A HOLOMORPHIC CHARACTERIZATION OF OPERATOR ALGEBRAS

(JOINT WORK WITH MATT NEAL)

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ABSTRACT

Unital operator algebras are characterized among operator spaces in terms of the holomorphic structure associated with the underlying Banach space.

INTRODUCTION

If an operator space A (i.e., a closed linear subspace of $B(H)$) is also a unital (not necessarily associative) Banach algebra with respect to a product which is completely contractive, then according to the theorem of Blecher,Ruan, and Sinclair (JFA 1990), it is completely isometric via an algebraic isomorphism to an operator algebra (i.e., an associative subalgebra of some $B(K)$).

Our main result drops the algebra assumption on A in favor of a holomorphic assumption. Using only natural conditions on holomorphic vector fields on Banach spaces, we are able to construct an algebra product on A which is completely contractive and unital, so that the Blecher-Ruan-Sinclair result can be applied.

Our result is thus an instance where the consideration of a ternary product, called the partial triple product, which arises from the holomorphic structure via the symmetric part of the Banach space, leads to results for binary products.

Examples of this phenomenon occurred in papers of Arazy and Solel (JFA 1990) and Arazy (Math. Scan. 1994) where this technique is used to describe the algebraic properties of isometries of certain operator algebras.

The technique was also used in by Kaup and Upmeier (PAMS 1978) to show that Banach spaces with holomorphically equivalent unit balls are linearly isometric .

Our main technique is to use a variety of elementary isometries on n by n matrices over A (most of the time, $n = 2$) and to exploit the fact that isometries of arbitrary Banach spaces preserve the partial triple product.

The first occurrence of this technique appears in the construction, for each $n \geq 1$, of a contractive projection P_n on $K\overline{\otimes}A$ ($K=$ compact operators on separable infinite dimensional Hilbert space) with range $M_n(A)$, as a convex combination of isometries.

We define the completely symmetric part of A to be the intersection of A (embedded in $K\overline{\otimes}A$) and the symmetric part of $K\overline{\otimes}A$ and show it is the image under P_1 of the symmetric part of $K\overline{\otimes}A$.

It follows from a result of Neal and Russo (PJM 2003) that the completely symmetric part of A is a TRO, which is a crucial tool in our work.

We note that if A is a subalgebra of $B(H)$ containing the identity operator I, then by the Arazy-Solel paper, the symmetric part of $K\overline{\otimes}A$ is the maximal C^*-subalgebra of $K\overline{\otimes}B(H)$ contained in $K\overline{\otimes}A$, namely $K\overline{\otimes}A \cap (K\overline{\otimes}A)^*$.

This shows that the completely symmetric part of A coincides with the symmetric part of A, and therefore contains I.

Our main result is the following theorem, in which for any element v in the symmetric part of a Banach space X, h_v denotes the corresponding complete holomorphic vector field on the open unit ball of X.

THEOREM

An operator space A is completely isometric to a unital operator algebra if and only there exists an element v in the completely symmetric part of A such that:

1. For $x \in A$,

$$h_v(x+v) - h_v(x) - h_v(v) + v = -2x$$

2. For $X \in M_n(A)$, $V = \mathrm{diag}\,(v, \ldots, v) \in M_n(A)$,

$$\|V - h_V(X)\| \leq \|X\|^2$$

Although we have phrased this theorem in holomorphic terms, it should be noted that the two conditions can be restated in terms of partial triple products as

$$\{xvv\} = x \text{ and } \|\{XVX\}\| \leq \|X\|^2.$$

As another example, if A is a TRO (i.e., a closed subspace of $B(H)$ closed under the ternary product ab^*c), then Since $K\overline{\otimes}B(H)$ is a TRO, hence a JC^*-triple, it is equal to its symmetric part, which shows that the completely symmetric part of A coincides with A.

Now suppose that the TRO A contains an element v satisfying $xv^*v = vv^*x = x$ for all $x \in X$. Then it is trivial that A becomes a unital C*-algebra for the product xv^*y, involution vx^*v, and unit v

By comparison, our main result starts only with an operator space A containing a distinguished element v in the completely symmetric part of A (defined below) having a unit-like property.

This is to be expected since the result of Blecher, Ruan, and Sinclair fails in the absence of a unit element.

STEPS IN THE PROOF

1. The completely symmetric part of an arbritary operator space A is defined.

2. The binary product $x \cdot y$ on A is constructed using properties of isometries on 2 by 2 matrices over A. The symmetrized product $x \cdot y + y \cdot x$ is expressed in terms of the partial Jordan triple product on A. (namely $= 2\{xvy\}$)

3. A formula is proved relating the matrix product $X \cdot Y$ induced by $x \cdot y$ to a product of 2 by 2 matrices containing X and Y as blocks. (first for X and Y 2 by 2, then n by n)

namely,

$$\left[\begin{array}{cc} 0 & Y \cdot X \\ 0 & 0 \end{array}\right] = 2\left[\begin{array}{cc} Y & 0 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} v \otimes I_n & 0 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} 0 & X \\ 0 & 0 \end{array}\right].$$

According to a paper of Blecher and Zarikian (PNAS 2004),

"The one-sided multipliers of an operator space X are a key to the 'latent operator algebraic structure' in X."

