

V—COHOMOLOGY OF ALGEBRAS

ASSOCIATIVE ALGEBRAS

HOCHSCHILD

ANNALS OF MATHEMATICS 1945

Let M be an associative algebra and X a two-sided M-module. For $n \geq 1$, let
$L^n(M, X) =$ all n-linear maps
$(L^0(M, X) = X)$

Coboundary operator
$\partial : L^n \to L^{n+1}$ (for $n \geq 1$)

$$\partial\phi(a_1, \cdots, a_{n+1}) = a_1\phi(a_2, \cdots, a_{n+1})$$
$$+\sum(-1)^j\phi(a_1, \cdots, a_{j-1}, a_ja_{j+1}, \cdots, a_{n+1})$$
$$+(-1)^{n+1}\phi(a_1, \cdots, a_n)a_{n+1}$$

For $n = 0$,
$\partial : X \to L(M, X) \qquad \partial x(a) = ax - xa$

Since $\partial \circ \partial = 0$,
$\text{Im}(\partial : L^{n-1} \to L^n) \subset \ker(\partial : L^n \to L^{n+1})$

$H^n(M, X) = \ker\partial/\text{Im}\partial$ is a vector space.

For $n = 1$, ker$\partial =$
$\{\phi : M \to X : a_1\phi(a_2) - \phi(a_1a_2) + \phi(a_1)a_2 = 0\}$
= the space of derivations from M to X

$$\partial : X \to L(M, X) \qquad \partial x(a) = ax - xa$$

Im$\partial =$ the space of inner derivations

Thus $H^1(M, X)$ measures how close derivations are to inner derivations.

An associative algebra B is an extension of associative algebra A if there is a homomorphism σ of B onto A. The extension splits if $B = \ker\sigma \oplus A^*$ where A^* is an algebra isomorphic to A, and is singular if $(\ker\sigma)^2 = 0$.

PROPOSITION

There is a one to one correspondence between isomorphism classes of singular extensions of A and $H^2(A, A)$

LIE ALGEBRAS

JACOBSON

LIE ALGEBRAS 1962

If L is a Lie algebra, then an L-module is a vector space M and a mapping of $M \times L$ into M, $(m, x) \mapsto mx$, satisfying

$$(m_1 + m_2)x = m_1 x + m_2 x$$

$$\alpha(mx) = (\alpha m)x = m(\alpha x)$$

$$m[x_1, x_2] = (mx_1)x_2 - (mx_2)x_1.$$

Let L be a Lie algebra, M an L-module. If $i \geq 1$, an *i-dimensional M-cochain* for L is a skew symmetric i-linear mapping f of $L \times L \times \cdots \times L$ into M. Skew symmetric means that if two arguments in $f(x_1, \cdots, x_i)$ are interchanged, the value of f changes sign. A 0-dimensional cochain is a constant function from L to M.

The coboundary operator δ (for $i \geq 1$) is:

$$\delta(f)(x_1, \cdots, x_{i+1})$$
$$= \sum_{q=1}^{i+1} (-1)^{i+1} f(x_1, \cdots, \hat{x}_q, \cdots, x_{i+1}) x_q$$
$$+ \sum_{q<r=1}^{i+1} (-1)^{r+q} f(x_1, \cdots, \hat{x}_q, \cdots, \hat{x}_r, \cdots, x_{i+1}, [x_q, x_r]).$$

and for $i = 0$, $\delta(f)(x) = ux$ (module action), if f is the constant $u \in M$.

One verifies that $\delta^2 = 0$ giving rise to cohomology groups

$$H^i(L, M) = Z^i(L, M)/B^i(L, M)$$

If $i = 0$ we take $B^i = 0$ and $H^0(L, M) = Z^0(L, M) = \{u \in M : ux = 0, \forall x \in L\}$.

THEOREM (WHITEHEAD'S LEMMAS)

If L is a finite dimensional semisimple Lie algebra over a field of characteristic 0, then

$$H^1(L, M) = H^2(L, M) = 0$$

for every finite dimensional module M of L.

THEOREM (WHITEHEAD)

If L is a finite dimensional semisimple Lie algebra over a field of characteristic 0, then

$$H^i(L, M) = 0 \ (\forall i \geq 0)$$

for every finite dimensional irreducible module M of L such that $ML \neq 0$.

JORDAN ALGEBRAS

GERSTENHABER
PROCEEDING OF THE NATIONAL ACADEMY OF SCIENCES 1964

GLASSMAN/JACOBSON
JOURNAL OF ALGEBRA 1970

Let A be an algebra defined by a set of identities and let M be an A-module. A singular extension of length 2 is, by definition, a null extension of A by M. So we need to know what a null extension is.

It is simply a short exact sequence

$$0 \to M \xrightarrow{\alpha} E \xrightarrow{\beta} A \to 0$$

where, provisionally, M is an algebra (rather than an A-module) with $M^2 = 0$.

If $n > 2$, a singular extension of length n is an exact sequence of bimodules

$$0 \to M \to M_{n-1} \to \cdots \to M_2 \to E \to A \to 0$$

Morphisms, equivalences, addition, and scalar multiplication of equivalence classes of singular extensions can be defined.

Then for $n \geq 2$, $H^n(A, M) :=$ equivalence classes of singular extensions of length n

These definitions are equivalent to the classical ones in the associative and Lie cases.

Gerstenhaber (using generalized projective resolutions) showed that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-bimodules, then there are natural homomorphisms δ^n so that the long sequence

$$0 \to \text{Der}\,(A, M') \to \text{Der}\,(A, M) \to \text{Der}\,(A, M'') \stackrel{\delta^1}{\to}$$

$$H^2(A, M') \to H^2(A, M) \to H^2(A, M'') \stackrel{\delta^2}{\to}$$

$$H^3(A, M') \to H^3(A, M) \to H^3(A, M'') \to$$

$$\cdots \to H^n(A, M'') \stackrel{\delta^n}{\to} H^{n+1}(A, M') \to \cdots$$

is exact. In particular,

$$H^n(A, M) = \ker \delta^n / \text{im}\, \delta^{n-1} \quad (n \geq 2)$$

What about $H^0(A, M)$ and $H^1(A, M)$ and Jordan algebras?

For this we turn first to Glassman's thesis of 1968.

Given an algebra A, consider the functor $\mathcal{T}$ from the category $\mathcal{C}$ of A-bimodules and A-homomorphisms to the category $\mathcal{V}$ of vector spaces and linear maps:

$$M \in \mathcal{C} \mapsto \mathsf{Der}\,(A, M) \in \mathcal{V} \quad , \quad \eta \mapsto \tilde{\eta}$$

where $\eta \in \mathsf{Hom}_A(M_1, M_2)$ and $\tilde{\eta} \in \mathsf{Hom}(\mathsf{Der}\,(A, M_1), \mathsf{Der}\,(A, M_2))$ is given by

$$\tilde{\eta} = \eta \circ D.$$

$$\left(A \xrightarrow{D} M_1 \xrightarrow{\eta} M_2\right)$$

An inner derivation functor is a subfunctor $\mathcal{J}$ which respects epimorphisms, that is,

$$M \in \mathcal{C} \mapsto \mathcal{J}(A, M) \subset \mathrm{Der}\,(A, M) \in \mathcal{V}$$

and if $\eta \in \mathrm{Hom}_A(M_1, M_2)$ is onto, then so is
$\mathcal{J}(A, \eta) := \tilde{\eta}|\mathcal{J}(A, M)$

Relative to the choice of $\mathcal{J}$ one defines

$$H^1(A, M) = \mathrm{Der}\,(A, M)/\mathcal{J}(A, M)$$

(The definition of $H^0(A, M)$ is more involved)

Glassman then proves that

$$0 \to H^0(A, M') \to H^0(A, M) \to H^0(A, M'') \xrightarrow{\delta^0}$$

$$H^1(A, M') \to H^1(A, M) \to H^1(A, M'') \xrightarrow{\delta^1}$$

$$H^2(A, M') \to H^2(A, M) \to H^2(A, M'') \xrightarrow{\delta^2}$$

$$H^3(A, M') \to H^3(A, M) \to H^3(A, M'') \to$$

$$\cdots \to H^n(A, M'') \xrightarrow{\delta^n} H^{n+1}(A, M') \to \cdots$$

is exact.

