AUTOMATIC CONTINUITY OF DERIVATIONS AND WEAK AMENABILITY OF JB*-TRIPLES

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Report on joint work with
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JORDAN THEORY, ANALYSIS, AND RELATED TOPICS
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TWO BASIC QUESTIONS CONCERNING DERIVATIONS ON BANACH ALGEBRAS

$A \to A$ and $A \to M$ (MODULE)

1. AUTOMATIC CONTINUITY?

2. INNER?

   (IF NOT, WHY NOT?)

CONTEXTS

(i) C*-ALGEBRAS
(associative Banach algebras)

(ii) JC*-ALGEBRAS
(Jordan Banach algebras)

(iii) JC*-TRIPLES
(Banach Jordan triples)

(i') associative triple systems

(ii') Lie algebras

(iii') Lie triple systems

I. C*-ALGEBRAS

derivation: $D(ab) = a \cdot Db + Da \cdot b$

inner derivation: ad x$(a) = x \cdot a - a \cdot x$ $(x \in M)$

1. AUTOMATIC CONTINUITY RESULTS

   KAPLANSKY 1949: $C(X)$

   SAKAI 1960:

   RINGROSE 1972: (module)

2. INNER DERIVATION RESULTS

   SAKAI, KADISON 1966

   CONNES 1976 (module)

   HAAGERUP 1983 (module)

Irving Kaplansky (1917–2006)

Kaplansky made major contributions to group theory, ring theory, the theory of operator algebras and field theory.

THEOREM (Sakai 1960)
Every derivation from a C*-algebra into itself is continuous.

Soichiro Sakai (b. 1928)

THEOREM (Ringrose 1972)
Every derivation from a C*-algebra into a Banach A-**bimodule** is continuous.

John Ringrose (b. 1932)

John Ringrose is a leading world expert on non-self-adjoint operators and operator algebras. He has written a number of influential texts including Compact non-self-adjoint operators (1971) and, with R V Kadison, Fundamentals of the theory of operator algebras in four volumes published in 1983, 1986, 1991 and 1992.

Richard Kadison (b. 1925)

Richard V. Kadison is an American mathematician known for his contributions to the study of operator algebras.

THEOREM (1966-Sakai, Kadison)
EVERY DERIVATION OF A C^*-ALGEBRA IS OF THE FORM $x \mapsto ax - xa$ FOR SOME a IN THE WEAK CLOSURE OF THE C^*-ALGEBRA

POP QUIZ: WHO PROVED THIS FOR $M_n(C)$?

Gerhard Hochschild (1915–2010)

(Photo 1968)

Gerhard Paul Hochschild was an American mathematician who worked on Lie groups, algebraic groups, homological algebra and algebraic number theory.

Joseph Henry Maclagan Wedderburn
(1882–1948)

Scottish mathematician, who taught at Princeton University for most of his career. A significant algebraist, he proved that a finite division algebra is a field, and part of the Artin–Wedderburn theorem on simple algebras. He also worked on group theory and matrix algebra.

Amalie Emmy Noether (1882–1935)

Amalie Emmy Noether was an influential German mathematician known for her groundbreaking contributions to abstract algebra and theoretical physics. Described as the most important woman in the history of mathematics, she revolutionized the theories of rings, fields, and algebras. In physics, Noether's theorem explains the fundamental connection between symmetry and conservation laws.

Nathan Jacobson (1910–1999)

Nathan Jacobson was an American mathematician who was recognized as one of the leading algebraists of his generation, and he was also famous for writing more than a dozen standard monographs.

JACOBSON'S PROOF (1937)

If δ is a derivation, consider the two representations of $M_n(C)$

$$z \mapsto \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \text{ and } z \mapsto \begin{bmatrix} z & 0 \\ \delta(z) & z \end{bmatrix}$$

The first is a direct sum of two copies of the identity representation; but so is the second, since

$$\begin{bmatrix} 0 & 0 \\ \delta(z) & z \end{bmatrix} \text{ is equivalent to } \begin{bmatrix} 0 & 0 \\ 0 & z \end{bmatrix}$$

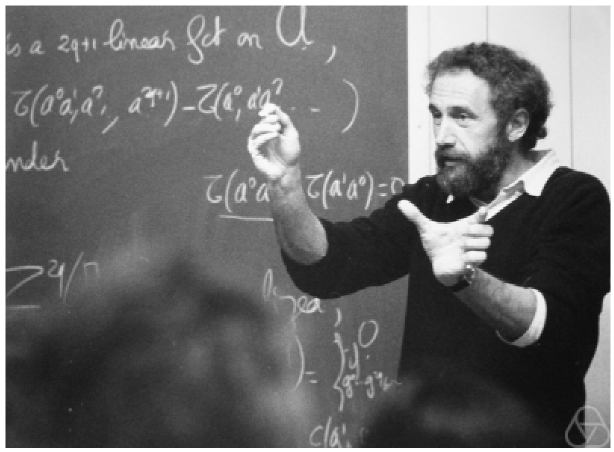
$$\text{so } \begin{bmatrix} z & 0 \\ \delta(z) & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}$$

Thus $az = za,\ bz = zb$

$\delta(z)a = cz - zc$ and $\delta(z)b = dz - zd.$

a and b are multiples of I and can't both be zero. QED

THEOREM (1976-Connes)
EVERY AMENABLE C^*-ALGEBRA IS NUCLEAR.



Alain Connes b. 1947

Alain Connes is the leading specialist on operator algebras.

In his early work on von Neumann algebras in the 1970s, he succeeded in obtaining the almost complete classification of injective factors.

Following this he made contributions in operator K-theory and index theory, which culminated in the Baum-Connes conjecture.

He also introduced cyclic cohomology in the early 1980s as a first step in the study of noncommutative differential geometry.

Connes has applied his work in areas of mathematics and theoretical physics, including number theory, differential geometry and particle physics.

THEOREM (1983-Haagerup)
EVERY NUCLEAR C^*-ALGEBRA IS AMENABLE.