The unified approach through multiplier operator algebras developed in that paper leads to simplifications of known results and applications to quantum M-ideal theory.

They also state
"With the extra structure consisting of the additional matrix norms on an operator algebra, one might expect to not have to rely as heavily on other structure, such as the product."

Our result is certainly in the spirit of this statement.

Another approach to operator algebras is in a paper of Kaneda (JFA 2007) in which the set of operator algebra products on an operator space is shown to be in bijective correspondence with the space of norm one quasi-multipliers on the operator space.

BACKGROUND

operator spaces, Jordan triples, and holomorphy in Banach spaces.

Operator spaces

By an **operator space**, sometimes called a quantum Banach space, we mean a closed linear subspace A of $B(H)$ for some complex Hilbert space H, equipped with the matrix norm structure obtained by the identification of $M_n(B(H))$ with $B(H \oplus H \oplus \cdots \oplus H)$.

Two operator spaces are **completely isometric** if there is a linear isomorphism between them which, when applied elementwise to the corresponding spaces of n by n matrices, is an isometry for every $n \geq 1$.

By an **operator algebra**, sometimes called a quantum operator algebra, we mean a closed associative subalgebra A of $B(H)$, together with its matrix norm structure as an operator space.

One important example of an operator space is a **ternary ring of operators**, or TRO, which is an operator space in $B(H)$ which contains ab^*c whenever it contains a, b, c.

A TRO is a special case of a JC^*-**triple**, that is, a closed subspace of $B(H)$ which contains the symmetrized ternary product $ab^*c + cb^*a$ whenever it contains a, b, c.

Jordan triples

More generally, a JB^*-**triple** is a complex Banach space equipped with a triple product $\{x, y, z\}$ which is linear in the first and third variables, conjugate linear in the second variable, satisfies the algebraic identities

$$\{x, y, z\} = \{z, y, x\}$$

and

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\}$$
$$-\{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$$

and the analytic conditions that the linear map $y \mapsto \{x, x, y\}$ is hermitian and positive and $\|\{x, x, x\}\| = \|x\|^3$.

The following two theorems are instrumental in what follows.

THEOREM
(Kaup MZ 1983)

The class of JB^*-triples coincides with the class of complex Banach spaces whose open unit ball is a bounded symmetric domain.

THEOREM
(Friedman-Russo JFA 1985,Kaup MS 1984,Stacho AM Szeged 1982)

The class of JB^*-triples is stable under contractive projections. More precisely, if P is a contractive projection on a JB^*-triple E with triple product denoted by $\{x,y,z\}_E$, then $P(E)$ is a JB^*-triple with triple product given by $\{a,b,c\}_{P(E)} = P\{a,b,c\}_E$ for $a,b,c \in P(E)$.

The following two theorems have already been mentioned above.

THEOREM
(Blecher,Ruan,Sinclair JFA 1990)

If an operator space supports a unital Banach algebra structure in which the product (not necessarily associative) is completely contractive, then the operator space is completely isometric to an operator algebra.

THEOREM
(Neal-Russo PJM 2003)

If an operator space has the property that the open unit ball of the space of n by n matrices is a bounded symmetric domain for every $n \geq 2$, then the operator space is completely isometric to a TRO.

Holomorphy in Banach spaces

Finally, we review the construction and properties of the partial Jordan triple product in an arbitrary Banach space.

Let X be a complex Banach space with open unit ball X_0.

Every holomorphic function $h : X_0 \to X$, also called a holomorphic vector field, is locally integrable, that is, the initial value problem

$$\frac{\partial}{\partial t}\varphi(t,z) = h(\varphi(t,z)) \ , \ \varphi(0,z) = z,$$

has a unique solution for every $z \in X_0$ for t in a maximal open interval J_z containing 0.

A **complete holomorphic vector field** is one for which $J_z = R$ for every $z \in X_0$.

It is a fact that every complete holomorphic vector field is the sum of the restriction of a skew-Hermitian bounded linear operator A on X and a function h_a of the form $h_a(z) = a - Q_a(z)$, where Q_a is a quadratic homogeneous polynomial on X.

The **symmetric part** of X is the orbit of 0 under the set of complete holomorphic vector fields, and is denoted by $S(X)$. It is a closed subspace of X and is equal to X precisely when X has the structure of a JB^*-triple (by the Theorem of Kaup).

If $a \in S(X)$, we can obtain a symmetric bilinear form on X, also denoted by Q_a via the polarization formula

$$Q_a(x, y) = \frac{1}{2}\left(Q_a(x + y) - Q_a(x) - Q_a(y)\right)$$

and then the partial Jordan triple product $\{\cdot,\cdot,\cdot\} : X \times S(X) \times X \to X$ is defined by $\{x, a, z\} = Q_a(x, z)$. The space $S(X)$ becomes a JB^*-triple in this triple product.

It is also true that the "main identity"

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\}$$
$$-\{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$$

holds whenever $a, y, b \in S(X)$ and $x, z \in X$.