We mention just one other result from Glassman's paper.

In a Jordan algebra, recall that
$\{xby\} = (xb)y + (by)x - (xy)b$.

The b-homotope of J, written $J^{(b)}$ is the Jordan algebra structure on the vector space J given by the multiplication $x \cdot_b y = \{xby\}$.

If M is a bimodule for J and b is invertible, the corresponding bimodule $M^{(b)}$ for $J^{(b)}$ is the vector space M with action
$a \cdot_b m = m \cdot_b a = \{abm\}$

LEMMA

$H^n(J, M) \sim H^n(J^{(b)}, M^{(b)})$

JORDAN ANALOGS OF WHITEHEAD'S LEMMAS

THEOREM (JACOBSON)

Let J be a finite dimensional semisimple Jordan algebra over a field of characteristic 0 and let M be a J-module. Let f be a linear mapping of J into M such that

$$f(ab) = f(a)b + af(b).$$

Then there exist $v_i \in M, b_i \in J$ such that

$$f(a) = \sum_i ((v_i a)b - v_i(ab_i)).$$

THEOREM (ALBERT-PENICO-TAFT) Let J be a finite dimensional semisimple (separable?) Jordan algebra and let M be a J-module. Let f be a bilinear mapping of $J \times J$ into M such that

$$f(a,b) = f(b,a)$$

and

$$f(a^2, ab) + f(a,b)a^2 + f(a,a)ab$$

$$= f(a^2b, a) + f(a^2, b)a + (f(a,a)b)a$$

Then there exist a linear mapping g from J into M such that

$$f(a,b) = g(ab) - g(b)a - g(a)b$$

1 AND 2 DIMENSIONAL COHOMOLOGY FOR QUADRATIC JORDAN ALGEBRAS

McCrimmon-TAMS 1971 pp.285-289

Seibt, Peter
Cohomology of algebras and triple systems.
Comm. Algebra 3 (1975), no. 12,
1097–1120.

From the author's introduction: "The classical cohomologies of unital associative algebras and of Lie algebras have both a double algebraic character:

They are embedded in all of the machinery of derived functors, and they allow full extension theoretic interpretations (**Yoneda**-interpretation) of the higher cohomology groups—which seems natural since the coefficient category for cohomology is actually definable in terms of singular extension theory.

If one wants to define a uniform cohomology theory for (linear) nonassociative algebras and triple systems which "generalizes" these two classical cohomologies one may proceed either via derived functors

[M. Barr and G. S. Rinehart, Trans. Amer. Math. Soc. 122 (1966), 416–426; B. Pareigis, Math. Z. 104 (1968), 281–336]

or via singular extension theory

[M. Gerstenhaber, Proc. Nat. Acad. Sci. U.S.A. 51 (1964), 626–629; N. D. Glassman, Pacific J. Math. 33 (1970), 617–634].

The purpose of this paper which adopts the first point of view is to discuss compatibility questions with the second one."

1 AND 2 DIMENSIONAL COHOMOLOGY FOR JORDAN TRIPLE SYSTEMS

McCrimmon PacJM 1982 pp 92-102

1,2 and 3 DIMENSIONAL COHOMOLOGY FOR BANACH JORDAN ALGEBRAS; PERTURBATION THEORY

Dosi-RMJM 2009 pp516-520

DEFINITION:

A BANACH ALGEBRA IS **STABLE** IF ANY TWO SUFFICIENTLY CLOSE BANACH ALGEBRA MULTIPLICATIONS ARE TOPOLOGICALLY ALGEBRAICALLY ISOMORPHIC

MORE PRECISELY

If m is a Banach algebra multiplication on A, then $\|m(x,y)\| \leq \|m\|\|x\|\|y\|$.

THEOREM

If $H^2(A,A) = H^3(A,A) = 0$, then there exists $\epsilon > 0$ such that if $\|m_1 - m_2\| < \epsilon$ then (A, m_1) and (A, m_2) are topologically algebraically isomorphic.

- Johnson, Proc. Lon. Math. Soc. 1977
- Raeburn and Taylor, Jour. Funct. Anal. 1977

The origin of perturbation theory is deformation theory.

Let c_{ij}^k be the structure constants of a finite dimensional Lie algebra L.

Let $c_{ij}^k(\epsilon) \rightarrow c_{ij}^k$

Stability means $(L, c_{ij}^k(\epsilon))$ is isomorphic to (L,c$_{ij}^k$) if ϵ is sufficiently small.

THEOREM

(Gerstenhaber, Ann. of Math. 1964)
Finite dimensional semisimple Lie algebras are stable.

ANAR DOSI (ALSO USES DOSIEV)
(Middle East Technical University, TURKEY)

THEOREM 1

IF L IS A BANACH LIE ALGEBRA AND $H^2(L,L) = H^3(L,L) = 0$, THEN L IS A STABLE BANACH LIE ALGEBRA

THEOREM 2

SIMILAR FOR BANACH JORDAN ALGEBRAS (WITH APPROPRIATE DEFINITIONS OF LOW DIMENSIONAL COHOMOLOGY GROUPS)

END OF SECTION V

VI—COHOMOLOGY OF OPERATOR ALGEBRAS

- Ringrose, presidential address
  Bull. LMS 1996
- Sinclair and Smith: Survey
  Contemporary Mathematics 2004

Hochschild cohomology involves an associative algebra A and A-bimodules X and gives rise to

- n-cochains $L^n(A, X)$,
- coboundary operators Δ_n,
- n-coboundaries B^n,
- n-cocycles Z^n and
- cohomology groups $H^n(A, X)$.

If A is a Banach algebra and X is a Banach A-bimodule (=Banach space with module actions jointly continuous) we have the continuous versions of the above concepts

$$L_c^n \ , \ B_c^n \ , \ Z_c^n \ , \ H_c^n(A, X).$$

Warning: B_c^n is not always closed, so H_c^n is still only a vector space.

Let A be a C*-algebra of operators acting on a Hilbert space H and let X be a dual normal A-module (X is a dual space and the module actions are separately ultra weakly-weak*-continuous). We now have

- normal n-cochains $L^n_w(A, X)$= bounded and separately weakly continuous n-cochains
- coboundary operators Δ_n,
- normal n-coboundaries B^n_w,
- normal n-cocycles Z^n_w and
- normal cohomology groups $H^n_w(A, X)$.

For a C*-algebra acting on a Hilbert space we thus have three possible cohomology theories:

- the purely algebraic Hochschild theory H^n
- the bounded theory H^n_c
- the normal theory H^n_w

THEOREM 1 (1971)

$$H^n_w(A, X) \sim H^n_w(R, X)$$

($R =$ultraweak closure of A)

THEOREM 2 (1972)

$$H^n_w(A, X) \sim H^c_w(A, X)$$

By Theorems 1 and 2, due to **Johnson-Kadison-Ringrose**, all four cohomology groups

$$H^n_w(A, X)\ ,\ H^n_w(R, X)\ ,\ H^n_c(R, X)\ ,\ H^n_w(R, X)$$

are isomorphic.

THEOREM 3 (1971)

(Johnson-Kadison-Ringrose)

$H_c^n(R, X) = 0 \ \forall n \geq 1$

(R =hyperfinite von Neumann algebra)

THEOREM 4 (1978)

(Connes)

If R is a von Neumann algebra with a separable predual, and $H_c^1(R, X) = 0$ for every dual normal R-bimodule X, then R is hyperfinite.

At this point, there were two outstanding problems of special interest;

Problem A

$$H_c^n(R, R) = 0 \ \forall n \geq 1?$$

for every von Neumann algebra R

Problem B

$$H_c^n(R, B(H)) = 0 \ \forall n \geq 1?$$

for every von Neumann algebra R acting on a Hilbert space H

(**Problem C** will come later)

ENTER COMPLETE BOUNDEDNESS

FAST FORWARD ONE DECADE

"The main obstacle to advance was a paucity of information about the general bounded linear (or multilinear) mapping between operator algebras. The major breakthrough, leading to most of the recent advances, came through the development of a rather detailed theory of completely bounded mappings." (Ringrose)

Let A be a C*-algebra and let S be a von Neumann algebra, both acting on the same Hilbert space H with $A \subset S$. We can view S as a dual normal A-module with A acting on S by left and right multiplication. We now have

- completely bounded n-cochains $L^n_{cb}(A, S)$
- coboundary operators Δ_n,
- completely bounded n-coboundaries B^n_{cb},
- completely bounded n-cocycles Z^n_{bc}
- completely bounded cohomology groups $H^n_{cb}(A, S)$.