THEOREM (1983-Haagerup)
EVERY C^*-ALGEBRA IS WEAKLY AMENABLE.

Uffe Haagerup b. 1950

Haagerup's research is in operator theory, and covers many subareas in the subject which are currently very active - random matrices, free probability, C*-algebras and applications to mathematical physics.

DIGRESSION
A BRIDGE TO JORDAN ALGEBRAS

A *Jordan derivation* from a Banach algebra A into a Banach A-module is a linear map D satisfying $D(a^2) = aD(a) + D(a)a$, $(a \in A)$, or equivalently,
$D(ab + ba) = aD(b) + D(b)a + D(a)b + bD(a)$,
$(a, b \in A)$.

Sinclair proved in 1970 that a **bounded** Jordan derivation from a semisimple Banach algebra to itself is a derivation, although this result fails for derivations of semisimple Banach algebras into a Banach bi-module.

Nevertheless, a celebrated result of B.E. Johnson in 1996 states that every **bounded** Jordan derivation from a C*-algebra A to a Banach A-bimodule is an associative derivation.

Alan M. Sinclair (retired)

Barry Johnson (1937–2002)

In view of the intense interest in automatic continuity problems in the past half century, it is therefore somewhat surprising that the following problem has remained open for fifteen years.

PROBLEM

Is every Jordan derivation from a C*-algebra A to a Banach A-bimodule automatically continuous (and hence a derivation, by Johnson's theorem)?

In 2004, J. Alaminos, M. Brešar and A.R. Villena gave a positive answer to the above problem for some classes of C*-algebras including the class of abelian C*-algebras

Combining a theorem of Cuntz from 1976 with the theorem just quoted yields

THEOREM
Every Jordan derivation from a C*-algebra A to a Banach A-module is continuous.

In the same way, using the solution in 1996 by Hejazian-Niknam in the commutative case we have

THEOREM
Every Jordan derivation from a C*-algebra A to a *Jordan* Banach A-module is continuous.

(Jordan module will be defined below)

These two results will also be among the consequences of our results on automatic continuity of derivations into Jordan triple modules.

(END OF DIGRESSION)

Pascual Jordan (1902–1980)

Pascual Jordan was a German theoretical and mathematical physicist who made significant contributions to quantum mechanics and quantum field theory.

II. JC*-ALGEBRA

derivation: $D(a \circ b) = a \circ Db + Da \circ b$

inner derivation: $\sum_i [L(x_i)L(a_i) - L(a_i)L(x_i)]$

$$(x_i \in M, a_i \in A)$$

$$b \mapsto \sum_i [x_i \circ (a_i \circ b) - a_i \circ (x_i \circ b)]$$

1. AUTOMATIC CONTINUITY RESULTS

   UPMEIER 1980

   HEJAZIAN-NIKNAM 1996 (module)

   ALAMINOS-BRESAR-VILLENA 2004 (module)

2. INNER DERIVATION RESULTS

   JACOBSON 1951 (module)

   UPMEIER 1980

THEOREM (1951-Jacobson)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN ALGEBRA INTO A (JORDAN) **MODULE** IS INNER

(Lie algebras, Lie triple systems)

THEOREM (1980-Upmeier)
EVERY DERIVATION OF A REVERSIBLE JC*-ALGEBRA EXTENDS TO A DERIVATION OF ITS ENVELOPING C*-ALGEBRA. (IMPLIES SINCLAIR)

THEOREM (1980-Upmeier)

1. Purely exceptional JBW-algebras have the inner derivation property
2. Reversible JBW-algebras have the inner derivation property
3. $\oplus L^\infty(S_j, U_j)$ has the inner derivation property if and only if $\sup_j \dim U_j < \infty$, U_j spin factors.

Nathan Jacobson (1910-1999)

Harald Upmeier (b. 1950)

JACOBSON'S PROOF
(1949)

First note that for any algebra, D is a derivation if and only if $[R_a, D] = R_{Da}$.

If you polarize the Jordan axiom $(a^2b)a = a^2(ba)$, you get $[R_a, [R_b, R_c]] = R_{A(b,a,c)}$ where $A(b,a,c) = (ba)c - b(ac)$ is the "associator".

From the commutative law $ab = ba$, you get

$$A(b, a, c) = [R_b, R_c]a$$

and so $[R_b, R_c]$ is a derivation, sums of which are called **inner**, forming an ideal in the Lie algebra of all derivations.

The **Lie multiplication algebra** L of the Jordan algebra A is the Lie algebra generated by the multiplication operators R_a. It is given by

$$L = \{R_a + \sum_i [R_{b_i}, R_{c_i}] : a, b_i, c_i \in A\}$$

so that L is the sum of a Lie triple system and the ideal of inner derivations.

Now let D be a derivation of a semisimple finite dimensional unital Jordan algebra A. Then $\tilde{D}$: $X \mapsto [X, D]$ is a derivation of L.

It is well known to algebraists that $L = L' + C$ where L' (the derived algebra $[L, L]$) is semisimple and C is the center of L. Also $\tilde{D}$ maps L' into itself and C to zero.

By the Cartan-Zassenhaus-Hochschild (?) Theorem, $\tilde{D}$ is an inner derivation of L' and hence also of L, so there exists $U \in L$ such that $[X, D] = [X, U]$ for all $X \in L$ and in particular $[R_a, D] = [R_a, U]$.