The following lemma is an elementary consequence of the definitions.

LEMMA 0

If ψ is a linear isometry of a Banach space X onto itself, then

(a) For every complete holomorphic vector field h on X_0, $\psi \circ h \circ \psi^{-1}$ is a complete holomorphic vector field. In particular, for $a \in S(X)$, $\psi \circ h_a \circ \psi^{-1} = h_{\psi(a)}$.

(b) $\psi(S(X)) = S(X)$ and ψ preserves the partial Jordan triple product:

$$\psi\{x, a, y\} = \{\psi(x), \psi(a), \psi(y)$$

for $a \in S(X),\ x, y \in X$.

The symmetric part of a Banach space behaves well under contractive projections

THEOREM (Stacho AM Szeged 1982)

If P is a contractive projection on a Banach space X and h is a complete holomorphic vector field on X_0, then $P \circ h|_{P(X)_0}$ is a complete holomorphic vector field on $P(X)_0$. In addition $P(S(X)) \subset S(X)$ and the partial triple product on $P(S(X))$ is given by $\{x, y, z\} = P\{x, y, z\}$ for $x, z \in P(X)$ and $y \in P(S(X))$.

Examples of the symmetric part $S(X)$ of a Banach space X

- $X = L_p(\Omega, \Sigma, \mu)$, $1 \leq p < \infty$, $p \neq 2$;
  $S(X) = 0$

- $X =$ (classical) H_p, $1 \leq p < \infty$, $p \neq 2$;
  $S(X) = 0$

- $X = H_\infty$ (classical) or the disk algebra;
  $S(X) = \mathbf{C}$

- $X =$ a uniform algebra $A \subset C(K)$;
  $S(A) = A \cap \overline{A}$

  (Braun-Kaup-Upmeier 1978)

- $X =$ unital subalgebra of $B(H)$
  $S(A) = A \cap A^*$

  (Arazy-Solel 1994)

More examples, due primarily to Stacho, and involving Reinhardt domains are recited in Arazy's survey paper, along with the following unpublished example due to Vigue

PROPOSITION

There exists an equivalent norm on ℓ_∞ so that ℓ_∞ in this norm has symmetric part equal to c_0

PROBLEM 1

Is the symmetric part of the predual of a JBW*-triple equal to 0?. What about the predual of a von Neumann algebra?

1. Completely symmetric part of an operator space

Let $A \subset B(H)$ be an operator space. We let K denote the compact operators on a separable infinite dimensional Hilbert space, say ℓ_2. Then $K = \overline{\cup_{n=1}^{\infty} M_n(\mathbf{C})}$ and thus

$$K \overline{\otimes} A = \overline{\cup_{n=1}^{\infty} M_n \otimes A} = \overline{\cup_{n=1}^{\infty} M_n(A)}$$

By an abuse of notation, we shall use $K \otimes A$ to denote $\cup_{n=1}^{\infty} M_n(A)$. We tacitly assume the embeddings $M_n(A) \subset M_{n+1}(A) \subset K \overline{\otimes} A$ induced by adding zeros.

The *completely symmetric part* of A is defined by $CS(A) = A \cap S(K \overline{\otimes} A)$., where we have identified A with $M_1(A) \subset K \overline{\otimes} A$.

For $1 \leq m < N$ let $\psi_{1,m}^N : M_N(A) \to M_N(A)$ and $\psi_{2,m}^N : M_N(A) \to M_N(A)$ be the isometries of order two defined on

$$\left[\begin{array}{cc} M_m(A) & M_{m,N-m}(A) \\ M_{N-m,m}(A) & M_{N-m}(A) \end{array} \right]$$

by

$$\psi_{1,m}^N : \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \to \left[\begin{array}{cc} a & -b \\ -c & d \end{array} \right]$$

and

$$\psi_{2,m}^N : \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \to \left[\begin{array}{cc} a & -b \\ c & -d \end{array} \right].$$

These two isometries give rise in an obvious way to two isometries $\tilde{\psi}_{1,m}$ and $\tilde{\psi}_{2,m}$ on $K \otimes A$, which extend to isometries $\psi_{1,m}, \psi_{2,m}$ of $K \overline{\otimes} A$ onto itself, of order 2 and fixing elementwise $M_m(A)$. The same analysis applies to the isometries defined by, for example,

$$\left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \to \left[\begin{array}{cc} a & b \\ -c & -d \end{array} \right], \left[\begin{array}{cc} -a & -b \\ c & d \end{array} \right], \left[\begin{array}{cc} -a & b \\ c & -d \end{array} \right].$$

We then can define a projection $\tilde{P}_m$ on $K \otimes A$ with range $M_m(A)$ via

$$\tilde{P}_m x = \frac{\tilde{\psi}_{2,m}\left(\frac{\tilde{\psi}_{1,m}(x)+x}{2}\right) + \frac{\tilde{\psi}_{1,m}(x)+x}{2}}{2}.$$