Digression #8—COMPLETELY BOUNDED MAPS

sneak preview of section IX

(QUANTUM FUNCTIONAL ANALYSIS)

(i.) BANACH SPACES

Why normed spaces?

$\mathbf{R}^n$ is a vector space with a norm, so you can take derivatives and integrals.

Normed spaces are important because you can do calculus.

Why completeness?

Three basic principles of functional analysis are based in some way on completeness:

- Hahn-Banach (Zorn)
- Open mapping (Baire)
- Uniform boundedness (Baire)

Examples of Interest to Us

- $C[0,1]$, L^p, ℓ^p, c_0
- $C(\Omega)$, $L^p(\Omega, \mu)$, $C_0(\Omega)$
- $B(X,Y)$, Banach algebras, Operator algebras $\mathcal{A} \subset B(X)$
- C*-algebras: group representations, quantum mechanics, Jordan algebras of self-adjoint operators
- Gelfand duality: Noncommutative topology, Noncommutative geometry
- Non-self-adjoint operator algebras: operator theory, several complex variables
- Operator spaces: C*-algebras, Banach spaces

Every Banach space is commutative

By the Hahn-Banach theorem, $X \subset C(X_1^*)$, a linear subspace of a **commutative** C*-algebra.

Hence: $C(\Omega)$ is the mother of all Banach spaces

Definition: An **operator space** (or noncommutative Banach space, or quantized Banach space) is a linear subspace of a C*-algebra (or $B(H)$).

Hence: $B(H)$ is the mother of all operator spaces

Injective Banach spaces

The Hahn-Banach theorem states that every bounded linear functional on a subspace of a Banach space has a bounded extension to the larger space with the same norm.

Definition: A Banach space is **injective** if every bounded linear operator from a subspace of a Banach space into it has a bounded extension to the larger space with the same norm.

Theorem: A Banach space is injective if and only if it is isometric to $C(\Omega)$, where Ω is extremely disconnected.

Remark: A Banach space is injective if and only if there is an injective Banach space containing it and a contractive projection of that space onto it.

(ii.) OPERATOR SPACES

An operator space may be viewed as a C*-algebra for which the multiplication and involution have been ignored.

You must replace these by some other structure.

What should this structure be and what can you do with it?

The answer lies in the morphisms.

In the category of Banach spaces, the morphisms are the bounded linear maps.

In the category of operator spaces, the morphisms are the **completely bounded maps**.

Operator spaces are intermediate between Banach spaces and C*-algebras

The advantage of operator spaces over C*-algebras is that they allow the use of finite dimensional tools and isomorphic invariants ("local theory").

C*-algebras are too rigid: morphisms are contractive, norms are unique.

Operator space theory opens the door to a massive transfer of technology coming from Banach space theory.

This quantization process has benefitted operator algebra theory mainly (as opposed to Banach space theory).

The Arveson-Wittstock-Paulsen Hahn-Banach theorem

Theorem: $B(H)$ is injective in the category of operator spaces and completely contractive maps.

Exercise: $X \subset B(H)$ is injective if and only if there is a completely contractive projection $P : B(H) \to B(H)$ with $P(B(H)) = X$.

Injective operator spaces

Theorem: An operator space is injective if and only if it is completely isometric to a corner of a C*-algebra.

In particular, it is completely isometric to a ternary ring of operators (TRO).

Theorem: Every operator space has an injective envelope.

(ternary envelope, shilov boundary)

Mixed injective operator spaces

Defintion: An operator space is a **mixed injective** if every completely contractive linear operator from a subspace of an operator space into it has a bounded contractive extension to the larger space.

Exercise: $X \subset B(H)$ is a mixed injective if and only if there is a contractive projection $P : B(H) \to B(H)$ with $P(B(H)) = X$.

(iii.) APPLICATIONS OF OPERATOR SPACE THEORY

1. Similarity problems (Pisier)

The Halmos problem:

Theorem: There exists a polynomially bounded operator which is not similar to a contraction.

The Kadison problem

Pisier, Lecture Notes in Math 1618.

2. $B(H)$ is not a nuclear pair (Junge-Pisier)

Theorem: $B(H) \otimes B(H)$ does not have a unique C*-norm.

Good case study: The proof is based on the solution to a problem which would be studied for its own sake:

whether the set of finite dimensional operator spaces is separable (it is not).

3.Operator amenable groups (Ruan)

Theorem: A group is amenable if and only if its group algebra is an amenable Banach algebra under convolution.

This is false for the Fourier algebra.

Theorem: A group is amenable if and only if its Fourier algebra is amenable in the operator space formulation.

(SEE EFFROS-RUAN BOOK).

4. Operator local reflexivity
(Effros-Junge-Ruan)

Theorem: Every Banach space is locally reflexive (Finite dimensional subspaces of the second dual can be approximated by finite dimensional subspaces of the space itself).

Application: The second dual of a JB*-triple is a JB*-triple.

Consequence: Structure theory of JB*-triples.

Not all operator spaces are locally reflexive (in the appropriate operator space sense)

Theorem: The dual of a C*-algebra is a locally reflexive operator space.

END OF DIGRESSION

Let A be a C*-algebra and let S be a von Neumann algebra, both acting on the same Hilbert space H with $A \subset S$. We can view S as a dual normal A-module with A acting on S by left and right multiplication. We now have

- completely bounded n-cochains $L^n_{cb}(A, S)$
- coboundary operators Δ_n,
- completely bounded n-coboundaries B^n_{cb},
- completely bounded n-cocycles Z^n_{bc}
- completely bounded cohomology groups $H^n_{cb}(A, S)$.

For a C*-algebra A and a von Neumann algebra S with $A \subset S \subset B(H)$ we thus have two new cohomology theories:

- the completely bounded theory H^n_{cb}
- the completely bounded normal theory H^n_{cbw}

By straightforward analogues of Theorems 1 and 2, all four cohomology groups

$$H^n_{cb}(A,S)\ ,\ H^n_{cbw}(A,S)\ ,\ H^n_{cb}(R,S)\ ,\ H^n_{cbw}(R,S)$$

are isomorphic, where R is the ultraweak closure of A.

THEOREM 5 (1987)
(Christensen-Effros-Sinclair)

$$H^n_{cb}(R,B(H))=0\ \forall n\geq 1$$

(R =any von Neumann algebra acting on H)

THEOREM 6 (1987)¶
(Christensen-Sinclair)

$$H^n_{cb}(R,R)=0\ \forall n\geq 1$$

(R =any von Neumann algebra)

¶unpublished as of 2004

"Cohomology and complete boundedness have enjoyed a symbiotic relationship wher advances in one have triggered progress in the other" (Sinclair-Smith)

Theorems 7 and 8 are due to Christensen-Effros-Sinclair.

THEOREM 7 (1987)

$H_c^n(R, R) = 0 \ \forall n \geq 1$

($R =$ von Neumann algebra of type I, II_∞, III, or of type II_1 and stable under tensoring with the hyperfinite factor)

THEOREM 8 (1987)

$H_c^n(R, B(H)) = 0 \ \forall n \geq 1$

($R =$ von Neumann algebra of type I, II_∞, III, or of type II_1 and stable under tensoring with the hyperfinite factor, acting on a Hilbert space H)

THEOREM 9 (1998)

(Sinclair-Smith based on earlier work of Christensen,Pop,Sinclair,Smith)

$$H_c^n(R,R)=0\ \forall n\geq 1$$

($R=$ von Neumann algebra of type II_1 with a Cartan subalgebra and a separable$^{\|}$ predual)

THEOREM 10 (2003)

(Christensen-Pop-Sinclair-Smith $n\geq 3$)

$$H_c^n(R,R)=H_c^n(R,B(H))=0\ \forall n\geq 1$$

($R=$ von Neumann algebra factor of type II_1 with property Γ, acting on a Hilbert space H)

($n=1$:Kadison-Sakai '66 and Christensen '86
$n=2$: Christensen-Sinclair '87, '01)

$^{\|}$The separability assumption was removed in 2009—Jan Cameron

We can now add a third problem (C) to our previous two (A,B)

Problem A

$H_c^n(R, R) = 0 \ \forall n \geq 1$?

for every von Neumann algebra R

Problem B

$H_c^n(R, B(H)) = 0 \ \forall n \geq 1$?

for every von Neumann algebra R acting on a Hilbert space H

Problem C

$H_c^n(R, R)) = 0 \ \forall n \geq 2$?