Then $Da = R_{Da}1 = [R_a, D]1 = [R_a, U]1 = (R_aU - UR_a)1 = a \cdot U1 - Ua$ so that $D = R_{U1} - U \in L$. Thus, $D = R_a + \sum[R_{b_i}, R_{c_i}]$ and so

$$0 = D1 = a + 0 = a \quad \text{QED}$$

Jordan triple structures

Kevin McCrimmon b. 1941

Wilhelm Kaup

Ottmar Loos + Erhard Neher

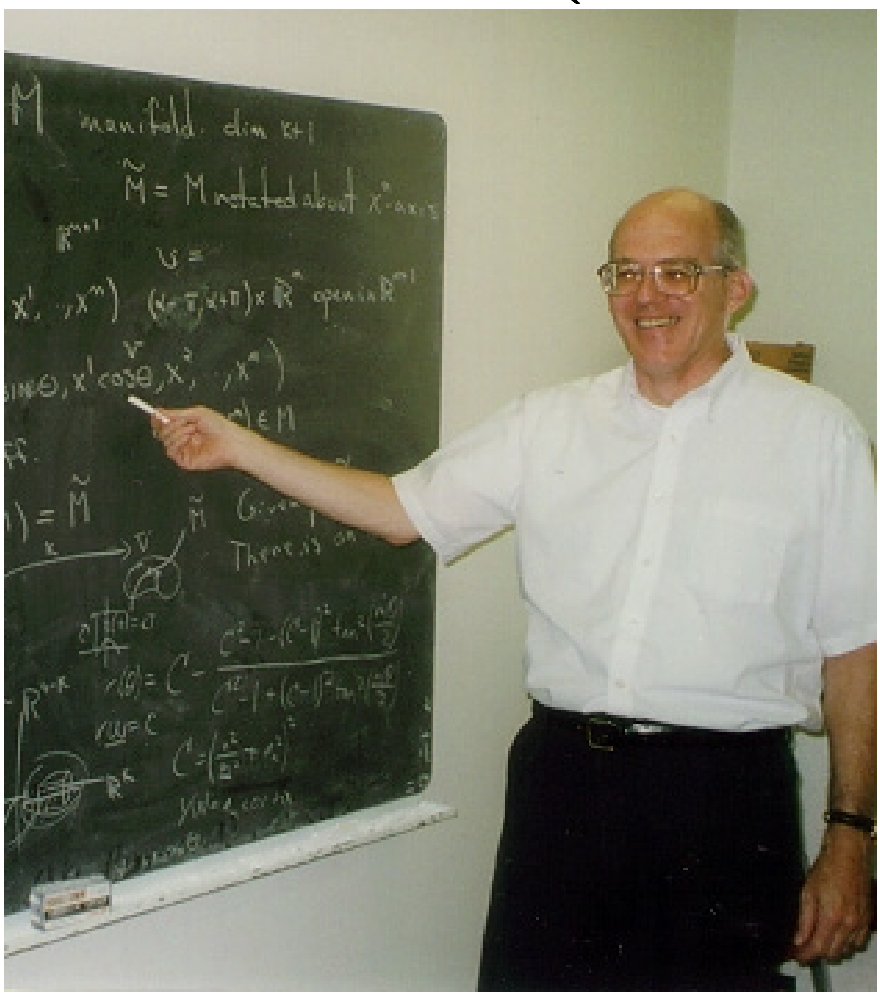
Max Koecher (1924–1990)

Max Koecher was a German mathematician. His main research area was the theory of Jordan algebras, where he introduced the Kantor-Koecher-Tits construction.

III. JC*-TRIPLE

KUDOS TO:

Lawrence A. Harris (PhD 1969)



1974 (infinite dimensional holomorphy)

1981 (spectral and ideal theory)

$$\{x, y, z\} = (xy^*z + zy^*x)/2$$

derivation:

$$D\{a,b,c\} = \{Da.b,c\} + \{a,Db,c\} + \{a,b,Dc\}$$

inner derivation: $\sum_i [L(x_i,a_i) - L(a_i,x_i)]$

$$(x_i \in M, a_i \in A)$$

$$b \mapsto \sum_i [\{x_i,a_i,b\} - \{a_i,x_i,b\}]$$

1. AUTOMATIC CONTINUITY RESULTS

   BARTON-FRIEDMAN 1990

   (**NEW**) PERALTA-RUSSO 2010 (module)

2. INNER DERIVATION RESULTS

   HO-MARTINEZ-PERALTA-RUSSO 2002

   MEYBERG 1972

   KÜHN-ROSENDAHL 1978 (module)

   (**NEW**) HO-PERALTA-RUSSO 2011 (module) weak amenability

AUTOMATIC CONTINUITY RESULTS

THEOREM (1990 Barton-Friedman)
EVERY DERIVATION OF A JB*-TRIPLE IS CONTINUOUS

THEOREM (2010 Peralta-Russo)
NECESSARY AND SUFFICIENT CONDITIONS UNDER WHICH A DERIVATION OF A JB*-TRIPLE INTO A JORDAN TRIPLE **MODULE** IS CONTINUOUS

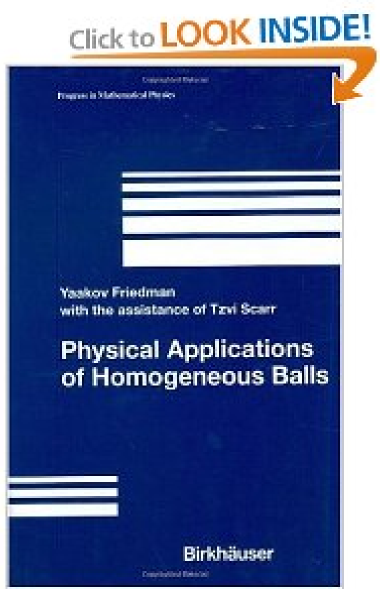
(JB*-triple and Jordan triple module are defined below)