The projection $\tilde{P}_m$ on $K \otimes A$ extends to a projection P_m on $K \overline{\otimes} A$, with range $M_m(A)$ given by

$$P_m x = \frac{\psi_{2,m}\left(\frac{\psi_{1,m}(x)+x}{2}\right) + \frac{\psi_{1,m}(x)+x}{2}}{2},$$

or

$$P_m = \frac{1}{4}(\psi_{2,m}\psi_{1,m} + \psi_{2,m} + \psi_{1,m} + \mathrm{Id}).$$

PROPOSITION 1

With the above notation,

(a) $P_n(S(K\overline{\otimes}A)) = M_n(CS(A))$

(b) * $M_n(CS(A))$ is a JB*-subtriple of $S(K\overline{\otimes}A)$, that is,

$$\{M_n(CS(A)), M_n(CS(A)), M_n(CS(A))\}$$

$\subset M_n(CS(A))$; Moreover,

$$\{M_n(A), M_n(CS(A)), M_n(A)\} \subset M_n(A).$$

(c) $CS(A)$ is completely isometric to a TRO.

note that in the first displayed formula of (b), the triple product is the one on the JB-triple $M_n(CS(A))$, namely, $\{xyz\}_{M_n(CS(A))} = P_n(\{xyz\}_{S(K\overline{\otimes}A)})$, which, it turns out, is actually the restriction of the triple product of $S(K\overline{\otimes}A)$: whereas in the second displayed formula, the triple product is the partial triple product on $K\overline{\otimes}A$

Proof. Since P_n is a linear combination of isometries of $K\overline{\otimes}A$, and since isometries preserve the symmetric part, $P_n(S(K\overline{\otimes}A)) \subset S(K\overline{\otimes}A)$.

Suppose $x = (x_{ij}) \in P_n(S(K\overline{\otimes}A))$. Write $x = (R_1, \cdots, R_n)^t = (C_1, \cdots, C_n)$ where R_i, C_j are the rows and columns of x. Let $\psi_1 = \psi_1^n$ and $\psi_2 = \psi_2^n$ be the isometries on $K\overline{\otimes}A$ whose action is as follows: for $x \in M_n(A)$,

$$\psi_1^n(x) = (R_1, -R_2, \cdots, -R_n)^t$$

$$\psi_2^n(x) = (-C_1, \cdots, -C_{n-1}, C_n),$$

and for an arbitrary element $y = [y_{ij}] \in K \otimes A$, say $y \in M_N \otimes A$, where without loss of generality $N > n$, and for $k = 1, 2$, ψ_k^n maps y into $\begin{bmatrix} \psi_k^n[y_{ij}]_{n\times n} & 0 \\ 0 & [y_{ij}]_{n<i,j\leq N-n} \end{bmatrix}$.

Then $x_{1n}\otimes e_{1n} = \frac{\psi_2\left(\frac{\psi_1(x)+x}{2}\right)+\frac{\psi_1(x)+x}{2}}{2} \in S(K\overline{\otimes}A)$.

Now consider the isometry ψ_3 given by $\psi_3(C_1, \cdots, C_n) = (C_n, C_2, \cdots, C_{n-1}, C_1)$. Then $x_{1,n} \otimes e_{11} = \psi_3(x_{1n} \otimes e_{1n}) \in S(K\overline{\otimes}A))$, and by definition, $x_{1n} \in CS(A)$. Continuing in this way, one sees that each $x_{ij} \in CS(A)$, proving that $P_n(S(K\overline{\otimes}A)) \subset M_n(CS(A))$

Conversely, suppose that $x = (x_{ij}) \in M_n(CS(A))$. Since each $x_{ij} \in CS(A)$, then by definition, $x_{ij} \otimes e_{11} \in S(K\overline{\otimes}A)$. By using isometries as in the first part of the proof, it follows that $x_{ij} \otimes e_{ij} \in S(K\overline{\otimes}A)$, and $x = \sum_{i,j} x_{ij} \otimes e_{ij} \in S(K\overline{\otimes}A)$. This proves (a).

As noted above, P_n is a contractive projection on the JB*-triple $S(K\overline{\otimes}A)$, so that by the Theorem of Frieman-Russo-Kaup-Stacho, the range of P_n, namely $M_n(CS(A))$, is a JB*-triple with triple product

$$\{xyz\}_{M_n(CS(A))} = P_n(\{xyz\}_{S(K\overline{\otimes}A)}),$$

for $x, y, z \in M_n(CS(A))$. This proves (c) by the Theorem of Neal-Russo.

However, P_n is a linear combination of isometries of $K\overline{\otimes}A$ which fix $M_n(A)$ elementwise, and any isometry ψ of $K\overline{\otimes}A$ preserves the partial triple product: $\psi\{abc\} = \{\psi(a)\psi(b)\psi(c)\}$ for $a, c \in K\overline{\otimes}A$ and $b \in S(K\overline{\otimes}A)$. This shows that

$$\{xyz\}_{M_n(CS(A))} = \{xyz\}_{S(K\overline{\otimes}A)}$$

for $x, y, z \in M_n(CS(A))$, proving the first part of (b). To prove the second part of (b), just note that if $x, z \in M_n(A)$ and $y \in M_n(CS(A))$, then P_n fixes $\{xyz\}$.