(R is a von Neumann algebra of type II_1)

A candidate is the factor arising from the free group on 2 generators.

Paulsen, Vern I.
Relative Yoneda cohomology for operator spaces.
J. Funct. Anal. 157 (1998), no. 2, 358–393.

Let A be a subalgebra in the algebra B(H) of bounded operators in a Hilbert space H, and X be a linear subspace in B(H) such that AXA⊂X.

Then X can be considered as an A-bimodule, and the Hochschild cohomology groups for the pair (A,X) can be constructed in the usual way, but with the cocycles being n-linear maps from A to X satisfying certain extra "complete boundedness" conditions.

Continuing the study of A. Ya. Khelemskii[The homology of Banach and topological algebras, Kluwer Acad. Publ., Dordrecht, 1989], himself and others, the author presents two new alternative presentations of these completely bounded Hochschild cohomologies, one of them as a relative Yoneda cohomology, i.e. as equivalence classes of relatively split resolutions, and the second as a derived functor, making it similar to EXT.

These presentations make clear the importance of the notions of relative injectivity, projectivity and amenability, which are introduced and studied.

Using the relative injectivity of von Neumann algebras, the author proves the triviality of completely bounded Hochschild cohomologies for all von Neumann algebras.

The Yoneda representation makes the proofs of a number of classical results more transparent.

Reviewed by G. V. Rozenblyum

Digression #9—PERTURBATION OF BANACH ALGEBRAS

Raeburn and Taylor 1977

Johnson 1977

Helemskii book 1986

Dosi 2009

Kadison and Kastler

END OF SECTION VI

VII—COHOMOLOGY OF TRIPLE SYSTEMS

1. Cohomology of Lie triple systems and lie algebras with involution, B. Harris, TAMS 1961

2. Cohomology of associative triple systems, Renate Carlsson, PAMS 1976

3. On the representation theory of Lie triple systems, T.L.Hodge and B.J. Parshall, Trans. A.M.S. 2002

WEDDERBURN DECOMPOSITION

4. Der Wedderburnsche Hauptsatz für alternative Tripelsysteme und Paare, Renate Carlsson, Math. Ann 1977

5. Wedderburnzerlegung für Jordan-Paare, Oda Kühn und Adelheid Rosendahl, Manus. Math 1978

1
Cohomology of Lie triple systems and lie algebras with involution
B. Harris, TAMS 1961

MATHEMATICAL REVIEWS

A Lie triple system T is a subspace of a Lie algebra L closed under the ternary operation $[xyz] = [x, [y, z]]$ or, equivalently, it is the subspace of L consisting of those elements x such that $\sigma(x) = -x$, where σ is an involution of L.

A T-module M is a vector space such that the vector-space direct sum $T \oplus M$ is itself a Lie triple system in such a way that

1. T is a subsystem
2. $[xyz] \in M$ if any of x, y, z is in M
3. $[xyz] = 0$ if two of x, y, z are in M.

A universal Lie algebra $L_u(T)$ and an $L_u(T)$-module M_s can be constructed in such a way that both are operated on by an involution σ and so that T and M consist of those elements of $L_u(T)$ and M_s which are mapped into their negatives by σ.

Now suppose L is a Lie algebra with involution σ and N is an L-σ module. Then σ operates on $H^n(L,N)$ so that

$$H^n(L,N) = H^n_+(L,N) \oplus H^n_-(L,N)$$

with both summands invariant under σ.

The cohomology of the Lie triple system is defined by $H^n(T,M) = H^n_+(L_u(T),M_s)$.

The author investigates these groups for $n = 0, 1, 2$.

- $H^0(T, M) = 0$ for all T and M
- $H^1(T, M) =$ derivations of T into M modulo inner derivations
- $H^2(T, M) =$ factor sets of T into M modulo trivial factor sets.

Turning to the case of finite-dimensional simple T and ground field of characteristic 0, one has the Whitehead lemmas

$$H^1(T, M) = 0 = H^2(T, M)$$

Weyl's theorem: Every finite-dimensional module is semi-simple.

The paper ends by showing that if in addition, the ground field Φ is algebraically closed, then $H^3(T, \Phi)$ is 0 or not 0, according as $L_u(T)$ is simple or not.

2
On the representation theory of Lie triple systems,
Hodge, Terrell L., Parshall, Brian, Trans. Amer. Math. Soc. 354 (2002), no. 11, 4359–4391

The authors of the paper under review study representations of Lie triple systems, both ordinary and restricted.

The theory is based on the connection between Lie algebras and Lie triple systems.

In addition, the authors begin the study of the cohomology theory for Lie triple systems and their restricted versions.

They also sketch some future applications and developments of the theory.

Reviewed by Plamen Koshlukov

3
Cohomology of associative triple systems, Renate Carlsson, PAMS 1976

MATHEMATICAL REVIEWS

A cohomology for associative triple systems is defined, with the main purpose to get quickly the cohomological triviality of finite-dimensional separable objects over fields of characteristic $\neq 2$, i.e., in particular the Whitehead lemmas and the Wedderburn principal theorem.

This is achieved by embedding an associative triple system A in an associative algebra $U(A)$ and associating with every trimodule M for A a bimodule M_u for $U(A)$ such that the cohomology groups $H^n(A, M)$ are subgroups of the classical cohomology groups $H^n(U(A), M_u)$.

Since $U(A)$ is chosen sufficiently close to A, in order to inherit separability, the cohomological triviality of separable A is an immediate consequence of the associative algebra theory.

The paper does not deal with functorialities, not even with the existence of a long exact cohomology sequence.

4
Der Wedderburnsche Hauptsatz für alternative Tripelsysteme und Paare, Renate Carlsson, Math. Ann 1977

MATHEMATICAL REVIEWS

The Wedderburn principal theorem, known for Lie triple systems, is proved for alternative triple systems and pairs.

If i is an involution of an alternative algebra B, then $\langle xyz\rangle := (x \cdot i(y)) \cdot z$ is an alternative triple $(x, y, z \in B)$.

A polarisation of an alternative triple A is a direct sum of two submodules $A^1 \oplus A^{-1}$ with

$$\langle A^1 A^{-1} A^1\rangle \subset A^1, \langle A^{-1} A^1 A^{-1}\rangle \subset A^{-1}$$

and

$$\langle A^1 A^1 A^1\rangle = \langle A^{-1} A^{-1} A^{-1}\rangle = \langle A^1 A^1 A^{-1}\rangle = \langle A^{-1} A^{-1} A^1\rangle = \langle A^{-1} A^1 A^1\rangle = \langle A^1 A^{-1} A^{-1}\rangle = \{0\}.$$

An alternative pair is an alternative triple with a polarisation.

THEOREM

If A is a finite-dimensional alternative triple system (or an alternative pair) over a field K, R the radical and A/R separable, then $A = B \oplus R$, where B is a semisimple subtriple (subpair) of A with $B = A/R$.

5
Wedderburnzerlegung für Jordan-Paare
Oda Kühn und Adelheid Rosendahl
Manus. Math 1978

AUTHOR'S ABSTRACT

In the first section we summarize some properties of Jordan pairs. Then we state some results about some groups defined by Jordan pairs.

In the next section we construct a Lie algebra to a Jordan pair. This construction is a generalization of the wellknown Koecher-Tits-construction. We calculate the radical of this Lie algebra in terms of the given Jordan pair.

In the last section we prove a Wedderburn decomposition theorem for Jordan pairs (and triples) in the characteristic zero case.

Some special cases in arbitrary characteristic have been shown by R. Carlsson.

Also we show that any two such decompositions are conjugate under a certain group of automorphism. Analogous theorems will be shown for Jordan Triples.