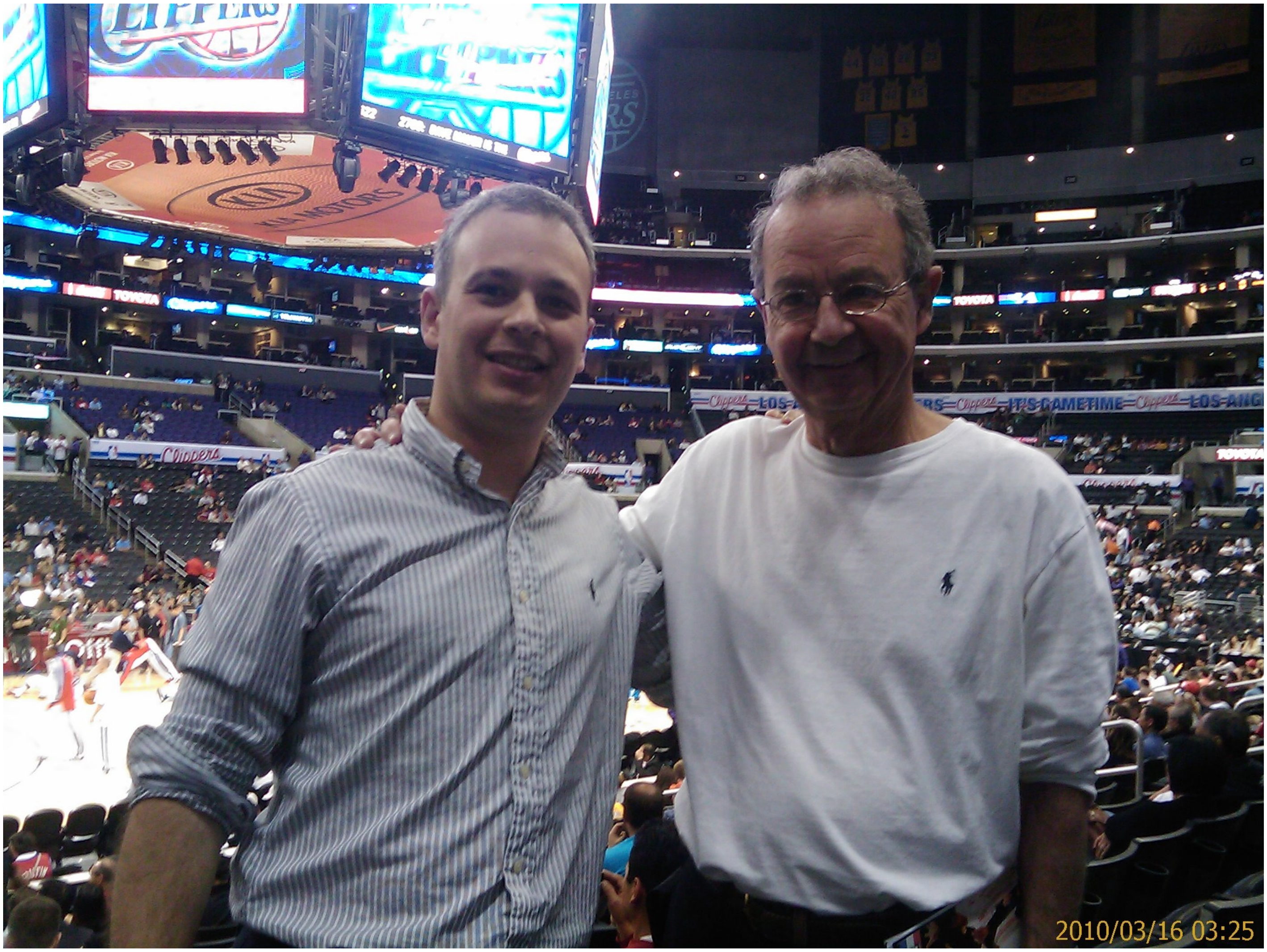
Tom Barton (b. 1955)

Tom Barton is Senior Director for Architecture, Integration and CISO at the University of Chicago. He had similar assignments at the University of Memphis, where he was a member of the mathematics faculty before turning to administration.

Yaakov Friedman (b. 1948)

Yaakov Friedman is director of research at Jerusalem College of Technology.





Antonio Peralta (b. 1974)

Bernard Russo (b. 1939)

GO LAKERS! 2010

1999 Pomona

PREVIOUS INNER DERIVATION RESULTS

FINITE DIMENSIONS

THEOREM (1972 Meyberg)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN TRIPLE SYSTEM IS INNER
(Lie algebras, Lie triple systems)

THEOREM (1978 Kühn-Rosendahl)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN TRIPLE SYSTEM INTO A MODULE IS INNER
(Lie algebras, Lie triple systems)

Kurt Meyberg

INFINITE DIMENSIONS

THEOREM 2002
(Ho-Martinez-Peralta-Russo)
CARTAN FACTORS OF TYPE $I_{n,n}$,
II (even or ∞), and III HAVE THE INNER DERIVATION PROPERTY

THEOREM 2002
(Ho-Martinez-Peralta-Russo)
INFINITE DIMENSIONAL CARTAN FACTORS OF TYPE $I_{m,n}, m \neq n$, and IV DO NOT HAVE THE INNER DERIVATION PROPERTY.

Juan Martinez Moreno

SOME CONSEQUENCES FOR JB*-TRIPLES OF OUR WORK ON AUTOMATIC CONTINUITY

1. AUTOMATIC CONTINUITY OF DERIVATION ON JB*-TRIPLE

(BARTON-FRIEDMAN)

2. AUTOMATIC CONTINUITY OF DERIVATION OF JB*-TRIPLE INTO DUAL

(SUGGESTS WEAK AMENABILITY)

3. AUTOMATIC CONTINUITY OF DERIVATION OF JB*-ALGEBRA INTO A JORDAN MODULE

(HEJAZIAN-NIKNAM)

SOME CONSEQUENCES FOR C*-ALGEBRAS OF OUR WORK ON AUTOMATIC CONTINUITY

1. AUTOMATIC CONTINUITY OF DERIVATION OF C*-ALGEBRA INTO A MODULE (RINGROSE)

2. AUTOMATIC CONTINUITY OF JORDAN DERIVATION OF C*-ALGEBRA INTO A MODULE (JOHNSON)

3. AUTOMATIC CONTINUITY OF JORDAN DERIVATION OF C*-ALGEBRA INTO A JORDAN MODULE

(HEJAZIAN-NIKNAM)

PRELIMINARY WORK ON TERNARY WEAK AMENABILITY FOR C*-ALGEBRAS AND JB*-TRIPLES (HO-PERALTA-RUSSO)

1. COMMUTATIVE C*-ALGEBRAS ARE TERNARY WEAKLY AMENABLE (TWA)
2. COMMUTATIVE JB*-TRIPLES ARE APPROXIMATELY WEAKLY AMENABLE
3. $B(H), K(H)$ ARE TWA IF AND ONLY IF FINITE DIMENSIONAL
4. CARTAN FACTORS $I_{n,1}$, IV ARE TWA IF AND ONLY IF FINITE DIMENSIONAL

SAMPLE LEMMA
The C*-algebra $A = K(H)$ of all compact operators on an infinite dimensional Hilbert space H is not Jordan weakly amenable.