COROLLARIES

1. $CS(A) = M_1(CS(A)) = P_1(S(K\overline{\otimes}A))$

2. $CS(A) \subset S(A)$ and $P_n\{yxy\} = \{yxy\}$ for $x \in M_n(CS(A))$ and $y \in A$.

Proof. For $x \in CS(A)$, let $\tilde{x} = x \otimes e_{11}$. Then $\tilde{x} \in S(K\overline{\otimes}A)$ and so there exists a complete holomorphic vector field $h_{\tilde{x}}$ on $(K\overline{\otimes}A)_0$. Since P_1 is a contractive projection of $K\overline{\otimes}A$ onto A, by the Theorem of Stacho, $P_1 \circ h_{\tilde{x}}|_{A_0}$ is a complete holomorphic vector field on A_0. But $P_1 \circ h_{\tilde{x}}|_{A_0}(0) = P_1 \circ h_{\tilde{x}}(0) = P_1(\tilde{x}) = x$, proving that $x \in S(A)$.

Recall from the proof of the second part of (b) that if $x, z \in M_n(A)$ and $y \in M_n(CS(A))$, then P_n fixes $\{xyz\}$.

The Cartan factors of type 1 are TROs, which we have already observed are equal to their completely symmetric parts.

The symmetric part of a JC*-triple coincides with the triple.

PROBLEM 2

What is the completely symmetric part of a JC*-triple? In particular, of a Cartan factor of type 2,3, or 4.

2. Definition of the algebra product

REMARK 1

In the rest of this talk, we shall assume that A is an operator space and $v \in CS(A)$ satisfies $\{xvv\} = x$ for every $x \in A$. In what follows, we work only with $M_2(A)$, which it turns out will be sufficient for our result.

LEMMA 1

$$\left\{\begin{bmatrix} x & \pm x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & \pm v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & \pm x \\ 0 & 0 \end{bmatrix}\right\} = 2\begin{bmatrix} \{xvx\} & \pm\{xvx\} \\ 0 & 0 \end{bmatrix}$$

Proof. Let ψ be the isometry defined by

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} a/\sqrt{2} & \pm a/\sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

LEMMA 2

$$\begin{bmatrix} \{xvx\} & 0 \\ 0 & 0 \end{bmatrix} = \{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\} + 2\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\}$$

Proof. By Lemma 1,

$$\begin{aligned} 4\begin{bmatrix} \{xvx\} & 0 \\ 0 & 0 \end{bmatrix} &= 2\begin{bmatrix} \{xvx\} & \{xvx\} \\ 0 & 0 \end{bmatrix} \\ &+ 2\begin{bmatrix} \{xvx\} & -\{xvx\} \\ 0 & 0 \end{bmatrix} \\ &= \{\begin{bmatrix} x & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & x \\ 0 & 0 \end{bmatrix}\} \\ &+ \{\begin{bmatrix} x & -x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & -v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & -x \\ 0 & 0 \end{bmatrix}\}. \end{aligned}$$

LEMMA 3

$$\{\begin{bmatrix} a & 0 \\ 0 & \pm a \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & \pm v \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & \pm b \end{bmatrix}\} = \begin{bmatrix} \{avb\} & 0 \\ 0 & \pm\{avb\} \end{bmatrix}$$

Proof. Let ψ be the isometry defined by

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ 0 & \pm a \end{bmatrix}.$$

LEMMA 4

$$\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\}+\{\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\}$$

and

$$\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\}$$

are both equal to 0.

Proof. By Lemma 3

$$\begin{aligned} 2\begin{bmatrix} \{xvx\} & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} \{xvx\} & 0 \\ 0 & \{xvx\} \end{bmatrix} \\ &+ \begin{bmatrix} \{xvx\} & 0 \\ 0 & -\{xvx\} \end{bmatrix} \\ &= \{\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}\} \\ &+ \{\begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & -v \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix}\}. \end{aligned}$$

LEMMA 5

$$\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}\} = 0$$

and[†]

$$\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\} = 0,$$

Equivalently,

$$\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\} = 0$$

and

$$\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\} = 0,$$

[†]This is true with the second v replaced by an arbitrary element of A. On the other hand, this fact is not needed in the proof of the main result.

Proof. The second statement follows from the first by using the isometry

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} b & a \\ d & c \end{bmatrix}.$$

Using Lemma 4 and an appropriate isometry (interchange both rows and columns simultaneously) yields

$$\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\} = 0.$$

Next, the isometry

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} -a & -b \\ c & d \end{bmatrix}.$$

shows that

$$\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\} = \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix},$$

for some $C, D \in A$.