MATHEMATICAL REVIEWS

The authors generalize the "Koecher-Tits-construction" for Jordan algebra to Jordan pairs (and in a parallel manner to Jordan triples). The functor obtained goes from Jordan pairs [or Jordan triples] to Lie algebras. An observation of Koecher that the theorem of Levi for Lie algebras of characteristic 0 implies (via the functor) the Wedderburn principal theorem for Jordan algebras is extended to Jordan pairs (and Jordan triples) V over a field of characteristic 0. In addition, the authors show that any two Wedderburn splittings of V are conjugate under a certain normal subgroup of the automorphism group of V.

END OF SECTION VII

VIII—COHOMOLOGY OF BANACH TRIPLE SYSTEMS (PROSPECTUS)

- Lie derivations into a module; automatic continuity and weak amenability (Harris,Miers,Mathieu,Villena)
- Cohomology of commutative JB*-triples (Kamowitz,Carlsson)
- Cohomology of TROs (Zalar,Carlsson)
- Wedderburn decompositions for JB*-triples (Kühn-Rosendahl)
- Low dimensional cohomology for JBW*-triples and algebras-perturbation (Dosi,McCrimmon)
- Structure group of JB*-triple (McCrimmon—derivations)
- Alternative Banach triples (Carlsson,Braun)
- Completely bounded triple cohomology (Timoney et.al.,Christensen et.al)
- Local derivations on JB*algebras and triples (Kadison,Johnson,Ajupov,. . .)
- Chu's work on Koecher-Kantor-Tits construction

END OF SECTION VIII

IX—QUANTUM FUNCTIONAL ANALYSIS

I. CLASSICAL OPERATOR SPACES

Gilles Pisier
Introduction to Operator Space Theory
Cambridge University Press 2003

("NEOCLASSICAL" AND "MODERN" OPERATOR SPACES WILL APPEAR IN SECTION X)

OPERATOR SPACE THEORY =
NON-COMMUTATIVE BANACH SPACE THEORY

AN OPERATOR SPACE IS A BANACH SPACE TOGETHER WITH AN ISOMETRIC EMBEDDING IN $B(H)$

OBJECTS = BANACH SPACES;
MORPHISMS = COMPLETELY BOUNDED MAPS

ORIGINS

- STINESPRING 1955
- ARVESON 1969
- RUAN 1988

EXAMPLES OF CB MAPS

- RESTRICTION OF *-HOMOMORPHISMS
- MULTIPLICATION OPERATORS

  THEOREM: EVERY CB-MAP IS THE PRODUCT OF THESE TWO.

- COMPLETE CONTRACTION
- COMPLETE ISOMETRY,
- COMPLETE ISOMORPHISM

complete semi-isometry
(Oikhberg-Rosenthal)

$CB(X,Y),\ \|\cdot\|_{\mathrm{cb}}$ IS A BANACH SPACE

COMPLETELY BOUNDED
BANACH-MAZUR DISTANCE

$$\mathrm{d}_{\mathrm{cb}}(E,F) = \inf\{\|u\|_{\mathrm{cb}} \cdot \|u^{-1}\|_{\mathrm{cb}}\}.$$

($u : E \to F$ complete isomorphism)

ROW AND COLUMN HILBERT SPACE

$$R,\ C,\ R_n,\ C_n$$

$$\mathrm{d}_{\mathrm{cb}}(R_n, C_n) = n$$
$$\mathrm{d}_{\mathrm{cb}}(R, C) = \infty$$

$$\mathrm{d}(R_n, C_n) = 1$$
$$\mathrm{d}(R, C) = 1$$

(PROOF FOR $\mathrm{d}_{\mathrm{cb}}(R_n, C_n) = n$)

$R,\ C,\ R_n,\ C_n$

ARE HOMOGENEOUS OPERATOR SPACES

$(\forall u : E \to E,\ \|u\|_{\mathrm{cb}} = \|u\|)$

SO ARE

min(E) and max(E)

AND

$\Phi(I)$

(PROOF FOR C)

min(E) IS DEFINED BY

$$E \subset C(T) \subset B(H)$$

$$\|(a_{ij})\|_{M_n(\min(E))} = \sup_{\xi \in B_{E^*}} \|(\xi(a_{ij}))\|_{M_n}$$

max(E) IS DEFINED BY

$$\|(a_{ij})\|_{M_n(\max(E))} = \sup\{\|(u(a_{ij}))\|_{M_n(B(H_u))} : u : E \to B(H_u),\ \|u\| \leq 1\}$$

THE IDENTITY MAP IS COMPLETELY CONTRACTIVE:

$$F \xrightarrow{u} \min(E) \to E \to \max(E) \xrightarrow{v} G$$

$$(\|u\|_{\text{cb}} = \|u\|, \quad \|v\|_{\text{cb}} = \|v\|)$$

EXAMPLE

$$\Phi(I) = \overline{\mathsf{sp}}\{V_i : i \in I\}$$

V_i SATISFYING THE CAR

$$\Phi_n := \Phi(\{1, 2, \ldots, n\}),$$

$$\Phi = \Phi(\{1, 2, \ldots\})$$

PROPOSITION

LET E BE A HILBERTIAN OPERATOR SPACE. THEN E IS HOMOGENEOUS IF AND ONLY IF $\|U\|_{\text{cb}} = 1\ \forall$ UNITARY $U : E \to E$

(application of Russo-Dye theorem)

CLASSICAL BANACH SPACES

ℓ_p, c_0, L_p, $C(K)$

"SECOND GENERATION"

(neoclassical?)

ORLICZ, SOBOLEV, HARDY, DISC ALGEBRA, SCHATTEN p-CLASS

CLASSICAL OPERATOR SPACES

R, C, $\min(\ell_2)$, $\max(\ell_2)$, OH, Φ

FINITE DIMENSIONAL VERSIONS

R_n, C_n, $\min(\ell_2^n)$, $\max(\ell_2^n)$, OH_n, Φ_n

THE CLASSICAL OPERATOR SPACES ARE MUTUALLY COMPLETELY NON-ISOMORPHIC.

IF E_n, F_n ARE n-DIMENSIONAL VERSIONS, THEN $\mathrm{d_{cb}}(E_n, F_n) \to \infty$

END OF SECTION IX

X—PROJECTIONS ON OPERATOR SPACES

1. CLASSIFICATION OF ATOMIC MIXED INJECTIVE OPERATOR SPACES

Neal-Russo
(Trans. Amer. Math. Soc. 2000/2003)

2. NEOCLASSICAL OPERATOR SPACES

Neal-Russo (Proc. Amer. Math. Soc. 2004)

3. MODERN OPERATOR SPACES

Neal-Ricard-Russo (Jour. Funct. Anal. 2006)

4. OPERATOR SPACE CHARACTERIZATION OF TROs

Neal-Russo (Pac. J. Math 2003).

5. QUANTUM OPERATOR ALGEBRAS

Blecher,Ruan,Sinclair, etc.

1. CLASSIFICATION OF ATOMIC MIXED INJECTIVE OPERATOR SPACES

THEOREM 1

(Neal-Russo)

There is a family of 1-mixed injective Hilbertian operator spaces H_n^k, $1 \leq k \leq n$, of finite dimension n, with the following properties:

(a) H_n^k is a subtriple of the Cartan factor of type 1 consisting of all $\binom{n}{k}$ by $\binom{n}{n-k+1}$ complex matrices.

(b) Let Y be a JW^*-triple of rank 1 (necessarily atomic).

- **(i)** If Y is of finite dimension n then it is isometrically completely contractive to some H_n^k.
- **(ii)** If Y is infinite dimensional then it is isometrically completely contractive to $B(H, \mathbf{C})$ or $B(\mathbf{C}, K)$.

(c) H_n^n (resp. H_n^1) coincides with R_n (resp. C_n).

(d) For $1 < k < n$, H_n^k is not completely semi-isometric to R_n or C_n.

Example 1:

H_3^2 is the subtriple of $B(\mathbf{C}^3)$ consisting of all matrices of the form

$$\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$$

and hence in this case Y is actually completely semi-isometric to the Cartan factor $A(\mathbf{C}^3)$ of 3 by 3 anti-symmetric complex matrices.