By the theorems of Johnson and Haagerup, we have

$$\mathcal{D}_J(A, A^*) = \mathcal{D}_b(A, A^*) = \mathcal{I}nn_b(A, A^*).$$

We shall identify A^* with the trace-class operators on H.

Supposing that A were Jordan weakly amenable, let $\psi \in A^*$ be arbitrary. Then D_ψ ($= \operatorname{ad}\psi$) would be an inner Jordan derivation, so there would exist $\varphi_j \in A^*$ and $b_j \in A$ such that

$$D_\psi(x) = \sum_{j=1}^{n} [\varphi_j \circ (b_j \circ x) - b_j \circ (\varphi_j \circ x)]$$

for all $x \in A$.

For $x, y \in A$, a direct calculation yields

$$\psi(xy - yx) = -\frac{1}{4}\left(\sum_{j=1}^{n} b_j\varphi_j - \varphi_j b_j\right)(xy - yx).$$

It is known (Pearcy-Topping 1971) that every compact operator on a separable (which we may assume WLOG) infinite dimensional Hilbert space is a finite sum of commutators of compact operators.

By the just quoted theorem of Pearcy and Topping, every element of $K(H)$ can be written as a finite sum of commutators $[x, y] = xy - yx$ of elements x, y in $K(H)$. Thus, it follows that the trace-class operator

$$\psi = -\frac{1}{4}\left(\sum_{j=1}^{n} b_j\varphi_j - \varphi_j b_j\right)$$

is a finite sum of commutators of compact and trace-class operators, and hence has trace zero. This is a contradiction, since ψ was arbitrary.

PROPOSITION
The JB*-triple $A = M_n(C)$ is ternary weakly amenable.

By a Proposition which is a step in the proof that commutative C*-algebras are ternary weakly amenable,

$$\mathcal{D}_t(A, A^*) = \mathcal{I}nn_b^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*),$$

so it suffices to prove that

$$\mathcal{I}nn_b^*(A, A^*) \circ * \subset \mathcal{I}nn_t(A, A^*).$$

As in the proof of the Lemma, if $D \in \mathcal{I}nn_b^*(A, A^*)$ so that $Dx = \psi x - x\psi$ for some $\psi \in A^*$, then

$$\psi = [\varphi_1, b_1] - [\varphi_2, b_2] + \frac{\mathrm{Tr}\,(\psi)}{n} I,$$

where b_1, b_2 are self adjoint elements of A and φ_1 and φ_2 are self adjoint elements of A^*. It is easy to see that, for each $x \in A$, we have $D(x^*) =$

$$\{\varphi_1, 2b_1, x\} - \{2b_1, \varphi_1, x\} - \{\varphi_2, 2b_2, x\} + \{2b_2, \varphi_2, x\},$$

so that $D \circ * \in \mathcal{I}nn_t(A, A^*)$.

APPENDIX
MAIN AUTOMATIC CONTINUITY RESULT
(Jordan triples, Jordan triple modules, Quadratic annihilator, Separating spaces)

Jordan triples

A complex (resp., real) *Jordan triple* is a complex (resp., real) vector space E equipped with a non-trivial triple product

$$E \times E \times E \to E$$

$$(x, y, z) \mapsto \{xyz\}$$

which is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one satisfying the so-called "*Jordan Identity*":

$$L(a,b)L(x,y) - L(x,y)L(a,b) =$$

$$L(L(a,b)x,y) - L(x,L(b,a)y),$$

for all a, b, x, y in E, where $L(x,y)z := \{xyz\}$.

A JB*-algebra is a complex Jordan Banach algebra A equipped with an algebra involution * satisfying $\| \{a, a^*, a\} \| = \|a\|^3$, $a \in A$. (Recall that $\{a, a^*, a\} = 2(a \circ a^*) \circ a - a^2 \circ a^*)$.

A *(complex) JB*-triple* is a complex Jordan Banach triple E satisfying the following axioms:

(a) For each a in E the map $L(a, a)$ is an hermitian operator on E with non negative spectrum.

(b) $\|\{a, a, a\}\| = \|a\|^3$ for all a in A.

Every C*-algebra (resp., every JB*-algebra) is a JB*-triple with respect to the product $\{a, b, c\} = \frac{1}{2}\,(ab^*c + cb^*a)$ (resp., $\{a, b, c\} := (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*)$.

Jordan triple modules

If A is an associative algebra, an A-*bimodule* is a vector space X, equipped with two bilinear products $(a, x) \mapsto ax$ and $(a, x) \mapsto xa$ from $A \times X$ to X satisfying the following axioms:

$$a(bx) = (ab)x, \;\; a(xb) = (ax)b, \text{ and, } (xa)b = x(ab),$$

for every $a, b \in A$ and $x \in X$.

If J is a Jordan algebra, a *Jordan J-module* is a vector space X, equipped with two bilinear products $(a, x) \mapsto a \circ x$ and $(x, a) \mapsto x \circ a$ from $J \times X$ to X, satisfying:

$$a \circ x = x \circ a, \quad a^2 \circ (x \circ a) = (a^2 \circ x) \circ a, \text{ and,}$$

$$2((x \circ a) \circ b) \circ a + x \circ (a^2 \circ b) = 2(x \circ a) \circ (a \circ b) + (x \circ b) \circ a^2,$$

for every $a, b \in J$ and $x \in X$

If E is a complex Jordan triple, a *Jordan triple E-module* (also called *triple E-module*) is a vector space X equipped with three mappings

$$\{.,.,.\}_1 : X \times E \times E \to X$$
$$\{.,.,.\}_2 : E \times X \times E \to X$$
$$\{.,.,.\}_3 : E \times E \times X \to X$$ satisfying:

1. $\{x, a, b\}_1$ is linear in a and x and conjugate linear in b, $\{abx\}_3$ is linear in b and x and conjugate linear in a and $\{a, x, b\}_2$ is conjugate linear in a, b, x

2. $\{x, b, a\}_1 = \{a, b, x\}_3$, and $\{a, x, b\}_2 = \{b, x, a\}_2$ for every $a, b \in E$ and $x \in X$.