Similarly, the isometry

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & -b \\ c & -d \end{bmatrix}$$

shows that

$$\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\} = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}.$$

Applying the isometry of multiplication of the second row by the imaginary unit shows that $C = 0$. Hence

$$\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\} = 0.$$

By appropriate use of isometries as above,

$$\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\} = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$$

for some $B \in A$. Applying the isometry of multiplication of the second column by the imaginary unit shows that $B = 0$. Hence

$$\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\} = 0.$$

It remains to show that

$$\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\right\} = 0,$$

To this end, by the main identity,

$$\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right\}\right\}$$

$$= R - S + T \qquad (1)$$

where

$$R = \left\{\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\right\}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right\},$$

$$S = \left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\right\}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right\}$$

and

$$T = \left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right\}\right\}.$$

Since

$$\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right\} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \qquad (2)$$

for some $A \in A$, the left side of (1) is equal to

$$\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right\} \qquad (3)$$

$$= \begin{bmatrix} \{vvA\} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

This term is also equal to R since

$$\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\right\} = \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $S = 0$, we have $T = 0$.

We next apply the main identity to get $0 = T = R' - S' + T'$, where

$$R' = \{\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\},$$

$$S' = \{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \{\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\} \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\}$$

and

$$T' = \{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\}\}.$$

By direct calculation, $R' = 0$ and $S' = 0$, and since $T = 0$ we have $T' = 0$ so that by (2) and (3),

$$\begin{aligned} 0 &= \{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\}\} \\ &= \{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\}. \end{aligned}$$

LEMMA 6

$$\begin{bmatrix} \{xvy\} & 0 \\ 0 & 0 \end{bmatrix} = \{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\} + \{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\} + \{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}\}$$

Proof. Replace x in Lemma 2 by $x+y$.

LEMMA 7

$$\left\{\begin{bmatrix} 0 & x \\ 0 & \pm x \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & \pm v \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & \pm x \end{bmatrix}\right\} = \begin{bmatrix} 0 & 2\{xvx\} \\ 0 & \pm 2\{xvx\} \end{bmatrix}$$

Proof. Let ψ be the isometry defined by

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & a/\sqrt{2} \\ 0 & \pm a/\sqrt{2} \end{bmatrix}.$$

The proof of the following lemma parallels exactly the proof of Lemma 2

LEMMA 8

$$\begin{bmatrix} 0 & \{xvx\} \\ 0 & 0 \end{bmatrix} = \{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\} + 2\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\}$$

As in Lemma 6, polarization of Lemma 8 yields the following lemma.

LEMMA 9

$$\begin{bmatrix} 0 & \{xvy\} \\ 0 & 0 \end{bmatrix} = \{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\} + \{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\} + \{\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\}$$

LEMMA 10

$$\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\} = 0.$$

Proof. Set $y = v$ in Lemma 9 and apply

$$D\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\right)$$

to each side of the equation in that lemma.

The three terms on the right each vanish, as is seen by applying the main identity to each term and making use of Lemma 5, and the fact that $CS(A)$ is a TRO, and hence $M_2(CS(A))$ is a JB*-triple.

LEMMA 11

$$\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\} = 0.$$

Proof. By applying the isometries of multiplication of the second column and second row by -1, we see that

$$\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

and that

$$\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}\} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}.$$

By Lemma 6

$$\begin{aligned} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} \{avv\} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \{\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\} + \end{aligned}$$

$$+ \ \{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\}$$

$$+ \ \{\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\} \quad (4)$$

The first term on the right side of (4) is zero by Lemma 10.

Let us write the second term on the right side of (4) as

$$\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\} =$$

$$\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\}\}$$

and apply the main Jordan identity to the right side, which we write symbolically as $\{AB\{CDE\}\}$ to obtain

$$\{AB\{CDE\}\} = \{\{ABC\}DE\} \qquad (5)$$

$$-\{C\{BAD\}E\} + \{CD\{ABE\}\}$$

We then calculate each term on the right side of (5) to obtain

$$\{\{ABC\}DE\} = \{\begin{bmatrix} 0 & \{vvx\} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\} =$$

$$\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\}$$

$$\{C\{BAD\}E\} = \frac{1}{2}\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} vv^*v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\} =$$

$$\frac{1}{2}\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\}$$

$$\{CD\{ABE\}\} =$$

$$\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\}\} =$$

$$= \{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \}.$$

The second term on the right side of (4) is therefore equal to

$$\frac{3}{2} \{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \}$$

Let us write the third term on the right side of (4) as

$$\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \} =$$

$$\{ \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} \} \}$$

and apply the main Jordan identity to the right side, which we again write symbolically as $\{A'B'\{C'D'E'\}\}$ to obtain $\{A'B'\{C'D'E'\}\} = \{\{A'B'C'\}D'E'\} - \{C'\{B'A'D'\}E'\} + \{C'D'\{A'B'E'\}\}$ We then calculate each term on the right side and find that each of these terms vanishes, the first and third by Lemma 10 and the second by the fact that $CS(A)$ is a TRO.

We have thus shown that

$$\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\} =$$

$$\frac{3}{2}\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\},$$

proving the lemma.