Example 2:

H_4^3 is the subtriple of $B(\mathbf{C}^6, \mathbf{C}^4)$ consisting of all matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 & -d & c & -b \\ 0 & d & -c & 0 & 0 & a \\ -d & 0 & b & 0 & -a & 0 \\ c & -b & 0 & a & 0 & 0 \end{bmatrix}.$$

THEOREM 2

(Neal-Russo)

Let X be a 1-mixed injective operator space which is atomic. Then X is completely semi-isometric to a direct sum of Cartan factors of types 1 to 4 and the spaces H_n^k.

THEOREM 3

(Neal-Russo)

Let Y be an atomic w*-closed JW^*-subtriple of a W^*-algebra.

(a) If Y is irreducible and of rank at least 2, then it is completely isometric to a Cartan factor of type 1–4 or the space Diag $(B(H,K), B(K,H))$.

(b) If Y is of finite dimension n and of rank 1, then it is completely isometric to Diag $(H_n^{k_1}, \ldots, H_n^{k_m})$, for appropriately chosen bases, and where $k_1 > k_2 > \cdots > k_m$.

(c) Y is completely semi-isometric to a direct sum of the spaces in (a) and (b). If Y has no infinite dimensional rank 1 summand, then it is completely isometric to a direct sum of the spaces in (a) and (b).

2.
THE HILBERTIAN (NEOCLASSICAL) OPERATOR SPACES H_n^k

Restatement of Theorem 2

A frequently mentioned result of Friedman and Russo states that if a subspace X of a C*-algebra A is the range of a contractive projection on A, then X is isometric to a JC*-triple, that is, a norm closed subspace of $B(H,K)$ stable under the triple product $ab^*c + cb^*a$.

If X is atomic (in particular, finite-dimensional), then it is isometric to a direct sum of Cartan factors of types 1 to 4.

This latter result fails, as it stands, in the category of operator spaces.

HOWEVER,

There exists a family of n-dimensional Hilbertian operator spaces H_n^k, $1 \leq k \leq n$, generalizing the row and column Hilbert spaces R_n and C_n such that, in the above result, if X is atomic, the word "isometric" can be replaced by "completely semi-isometric," provided the spaces H_n^k are allowed as summands along with the Cartan factors.

The space H_n^k is contractively complemented in some $B(K)$, and for $1 < k < n$, is not completely (semi-)isometric to either of the Cartan factors $B(\mathbf{C}, \mathbf{C}^n) = H_n^1$ or $B(\mathbf{C}^n, \mathbf{C}) = H_n^n$.

These spaces appeared in a slightly different form and context in a memoir of ARAZY and FRIEDMAN (1978).

CONSTRUCTION OF H_n^k

Let I denote a subset of $\{1,2,\ldots,n\}$ of cardinality $|I|=k-1$. The number of such I is $q:=\binom{n}{k-1}$.

Let J denote a subset of $\{1,2,\ldots,n\}$ of cardinality $|J|=n-k$. The number of such J is $p:=\binom{n}{n-k}$.

The space H_n^k is the linear span of matrices $b_i^{n,k}$, $1\le i\le n$, given by

$$b_i^{n,k}=\sum_{I\cap J=\emptyset,(I\cup J)^c=\{i\}}\epsilon(I,i,J)e_{J,I},$$

where

$$e_{J,I}=e_J\otimes e_I=e_Je_I^t\in M_{p,q}(\mathbf{C})=B(\mathbf{C}^q,\mathbf{C}^p),$$

and $\epsilon(I,i,J)$ is the signature of the permutation taking $(i_1,\ldots,i_{k-1},i,j_1,\ldots,j_{n-k})$ to $(1,\ldots,n)$.

Since the $b_i^{n,k}$ are the image under a triple isomorphism (actually ternary isomorphism) of a rectangular grid in a JC*-triple of rank one, they form an orthonormal basis for H_n^k.

HOMOGENEITY OF H_n^k AND THE COMPLETELY BOUNDED BANACH-MAZUR DISTANCE

THEOREM 1

H_n^k is a homogeneous operator space.

THEOREM 2

$\mathrm{d_{cb}}(H_n^k, H_n^1) = \sqrt{\frac{kn}{n-k+1}}$, for $1 \leq k \leq n$.

CREATION OPERATORS

Let $C_h^{n,k}$ denote the wedge (or creation) operator from $\wedge^{k-1}\mathbf{C}^n$ to $\wedge^k\mathbf{C}^n$ given by

$$C_h^{n,k}(h_1 \wedge \cdots \wedge h_{k-1}) = h \wedge h_1 \wedge \cdots \wedge h_{k-1}.$$

Letting $\mathcal{C}^{n,k}$ denote the space sp$\{C_{e_i}^{n,k}\}$, we have the following.

PROPOSITION

H_n^k is completely isometric to $\mathcal{C}^{n,k}$.

PROOF

There exist unitaries such that

$$W_k^n U_{n-k}^n b_i^{n,k} = V_k^n C_{e_i}^{n,k} U_{k-1}^n.$$

$$\begin{array}{ccc}
\mathbf{C}^q & \xrightarrow{b_i^{n,k}} & \mathbf{C}^p \\
U_{k-1}^n \downarrow & & \downarrow U_{n-k}^n \\
\wedge^{k-1}\mathbf{C}^n & & \wedge^{n-k}\mathbf{C}^n \\
C_{e_i}^{n,k} \downarrow & & \downarrow W_k^n \\
\wedge^{k}\mathbf{C}^n & \xrightarrow{V_k^n} & \wedge^{n-k}\mathbf{C}^n
\end{array}$$

- $U^n_{k-1}(e_I) = e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}$, where $I = \{i_1 < \cdots < i_{k-1}\}$. U^n_{n-k} is similar.

- $V^n_k(e_{i_1} \wedge \cdots \wedge e_{i_k}) = e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$, where $\{j_1 < \cdots < j_{n-k}\}$ is the complement of $\{i_1 < \cdots < i_k\}$.

- $W^n_k(e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}) = \epsilon(i, I)\epsilon(I, i, J) e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$ for any i and I such that $I \cap J = \emptyset$ and $(I \cup J)^c = \{i\}$ (which is independent of the choice of i or I).

SOME PROPERTIES OF CREATION OPERATORS

- $C_h^{n,k*} C_h^{n,k}(h_1 \wedge \cdots \wedge h_{k-1}) =$
  $(h|h) h_1 \wedge \cdots \wedge h_{k-1} - (h_1|h) h \wedge h_2 \wedge \cdots \wedge h_{k-1}$
  $$+ \cdots \pm (h_{k-1}|h) h \wedge h_1 \wedge \cdots \wedge h_{k-2}. \quad (5)$$

- $C_h^{n,k} C_h^{n,k*}(h_1 \wedge \cdots \wedge h_k) =$
  $\sum_{j=1}^k (h_j|h) h_1 \wedge \cdots \wedge h'_j \wedge \cdots \wedge h_k \quad (h'_j = h).$
  In particular, $C_h^{n,1} C_h^{n,1*} = h \otimes \overline{h}$, for $h \in \mathbf{C}^n$.

- $\mathrm{tr}(C_h^{n,k*} C_h^{n,k}) = \binom{n-1}{k-1} \|h\|^2$.
  In particular, $C_h^{n,1*} C_h^{n,1} = \|h\|^2$.

- The eigenvalues of $\sum_{i=1}^m C_{h_i}^{n,k} C_{h_i}^{n,k*}$ are sums of k eigenvalues of $\sum_{i=1}^m C_{h_i}^{n,1} C_{h_i}^{n,1*}$.

3. MODERN OPERATOR SPACES

- Classify all infinite dimensional rank 1 JC*-triples up to complete isometry (**algebraic**)

ANSWER:

$$\Phi, \quad H_\infty^{m,R}, \quad H_\infty^{m,L}, \quad H_\infty^{m,R} \cap H_\infty^{m,L}$$

- Give a suitable "classification" of all Hilbertian operator spaces which are contractively complemented in a C*-algebra or normally contractively complemented in a W*-algebra (**analytic**)

ANSWER: $\Phi, \quad \mathbf{C}, \quad \mathbf{R}, \quad \mathbf{C} \cap \mathbf{R}$

Only $\mathbf{R}$ and $\mathbf{C}$ for **completely** contractively complemented Hilbertian operator spaces (Robertson 1991)

To define the two intersections, we need a new construction in operator space theory involving interpolation theory**.

operator spaces—re-revisited

(the morphisms)

- An operator space is a subspace X of $B(H)$.
- Its *operator space structure* is given by the sequence of norms on the set of matrices $M_n(X)$ with entries from X, determined by the identification $M_n(X) \subset M_n(B(H)) = B(H \oplus H \oplus \cdots \oplus H)$.
- A linear mapping $\varphi : X \to Y$ between two operator spaces is *completely bounded* if the induced mappings $\varphi_n : M_n(X) \to M_n(Y)$ defined by $\varphi_n([x_{ij}]) = [\varphi(x_{ij})]$ satisfy $\|\varphi\|_{\text{cb}} := \sup_n \|\varphi_n\| < \infty$.