3. Denoting by $\{.,.,.\}$ any of the products $\{.,.,.\}_1$, $\{.,.,.\}_2$ and $\{.,.,.\}_3$, the identity $\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\}$, holds whenever one of the elements a, b, c, d, e is in X and the rest are in E.

It is a little bit laborious to check that the dual space, E^*, of a complex (resp., real) Jordan Banach triple E is a complex (resp., real) triple E-module with respect to the products:

$$\{a, b, \varphi\}(x) = \{\varphi, b, a\}(x) := \varphi\{b, a, x\} \qquad (1)$$

and

$$\{a, \varphi, b\}(x) := \overline{\varphi\{a, x, b\}}, \forall \varphi \in E^*, a, b, x \in E. \qquad (2)$$

For each submodule S of a triple E-module X, we define its *quadratic annihilator*, $\text{Ann}_E(S)$, as the set

$$\{a \in E : Q(a)(S) = \{a, S, a\} = 0\}.$$

Separating spaces

Separating spaces have been revealed as a useful tool in results of automatic continuity.

Let $T : X \to Y$ be a linear mapping between two normed spaces. The *separating space*, $\sigma_Y(T)$, of T in Y is defined as the set of all z in Y for which there exists a sequence $(x_n) \subseteq X$ with $x_n \to 0$ and $T(x_n) \to z$.

A straightforward application of the closed graph theorem shows that a linear mapping T between two Banach spaces X and Y is continuous if and only if $\sigma_Y(T) = \{0\}$

Main Result

THEOREM Let E be a complex JB*-triple, X a Banach triple E-module, and let $\delta : E \to X$ be a triple derivation. Then δ is continuous if and only if $\mathrm{Ann}_E(\sigma_X(\delta))$ is a (norm-closed) linear subspace of E and

$$\{\mathrm{Ann}_E(\sigma_X(\delta)), \mathrm{Ann}_E(\sigma_X(\delta)), \sigma_X(\delta)\} = 0.$$

COROLLARY Let E be a real or complex JB*-triple. Then

(a) Every derivation $\delta : E \to E$ is continuous.

(b) Every derivation $\delta : E \to E^*$ is continuous.

Elie Cartan 1869–1951

Elie Joseph Cartan was an influential French mathematician, who did fundamental work in the theory of Lie groups and their geometric applications. He also made significant contributions to mathematical physics, differential geometry, and group theory. He was the father of another influential mathematician, Henri Cartan.

II' LIE ALGEBRAS

(From Meyberg Notes 1972—Chapter 5)

An algebra L with multiplication $(x, y) \mapsto [x, y]$ is a Lie algebra if $[xx] = 0$ and

$$[[xy]z] + [[yz]x] + [[zx]y] = 0.$$

Left multiplication in a Lie algebra is denoted by ad(x): ad$(x)(y) = [x, y]$. An associative algebra A becomes a Lie algebra A^- under the product, $[xy] = xy - yx$.

The first axiom implies that $[xy] = -[yx]$ and the second (called the *Jacobi identity*) implies that $x \mapsto \text{ad}x$ is a homomorphism of L into the Lie algebra $(\text{End}\, L)^-$, that is, $\text{ad}\,[xy] = [\text{ad}\, x, \text{ad}\, y]$.

Assuming that L is finite dimensional, the Killing form is defined by $\lambda(x, y) = \mathbf{tr}\,\mathbf{ad}(x)\mathbf{ad}(y)$.

CARTAN CRITERION

A finite dimensional Lie algebra L over a field of characteristic 0 is semisimple if and only if the Killing form is nondegenerate.

A linear map D is a derivation if $D \cdot \mathrm{ad}(x) = \mathrm{ad}(Dx) + \mathrm{ad}(x) \cdot D$. Each $\mathrm{ad}(x)$ is a derivation, called an inner derivation.

THEOREM OF E. CARTAN

If the finite dimensional Lie algebra L over a field of characteristic 0 is semisimple, then every derivation is inner.

PROOF

Let D be a derivation of L. **Since $x \mapsto \mathbf{tr}\, D \cdot \mathbf{ad}\,(x)$ is a linear form, there exists $d \in L$ such that $\mathbf{tr}\, D \cdot \mathbf{ad}\,(x) = \lambda(d, x) = \mathbf{tr}\,\mathbf{ad}\,(d) \cdot \mathbf{ad}\,(x)$.** Let E be the derivation $E = D - \mathrm{ad}\,(d)$ so that

$$\mathrm{tr}\, E \cdot \mathrm{ad}\,(x) = 0. \qquad (3)$$

Note next that $E \cdot [\text{ad}\,(x), \text{ad}\,(y)] = E \cdot \text{ad}\,(x) \cdot \text{ad}\,(y) - E \cdot \text{ad}\,(y) \cdot \text{ad}\,(x) =$
$(\text{ad}\,(x) \cdot E + [E, \text{ad}\,(x)]) \cdot \text{ad}\,(y) - E \cdot \text{ad}\,(y) \cdot \text{ad}\,(x)$
so that
$[E, \text{ad}\,(x)] \cdot \text{ad}\,(y) = E \cdot [\text{ad}\,(x), \text{ad}\,(y)]$
$-\text{ad}\,(x) \cdot E \cdot \text{ad}\,(y) + E \cdot \text{ad}\,(y) \cdot \text{ad}\,(x)$
$= E \cdot [\text{ad}\,(x), \text{ad}\,(y)] + [E \cdot \text{ad}\,(y), \text{ad}\,(x)]$
and
$\text{tr}\,[E, \text{ad}\,(x)] \cdot \text{ad}\,(y) = \text{tr}\,E \cdot [\text{ad}\,(x), \text{ad}\,(y)]$.
However, since E is a derivation
$[E, \text{ad}\,(x)] \cdot \text{ad}\,(y)$
$= E \cdot \text{ad}\,(x) \cdot \text{ad}\,(y) - \text{ad}\,(x) \cdot E \cdot \text{ad}\,(y)$
$= (\text{ad}\,(Ex) + \text{ad}\,(x) \cdot E) \cdot \text{ad}\,(y) - \text{ad}\,(x) \cdot E \cdot \text{ad}\,(y)$
$= \text{ad}\,(Ex) \cdot \text{ad}\,(y)$.