DEFINITION

Let us now define a product $y \cdot x$ by

$$\begin{bmatrix} y \cdot x & 0 \\ 0 & 0 \end{bmatrix} = 2\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\}$$

and denote the corresponding matrix product by $X \cdot Y$. That is, if $X = [x_{ij}]$ and $Y = [y_{ij}]$, then $X \cdot Y = [z_{ij}]$ where

$$z_{ij} = \sum_k x_{ik} \cdot y_{kj}.$$

Note that

$$\{xvy\} = \frac{1}{2}(y \cdot x + x \cdot y).$$

since by Lemmas 6 and 11 we can write

$$\begin{bmatrix} \{xvy\} & 0 \\ 0 & 0 \end{bmatrix} = \{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\}$$

$$+\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}\}.$$

Formula for $X \cdot Y$ as block matrices

The following lemma, in which the right side is equal to $\frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & x \cdot y \end{bmatrix}$, is needed to prove Proposition 2 below.

LEMMA 12

$$\{\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\} = \{\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\}$$

Proof. Let ψ be the isometry

$$\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \mapsto \frac{1}{\sqrt{2}}\begin{bmatrix} x & y \\ x & y \end{bmatrix}.$$

LEMMA 13

$x \cdot v = v \cdot x = x$ for every $x \in A$.

Proof. Apply the main identity to write

$$\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\}\}$$

$$= R - S + T$$

where

$$\begin{aligned} R &= \{\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\} \\ &= \frac{1}{2}\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\} \end{aligned}$$

$$\begin{aligned} S &= \{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\} \\ &= \{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\} \end{aligned}$$

and

$$T = \{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\}\}$$

$$= \{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\}.$$

Thus

$$\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\} =$$

$$\frac{1}{2}\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\} = \frac{1}{4}\begin{bmatrix} x \cdot v & 0 \\ 0 & 0 \end{bmatrix}\}. \tag{7}$$

Apply the main identity again to write

$$\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\}\}$$

$$= R' - S' + T'$$

where

$$R' = \{\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\} =$$

$$= \{ \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} \},$$

$$S' = \{ \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \{ \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \} \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} \} =$$

$$= \frac{1}{2} \{ \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} \},$$

and $T' =$

$$\{ \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \{ \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} \} \} = 0$$

by Lemma 5. Thus

$$\{ \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \{ \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} \} \} =$$

$$= R' - S' + T' =$$

$$\frac{1}{2} \{ \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} \} = \frac{1}{4} \begin{bmatrix} v \cdot x & 0 \\ 0 & 0 \end{bmatrix}, \tag{8}$$

the last step by Lemma 12.

By Lemmas 6, 10, and 12

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} = \{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\} + \tag{9}$$

$$\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\}.$$

Adding (7) and (8) and using (9) results in

$$\frac{1}{2}\begin{bmatrix} v \cdot x & 0 \\ 0 & 0 \end{bmatrix} = \{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\}$$

$$= \frac{1}{4}\begin{bmatrix} x \cdot v & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} v \cdot x & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus $v \cdot x = x \cdot v$ and since $x \cdot v + v \cdot x = 2\{vvx\} = 2x$, the lemma is proved.

The following lemma and corollary are not needed for the main result, and are stated for the sake of completeness.

LEMMA 14

$$\{\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\} = \{\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$$

COROLLARY

$$\{\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\} = 0.$$

REMARK 2

Lemmas 1 through 11 remain valid, with the same proofs, with the elements x, a, b, c, d replaced by elements X, A, B, C, D of $M_n(A)$ and v replaced by any element V of $M_n(CS(A))$ satisfying $\{VVX\} = X$, in particular, with $V =$ diag $(v, v, \ldots, v) = v \otimes I_n$.

Thus, for $X, Y \in M_n(A)$ and $V =$ diag $(v, v, \ldots, v)$,

$X \cdot Y + Y \cdot X = 2\{XVY\}$ and $X \cdot X = \{XVX\}$

and (analog of Lemma 5):

LEMMA 15

For $X, A, B, C, D \in M_n(A)$ and $V =$ diag $(v, v, \ldots, v)$.

$$\{\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & V \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\} = 0.$$

PROPOSITION 2

For $X, Y \in M_n(A)$.

$$\begin{bmatrix} 0 & Y \cdot X \\ 0 & 0 \end{bmatrix} = 2 \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \otimes I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

Proof. We assume first that $n = 2$ ($n = 1$ corresponds to the definition). The left side expands into 8 terms: $\begin{bmatrix} 0 & Y \cdot X \\ 0 & 0 \end{bmatrix} =$

$$= \begin{bmatrix} 0 & \begin{bmatrix} y_{11} \cdot x_{11} & 0 \\ 0 & 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \begin{bmatrix} y_{12} \cdot x_{21} & 0 \\ 0 & 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \begin{bmatrix} 0 & y_{11} \cdot x_{12} \\ 0 & 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \begin{bmatrix} 0 & y_{12} \cdot x_{22} \\ 0 & 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \begin{bmatrix} 0 & 0 \\ y_{21} \cdot x_{11} & 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \begin{bmatrix} 0 & 0 \\ y_{22} \cdot x_{21} & 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \begin{bmatrix} 0 & 0 \\ 0 & y_{21} \cdot x_{12} \end{bmatrix} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \begin{bmatrix} 0 & 0 \\ 0 & y_{22} \cdot x_{22} \end{bmatrix} \\ 0 & 0 \end{bmatrix}$$

For the right side, we have

$$\{ \begin{bmatrix} \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \\ 0 & 0 \end{bmatrix} \}$$

which is the sum of 32 terms. We show now that 24 of these 32 terms are zero, and each of the other 8 terms is equal to one of the 8 terms in the expansion of the left side.