**interpolation theory is also used to define the operator space structure of the Schatten classes (noncommutative L^p-spaces) mentioned in connection with projective stability and projective rigidity back in section IV

(the history)

- Origins in the work of Stinespring in the 1950s, and Arveson in the 1960s.
- Many tools were developed in the 1970s and 1980s by a number of operator algebraists
- An abstract framework was developed in 1988 in the thesis of Ruan.
- References: books by Effros-Raun (2000), Paulsen (2002), Pisier (2003), Blecher-LeMerdy (2004)

(column and row revisited)

- Two important examples of Hilbertian operator spaces (:= operator spaces isometric to Hilbert space) are the row and column spaces R, C, and their finite-dimensional versions $R_n,\ C_n$
- In $B(\ell_2)$, *column Hilbert space* $C := \overline{\text{sp}}\{e_{i1} : i \geq 1\}$ and *row Hilbert space* $R := \overline{\text{sp}}\{e_{1j} : j \geq 1\}$
- R and C are Banach isometric, but not completely isomorphic ($\mathrm{d}_{\mathrm{cb}}(R, C) = \infty$)
- R_n and C_n are completely isomorphic, but not completely isometric
- $\mathrm{d}_{\mathrm{cb}}(R_n, C_n) = n$

(homogeneity revisited)

- R, C, R_n, C_n are examples of *homogeneous* operator spaces, that is, operator spaces E for which $\forall u : E \to E$, $\|u\|_{\text{cb}} = \|u\|$
- Another important example of an Hilbertian homogeneous operator space is $\Phi(I)$
- $\Phi(I) = \overline{\text{sp}}\{V_i : i \in I\}$, where the V_i are bounded operators on a Hilbert space satisfying the canonical anti-commutation relations
- In some special cases, the notations $\Phi_n := \Phi(\{1, 2, \ldots, n\})$, and $\Phi = \Phi(\{1, 2, \ldots\})$ are used.

(intersection)

- If $E_0 \subset B(H_0)$ and $E_1 \subset B(H_1)$ are operator spaces whose underlying Banach spaces form a compatible pair in the sense of interpolation theory, then the Banach space $E_0 \cap E_1$, with the norm

$$\|x\|_{E_0 \cap E_1} = \max(\|x\|_{E_0}, \|x\|_{E_1})$$

  equipped with the operator space structure given by the embedding $E_0 \cap E_1 \ni x \mapsto (x, x) \in E_0 \oplus E_1 \subset B(H_0 \oplus H_1)$ is called the *intersection* of E_0 and E_1 and is denoted by $E_0 \cap E_1$

- EXAMPLES
  $R \cap C,\ \Phi = \cap_0^\infty H_\infty^{m,L},\ \Phi_n = \cap_1^n H_n^k$

- The definition of intersection extends easily to arbitrary families of compatible operator spaces

Digression #10
RANK ONE JC*-TRIPLES

(JC*-triples revisited)

- A JC^*-triple is a norm closed complex linear subspace of $B(H,K)$ (equivalently, of a C^*-algebra) which is closed under the operation $a \mapsto aa^*a$.
- JC^*-triples were defined and studied (using the name J^*-algebra) as a generalization of C^*-algebras by Harris in connection with function theory on infinite dimensional bounded symmetric domains.
- By a polarization identity (involving $\sqrt{-1}$), any JC^*-triple is closed under the triple product

$$(a,b,c) \mapsto \{abc\} := \frac{1}{2}(ab^*c + cb^*a), \qquad (6)$$

under which it becomes a Jordan triple system.

- A linear map which preserves the triple product (6) will be called a *triple homomorphism*.
- Cartan factors are examples of JC^*-triples, as are C^*-algebras, and Jordan C^*-algebras.
- We shall only make use of Cartan factors of type 1, that is, spaces of the form $B(H,K)$ where H and K are complex Hilbert spaces.

(TROs revisited)

- A special case of a JC^*-triple is a *ternary algebra*, that is, a subspace of $B(H,K)$ closed under the *ternary product* $(a,b,c) \mapsto ab^*c$.
- A *ternary homomorphism* is a linear map ϕ satisfying $\phi(ab^*c) = \phi(a)\phi(b)^*\phi(c)$.
- These spaces are also called ternary rings of operators and abbreviated TRO.

- TROs have come to play a key role in operator space theory, serving as the algebraic model in the category.
- The algebraic models for the categories of order-unit spaces, operator systems, and Banach spaces, are respectively Jordan C^*-algebras, C^*-algebras, and JB^*-triples.
- For TROs, a ternary isomorphism is the same as a complete isometry.

(rank-one JC*-triples)

- Every JW^*-triple of rank one is isometric to a Hilbert space and every maximal collinear family of partial isometries corresponds to an orthonormal basis.

- Conversely, every Hilbert space with the abstract triple product $\{xyz\} := ((x|y)z + (z|y)x)/2$ can be realized as a JC^*-triple of rank one in which every orthonormal basis forms a maximal family of mutually collinear minimal partial isometries.

- collinear means:

$$vv^*w + wv^*v = w \text{ and } ww^*v + vw^*w = v$$

OPERATOR SPACE STRUCTURE OF HILBERTIAN JC*-TRIPLES

- The general setting: Y is a JC^*-subtriple of $B(H)$ which is Hilbertian in the operator space structure arising from $B(H)$, and $\{u_i : i \in \Omega\}$ is an orthonormal basis consisting of a maximal family of mutually collinear partial isometries of Y.

- We let T and A denote the TRO and the C^*-algebra respectively generated by Y. For any subset $G \subset \Omega$, $(uu^*)_G := \prod_{i \in G} u_i u_i^*$ and $(u^*u)_G := \prod_{i \in G} u_i^* u_i$. The elements $(uu^*)_G$ and $(u^*u)_G$ lie in the weak closure of A and more generally in the left and right linking von Neumann algebras of T.

Fix $m \geq 0$. To construct $H_\infty^{m,R}$ we make an assumption on the ternary envelope of Y.

- **Assume** $(u^*u)_G \neq 0$ for $|G| \leq m+1$ and $(u^*u)_G = 0$ for $|G| \geq m+2$.
- Define elements which are indexed by an arbitrary pair of subsets I, J of Ω satisfying

$$|\Omega - I| = m+1, \ |J| = m, \qquad (7)$$

as follows:
$u_{IJ} =$
$(uu^*)_{I-J} u_{c_1} u^*_{d_1} u_{c_2} u^*_{d_2} \cdots u_{c_s} u^*_{d_s} u_{c_{s+1}} (u^*u)_{J-I}$,
where
$I \cap J = \{d_1, \ldots, d_s\}$ and $(I \cup J)^c = \{c_1, \ldots, c_{s+1}\}$.

- In the special case where $I \cap J = \emptyset$, $u_{I,J}$ has the form

$$u_{I,J} = (uu^*)_I u_c (u^*u)_J,$$

where $I \cup J \cup \{c\} = \Omega$ is a partition of Ω. We call such an element a "one", and denote it also by $u_{I,c,J}$.

- LEMMA Fix $m \geq 0$.
Assume $(u^*u)_G \neq 0$ for $|G| \leq m+1$ and $(u^*u)_G = 0$ for $|G| \geq m+2$. For any $c \in \Omega$,

$$u_c = \sum_{I,J} u_{I,J} = \sum_{I,J} u_{I,c,J} \tag{8}$$

where the sum is taken over all disjoint I, J satisfying (7) and not containing c, and converges weakly in the weak closure of T.