Thus $\lambda(Ex, y) = \text{tr}\,\text{ad}\,(Ex) \cdot \text{ad}\,(y)$
$= \text{tr}\,[E, \text{ad}\,(x)] \cdot \text{ad}\,(y)$
$= \text{tr}\,E \cdot [\text{ad}\,(x), \text{ad}\,(y)] = 0$ **by** (3).

Since x and y are arbitrary, $E = 0$ and so $D - \text{ad}\,(d) = 0$. QED

Hans Zassenhaus (1912–1991)

Hans Julius Zassenhaus was a German mathematician, known for work in many parts of abstract algebra, and as a pioneer of computer algebra.

Sophus Lie (1842–1899)

Marius Sophus Lie was a Norwegian mathematician. He largely created the theory of continuous symmetry, and applied it to the study of geometry and differential equations.

(II") LIE OPERATOR ALGEBRAS

C. Robert Miers, Lie derivations of von Neumann algebras. DukeMath. J. 40 (1973), 403–409.

If M is a von Neumann algebra, $[M, M]$ the Lie algebra linearly generated by $\{[X, Y] = XY - YX : X, Y \in M\}$ and $L : [M, M] \to M$ a Lie derivation, i.e., L is linear and $L[X, Y] = [LX, Y] + [X, LY]$, then L has an extension $D : M \to M$ that is a derivation of the associative algebra.

The proof involves matrix-like computations.

Using the Sakai-Kadison theorem, Miers shows that if $L : M \to M$ is a Lie derivation, then $L = D + \lambda$, where D is an associative derivation and λ is a linear map into the center of M vanishing on $[M, M]$.

THEOREM (JOHNSON 1996)

EVERY CONTINUOUS LIE DERIVATION OF A C^*-ALGEBRA A INTO A BANACH BIMODULE X (IN PARTICULAR, $X = A$) IS THE SUM OF AN ASSOCIATIVE DERIVATION AND A "TRIVIAL" DERIVATION

(TRIVIAL=ANY LINEAR MAP WHICH VANISHES ON COMMUTATORS AND MAPS INTO THE "CENTER" OF THE MODULE).

"It is also an interesting open question whether or not every Lie derivation on a semisimple Banach algebra to itself has this form."

(H.G. Dales)

NOTE: Johnson's 1996 paper does not quote Miers's 1973 paper, which it partially but significantly generalizes.

The continuity assumption can be dropped in Johnson's result if $X = A$ and A is a C*-algebra or a semisimple symmetrically amenable Banach algebra

Mathieu, Martin; Villena, Armando R.
The structure of Lie derivations on C*-algebras.
J. Funct. Anal. 202 (2003), no. 2, 504–525.

Alaminos, J.; Mathieu, M.; Villena, A. R.
Symmetric amenability and Lie derivations.
Math. Proc. Cambridge Philos. Soc. 137 (2004), no. 2, 433–439.

III" LIE TRIPLE SYSTEMS

THEOREM

Every derivation of a finite dimensional semisimple Lie triple system is inner

PROOF

(From Meyberg Notes 1972—Chapter 6)

Let F be a finite dimensional semisimple Lie triple system over a field of characteristic 0 and suppose that D is a derivation of F.

Let L be the Lie algebra (Inder F) $\oplus F$ with product

$$[(H_1, x_1), (H_2, x_2)] =$$

$$([H_1, H_2] + L(x_1, x_2), H_1x_2 - H_2x_1).$$

A derivation of L is defined by $\delta(H \oplus a) = [D, H] \oplus Da$.

We take as a leap of faith that F semisimple implies L semisimple (IT'S TRUE!).

Thus there exists $U = H_1 \oplus a_1 \in L$ such that $\delta(X) = [U, X]$ for all $X \in L$.

Then $0 \oplus Da = \delta(0 \oplus a) = [H_1 + a_1, 0 \oplus a] = L(a_1, a) \oplus H_1 a$ so $L(a_1, a) = 0$ and $D = H_1 \in \operatorname{Inder} F$. QED

(III") LIE OPERATOR TRIPLE SYSTEMS

C. Robert Miers, Lie triple derivations of von Neumann algebras. Proc. Amer. Math. Soc. 71 (1978), no. 1, 57–61.

Authors summary: A Lie triple derivation of an associative algebra M is a linear map $L : M \to M$ such that

$$L[[X, Y], Z] = [[L(X), Y], Z] +$$
$$[[X, L(Y)], Z] + [[X, Y], L(Z)]$$

for all $X, Y, Z \in M$.

We show that if M is a von Neumann algebra with no central Abelian summands then there exists an operator $A \in M$ such that $L(X) = [A, X] + \lambda(X)$ where $\lambda : M \to Z_M$ is a linear map which annihilates brackets of operators in M.

III JORDAN TRIPLE SYSTEMS

(From Meyberg Notes 1972—Chapter 11)*

Let V be a Jordan triple and let $L(V)$ be its TKK Lie algebra (**Tits-Kantor-Koecher**)

$L(V) = V \oplus V_0 \oplus V$ and the Lie product is given by $[(x,h,y),(u,k,v)] = (hu - kx,\ [h,k] + x \,\square\, v - u \,\square\, y,\ k^\natural y - h^\natural v)$.