The proof for arbitrary n is carried out in the same way. There will be n^2 cases corresponding to the elements y_{ij} and each of these cases will have n subcases corresponding to the elements $v \otimes e_{ii}$. Each of these subcases will have n^2 further subcases corresponding to the elements x_{ij}. Many of these latter two subcases can be handled simultaneously, as illustrated in the cases n=2 (above) and $n = 3$.

We can now state and prove our main result.

THEOREM 1

An operator space A is completely isometric to a unital operator algebra if and only there exists an element v in the completely symmetric part of A such that:

1. For $x \in A$,

$$h_v(x+v) - h_v(x) - h_v(v) + v = -2x$$

2. For $X \in M_n(A)$, $V = \text{diag}\,(v, \ldots, v) \in M_n(A)$,

$$\|V - h_V(X)\| \leq \|X\|^2$$

Proof. The first assumption is equivalent to the condition $\{xvv\} = x$, so that all the machinery developed so far is available if we replace elements of A by elements of $M_n(A)$. In particular, for every $X \in M_2(X)$, $X \cdot X = \{XVX\}$.

We now apply Proposition 2 with $Y = X = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}$ for elements $x, y \in A$ of norm 1. The result is

$$\begin{bmatrix} \begin{bmatrix} x \cdot y & 0 \\ 0 & y \cdot x \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} = 2\{\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}\}$$

and therefore

$$\max(\|x \cdot y\|, \|y \cdot x\|) \leq \|X\|^2 = \max(\|x\|, \|y\|) = 1,$$

so the multiplication on A is contractive. The same argument shows that if $X, Y \in M_n(A)$, then $\|X \cdot Y\| \leq \|X\|\|Y\|$ so the multiplication is completely contractive. The result now follows from the theorem of Blecher-Ruan-Sinclair.

The following is a variant of our main result which may be of interest.

THEOREM 2

An operator space A is completely isometric to a unital operator algebra if and only there exists $v \in CS(A)$ such that:

1. $h_v(x+v) - h_v(x) - h_v(v) + v = -2x$ for all $x \in A$

2. Let $\tilde{V}$ denote the $2n$ by $2n$ matrix $\begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix}$, where $V = \text{diag}(v, \ldots, v) \in M_n(A)$. For all $X, Y \in M_n(A)$

$$\|h_{\tilde{V}}(\begin{bmatrix} Y & X \\ 0 & 0 \end{bmatrix}) - h_{\tilde{V}}(\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}) - h_{\tilde{V}}(\begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} + \tilde{V}\|$$

$$\leq \|X\|\|Y\|$$

These conditions can be rewritten as $\{xvv\} = x$ and

$$\|\{\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}\}\| \leq \frac{1}{2}\|X\|\|Y\|$$

Proof. We know that A is a unital algebra with unit v of norm 1. By the second condition and Proposition 2,

$$\|Y \cdot X\| \leq \|X\|\|Y\|.$$

By Lemma 13 A is a unital (with a unit of norm 1 and not necessarily associative) algebra. The result now follows from the theorem of Blecher-Ruan-Sinclair.

REMARK 3

The logic for the proof of Theorems 1 and 2 is the following.

1. Prove $\{VVX\} = X$ for $V = \text{diag}\,(v, v)$ and $X \in M_2(A)$

2. Prove Lemmas 1 to 15 for $M_2(A)$.
(AUTOMATIC ONCE YOU HAVE STEP 1)

3. Prove Prop. 2 for n=2.

4. The proofs of Theorems 1 and 2 now show that the multiplication is 2-contractive.

5. Prove $\{VVX\} = X$ for $V = \text{diag}\,(v, v, v)$ and $X \in M_3(A)$ (uses steps 1, 2 and 3). The same proof works for $V = \text{diag}\,(v, v, v, v)$ and $X \in M_4(A)$

6. Steps 2 and 3 are now valid for n=3 and 4. So in Theorems 1 and 2, the multiplication is 3-contractive and 4-contractive.

7. Continuing in this way we see that the multiplications in Theorems 1 and 2 are $(2n-1)$-contractive and $2n$-contractive for every n, hence completely contractive.

REMARK 4

It does not appear to be true that (2) of Theorem 1 implies (2) of Theorem 2 by polarization.

REMARK 5

Since application of the theorem of Blecher-Ruan-Sinclair forces the product to be associative,

$$\begin{bmatrix} (x\cdot y)\cdot z & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x\cdot(y\cdot z) & 0 \\ 0 & 0 \end{bmatrix}$$

implies

$$\{\begin{bmatrix} z & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\{\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\}\} =$$

$$\{\{\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z & 0 \\ 0 & 0 \end{bmatrix}\}$$

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