- The family $\{\epsilon(IJ)u_{I,J}\}$ forms a rectangular grid which satisfies the extra property

$$\epsilon(IJ)u_{IJ}[\epsilon(IJ')u_{IJ'}]^*\epsilon(I'J')u_{I'J'} = \epsilon(I'J)u_{I'J}. \tag{9}$$

- The map $\epsilon(IJ)u_{IJ} \to E_{JI}$ is a ternary isomorphism (and hence complete isometry) from the norm closure of $\mathrm{sp}_C\, u_{IJ}$ to the norm closure of $\mathrm{sp}_C\,\{E_{JI}\}$, where E_{JI} denotes an elementary matrix, whose rows and columns are indexed by the sets J and I, with a 1 in the (J, I)-position.

- This map can be extended to a ternary isomorphism from the w*-closure of $\mathrm{sp}_C\, u_{IJ}$ onto the Cartan factor of type I consisting of all $\aleph_0$ by $\aleph_0$ complex matrices which act as bounded operators on ℓ_2. By restriction to Y and (8), Y is completely isometric to a subtriple $\tilde{Y}$, of this Cartan factor of type 1.

- DEFINITION
  We shall denote the space $\tilde{Y}$ above by $H_\infty^{m,R}$.

- Explicitly, $H_\infty^{m,R} = \overline{\mathsf{sp}}_C\{b_i^m : i \in \mathbf{N}\}$, where

$$b_i^m = \sum_{I\cap J=\emptyset,(I\cup J)^c=\{i\},|J|=m} \epsilon(I,i,J)e_{J,I}.$$

- An entirely symmetric argument (with J infinite and I finite) under an entirely symmetric assumption on Y defines the space $H_\infty^{m,L}$.

- Explicitly, $H_\infty^{m,L} = \overline{\mathsf{sp}}_C\{\tilde{b}_i^m : i \in \mathbf{N}\}$, where

$$\tilde{b}_i^m = \sum_{I\cap J=\emptyset,(I\cup J)^c=\{i\},|I|=m} \epsilon(I,i,J)e_{J,I},$$

  with $\epsilon(I,i,J)$ defined in the obvious analogous way with I finite instead of J.

- Having constructed the spaces $H_\infty^{m,R}$ and $H_\infty^{m,L}$, we now assume that Y is an arbitrary rank-one JC*-subtriple of B(H)

  Our analysis will consider the following three mutually exhaustive and mutually exclusive possibilities (in each case, the set F is allowed to be empty):

  Case 1 $(uu^*)_{\Omega - F} \neq 0$ for some finite set $F \subset \Omega$;

  Case 2 $(u^*u)_{\Omega - F} \neq 0$ for some finite set $F \subset \Omega$;

  Case 3 $(uu^*)_{\Omega - F} = (u^*u)_{\Omega - F} = 0$ for all finite subsets F of Ω.

- In case 1,
  Y is completely isometric to an intersection $Y_1 \cap Y_2$ such that Y_1 is completely isometric to a space $H_\infty^{m,R}$, and Y_2 is a Hilbertian JC^*-triple.

- In case 2,
  Y is completely isometric to an intersection $Y_1 \cap Y_2$ such that Y_1 is completely isometric to a space $H_\infty^{m,L}$, and Y_2 is an Hilbertian JC^*-triple.

- In case 3,
  Y is completely isometric to Φ.

THEOREM 1

Let Y be a JC^*-subtriple of $B(H)$ which is a separable infinite dimensional Hilbertian operator space. Then Y is completely isometric to one of the following spaces:

$$\Phi, \quad H_\infty^{m,R}, \quad H_\infty^{m,L}, \quad H_\infty^{m,R} \cap H_\infty^{n,L}.$$

PROOF: Let $\equiv$ denote "completely isometric to."

1. $Y \equiv \Phi$, or $Y = Y_1 \cap Z$, where $Y_1 \equiv H_\infty^{m,R}$ or $H_\infty^{k,L}$.
2. $Y_1 \equiv H_\infty^{m_1,R} \Rightarrow Y = \mathcal{R} \cap Z$, where $\mathcal{R} = \cap H_\infty^{m_j,R}$, $m_j \uparrow$ (finite or infinite).
3. $Z \equiv \Phi$, or $Z = Z_1 \cap W$, where $Z_1 \equiv H_\infty^{k,L}$.
4. $Z = \mathcal{L} \cap W$, where $\mathcal{L} = \cap_j H_\infty^{k_j,L}$, $k_j \uparrow$. Hence $Y = \mathcal{R} \cap \mathcal{L} \cap \Phi$.
5. $H_\infty^{m',L} \to H_\infty^{m,L}$ is a complete contraction when $m' > m$ (and similarly for $H_\infty^{k,R}$).
6. Φ is completely isometric to $\cap_{m=0}^{\infty} H_\infty^{m,L}$ and to $\cap_{m=1}^{\infty} H_\infty^{m,R}$ and to $\cap_{j=1}^{\infty} H_\infty^{n_j,L}$ for any sequence $n_j \to \infty$ QED

(further properties of $H_\infty^{m,R}$, $H_\infty^{m,L}$)

Representation on the Fock space

H separable Hilbert space,
$l_m(h) : H^{\wedge m} \to H^{\wedge m+1}$ creation operator
$l_m(h)x = h \wedge x$.
Creation operators $\mathcal{C}^m = \overline{\text{sp}}\{l_m(e_i)\}$, $\{e_i\}$ an orthonormal basis.
Operator space structure from $B(H^{\wedge m}, H^{\wedge m+1})$.
Annihilation operators $\mathcal{A}^m$ consists of the adjoints of the creation operators on $H^{\wedge m-1}$.

LEMMA

$H_\infty^{m,R}$ is completely isometric to $\mathcal{A}^{m+1}$ and $H_\infty^{m,L}$ is completely isometric to $\mathcal{C}^m$.

REMARK

Every finite or infinite dimensional separable Hilbertian JC*-subtriple Y is completely isometric to a finite or infinite intersection of spaces of creation and annihilation operators, as follows

(a) If Y is infinite dimensional, then it is completely isometric to one of

$$\mathcal{A}^m, \quad \mathcal{C}^m, \quad \mathcal{A}^m \cap \mathcal{C}^k, \quad \cap_{k=1}^{\infty} \mathcal{C}^k$$

(b) If Y is of dimension n, then Y is completely isometric to $\cap_{j=1}^{m} \mathcal{C}^{k_j}$, where $n \geq k_1 > \cdots > k_m \geq 1$

Completely bounded Banach-Mazur distance

THEOREM 2

For $m,\ k \geq 1$,

(a) $d_{cb}(H_\infty^{m,R}, H_\infty^{k,R}) = d_{cb}(H_\infty^{m,L}, H_\infty^{k,L}) = \sqrt{\frac{m+1}{k+1}}$ when $m \geq k$

(b) $d_{cb}(H_\infty^{m,R}, H_\infty^{k,L}) = \infty$

(c) $d_{cb}(H_\infty^{m,R}, \Phi) = d_{cb}(H_\infty^{m,L}, \Phi) = \infty$

(contractively complemented Hilbertian operator spaces)

THEOREM 3

Suppose Y is a separable infinite dimensional Hilbertian operator space which is contractively complemented (resp. normally contractively complemented) in a C^*-algebra A (resp. W*-algebra A) by a projection P.

Then,

(a) $\{Y, A^{**}, P^{**}\}$ (resp. $\{Y, A, P\}$) is an expansion of its support $\{H, A^{**}, Q\}$ (resp. $\{H, A^{**}, Q\}$, which is essential)

(b) H is contractively complemented in A^{**} (resp. A) by Q and is completely isometric to either R, C, $R \cap C$, or Φ.

THEOREM 4 (converse)

The operator spaces $R, C, R \cap C$, and Φ are each essentially normally contractively complemented in a von Neumann algebra.

4. Operator space characterizations of C*-algebras and TRO's (Neal and Russo)

THEOREM 1

Let $A \subset B(H)$ be an operator space and suppose that $M_n(A)_0$ is a bounded symmetric domain for all $n \geq 2$, then A is ternary isomorphic and completely isometric to a TRO.

THEOREM 3

Let $A \subset B(H)$ be an operator space and suppose that $M_n(A)_0$ is a bounded symmetric domain for all $n \geq 2$ and A_0 is of tube type. Then A is completely isometric to a C*-algebra.

5. Quantum operator algebras