$V_0 = \text{span}\{V \,\square\, V\}$ is a Lie subalgebra of $L(V)$ and for $h = \sum_i a_i \,\square\, b_i \in V_0$, the map $h^\natural : V \to V$ is defined by $h^\natural = \sum_i b_i \,\square\, a_i$.

We can show the correspondence of derivations $\delta : V \to V$ and $D : L(V) \to L(V)$ for Jordan triple V and its TKK Lie algebra $L(V)$.

*slightly simplified by Chu and Russo 2012

Let $\theta : L(V) \to L(V)$ be the main involution
$$\theta(x \oplus h \oplus y) = y \oplus -h^\natural \oplus x$$

LEMMA 1

Let $\delta : V \to V$ be a derivation of a Jordan triple V, with TKK Lie algebra $(L(V), \theta)$. Then there is a derivation $D : L(V) \to L(V)$ satisfying

$$D(V) \subset V \quad \text{and} \quad D\theta = \theta D.$$

PROOF

Given $a, b \in V$, we define

$$\begin{aligned} D(a, 0, 0) &= (\delta a, 0, 0) \\ D(0, 0, b) &= (0, 0, \delta b) \\ D(0, a \,\square\, b, 0) &= (0,\ \delta a \,\square\, b + a \,\square\, \delta b,\ 0) \end{aligned}$$

and extend D linearly on $L(V)$. Then D is a derivation of $L(V)$ and evidently, $D(V) \subset V$.

It is readily seen that $D\theta = \theta D$, since

$$\begin{aligned} D\theta(0, a \square b, 0) &= D(0, -b \square a, 0) \\ &= (0,\ -\delta b \square a - b \square \delta a,\ 0) \\ &= \theta(0,\ \delta a \square b + a \square \delta b,\ 0) \\ &= \theta D(0, a \square b, 0). \text{QED} \end{aligned}$$

LEMMA 2

Let V be a Jordan triple with TKK Lie algebra $(L(V), \theta)$. Given a derivation $D : L(V) \to L(V)$ satisfying $D(V) \subset V$ and $D\theta = \theta D$, the restriction $D|_V : V \to V$ is a triple derivation.

THEOREM

Let V be a Jordan triple with TKK Lie algebra $(L(V), \theta)$. There is a one-one correspondence between the triple derivations of V and the Lie derivations $D : L(V) \to L(V)$ satisfying $D(V) \subset V$ and $D\theta = \theta D$.

LEMMA 3

Let V be a Jordan triple with TKK Lie algebra $(L(V), \theta)$. Let $D : L(V) \to L(V)$ be a Lie inner derivation such that $D(V) \subset V$. Then the restriction $D|_V$ is a triple inner derivation of V.

COROLLARY

Let δ be a derivation of a finite dimensional semisimple Jordan triple V. Then δ is a triple inner derivation of V.

PROOF

The TKK Lie algebra $L(V)$ is semisimple. Hence the result follows from the Lie result and Lemma 3

The proof of lemma 3 is instructive.

1. $D(x,k,y) = [(x,k,y),(a,h,b)]$ for some $(a,h,b) \in (V)$

2. $D(x,0,0) = [(x,0,0),(a,h,b)] = (-h(x), x \square b, 0)$

3. $\delta(x) = -h(x) = -\sum_i \alpha_i \square \beta_i(x)$

4. $D(0,0,y) = [(0,0,y),(a,h,b)] = (0, -a \square y, h^\natural(y))$

5. $\delta(x) = -h^\natural(x) = \sum_i \beta_i \square \alpha_i(x)$

6. $\delta(x) = \frac{1}{2}\sum_i(\beta_i \square \alpha_i - \alpha_i \square \beta_i)(x)$

QED

I ASSOCIATIVE TRIPLE SYSTEMS

EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE ASSOCIATIVE TRIPLE SYSTEM IS INNER

LISTER 1971

EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE ASSOCIATIVE TRIPLE SYSTEM INTO A MODULE IS INNER

CARLSSON 1976

I' ASSOCIATIVE OPERATOR TRIPLE SYSTEMS

Borut Zalar, On the structure of automorphism and derivation pairs of B*-triple systems. Topics in Operator Theory, operator algebras and applications (Timisoara,1994),265-271, Rom. Acad.,Bucharest, 1995

Let $W \subset B(H,K)$ be a TRO which contains all the compact operators. If D is a derivation of W with respect to the associative triple product ab^*c then there exist $a = -a^* \in B(K)$ and $b = -b^* \in B(H)$ such that $Dx = ax + xb$.

Extended to $B(X,Y)$ (X, Y Banach spaces) in

Maria Victoria Velasco and Armando R. Villena; Derivations on Banach pairs. Rocky Mountain J. Math 28 1998 1153–1187.

Magnus Hestenes (1906–1991)

Magnus Rudolph Hestenes was an American mathematician. Together with Cornelius Lanczos and Eduard Stiefel, he invented the conjugate gradient method.

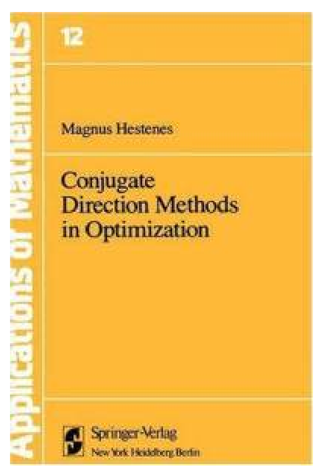


Table 1

ALGEBRAS

commutative algebras

$$ab = ba$$

associative algebras

$$a(bc) = (ab)c$$

Lie algebras

$$a^2 = 0$$

$$(ab)c + (bc)a + (ca)b = 0$$

Jordan algebras

$$ab = ba$$

$$a(a^2b) = a^2(ab)$$

Table 2

TRIPLE SYSTEMS

associative triple systems

$$(abc)de = ab(cde) = a(dcb)e$$

Lie triple systems

$$aab = 0$$

$$abc + bca + cab = 0$$

$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

Jordan triple systems

$$abc = cba$$

$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$