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[4] Fenn R, Rourke, C J Knot Th Ram 1 1992 no 4 343–406
~~forget it~~ (unavailable, not needed)

[7] Kinyon, M.K. J. Lie Th 17 2007 no 1 99–114
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§2 Definition: (real) Leibniz algebra $\mathfrak{h}$ is vector space / $\mathbb{R}$

with linear map $[\,;\,] : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ (ignore $\otimes$ for now)

$(=$ bilinear map $[\,\cdot\,,\,\cdot\,] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h})$

left Leibniz identity holds i.e. $y \to [x,y]$ is a derivation:

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]]$$

Two canonical subspaces:

$Q(\mathfrak{h})$ = linear combinations of brackets $[x,x]$

$$= \left\{ \sum_{r=1}^{N} \lambda_r [x_r, x_r] : N \geq 1,\ \lambda_1,\dots,\lambda_N \in \mathbb{R},\ x_1,\dots,x_N \in \mathfrak{h} \right\}$$

$$\mathfrak{z}(\mathfrak{h}) = \{ x \in H : [x,y] = 0\ \forall y \in \mathfrak{h} \}$$

Remark $\bar{\mathfrak{h}} := \mathfrak{h}/Q(\mathfrak{h})$ is a Lie algebra

$\mathfrak{h}/\mathfrak{z}(\mathfrak{h})$ is a Lie algebra

(we mentioned this in class) I called $Q(\mathfrak{h})$, I

(see my correction on this point at the end of this note)

Def 2.1 $(\mathfrak{h}, p, \mathfrak{g}, [\,,\,]_{\mathfrak{g}}, \dot{\rho})$ is a <u>$\mathfrak{g}$-augmented Leibniz algebra</u> if

(it is not ~~an~~ Leibniz algebra yet)

1. $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is a real Lie algebra

2. $\mathfrak{h}$ is a left $\mathfrak{g}$-module via the map $\dot{\rho} : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$, $\dot{\rho}_{\xi}(x) = \xi \cdot x \quad \xi \in \mathfrak{g}, x \in \mathfrak{h}$

3. $p : \mathfrak{h} \to \mathfrak{g}$ is a homomorphism of $\mathfrak{g}$-modules

i.e. $p(\xi \cdot x) = [\xi, p(x)]_{\mathfrak{g}}$

<u>Picture</u>: for $\xi \in \mathfrak{g}$, this diagram commutes:

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, "\dot{\rho}(\xi)"] \arrow[d, "p"'] & \mathfrak{h} \arrow[d, "p"] \\
\mathfrak{g} \arrow[r, "\mathrm{ad}(\xi)"] & \mathfrak{g}
\end{tikzcd}$$

$\dot{\rho}(\xi)(x) = \xi \cdot x$

Definition of a morphism of $\mathfrak{g}$-augmented leibniz algs

$(\mathfrak{h}, p, \mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \dot{\rho})$ and $(\mathfrak{h}', p', \mathfrak{g}', [\cdot,\cdot]'_{\mathfrak{g}'}, \dot{\rho}')$

is (ϕ, Φ) where $\boxed{\begin{array}{l} \text{(1) } \phi : \mathfrak{g} \to \mathfrak{g}' \text{ is a Lie alg. homo} \\ \Phi : \mathfrak{h} \to \mathfrak{h}' \text{ is a Lie-module homo over } \phi \\ \text{(i.e. } \Phi(\xi . x) = \phi(\xi) . \Phi(x) \text{, } x \in \mathfrak{h}, \ \xi \in \mathfrak{g} \text{)} \end{array}}$

and $\boxed{\text{(2) } p' \circ \Phi = \phi \circ p.}$

Picture for (1) $\Phi(\xi . x) = \phi(\xi) . \Phi(x)$ (the diagrams commute)

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, "\dot{\rho}(\xi)"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "/\!/"] & \mathfrak{h} \arrow[d, "\Phi"] \\
\mathfrak{h}' \arrow[r, "\dot{\rho}'(\phi(\xi))"'] & \mathfrak{h}'
\end{tikzcd}$$

Picture for (2) $p' \circ \Phi = \phi \circ p$

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, "\dot{\rho}(\xi)"] \arrow[d, "p"'] \arrow[dr, phantom, "/\!/"] & \mathfrak{h} \arrow[r, "\Phi"] \arrow[d, "p"] \arrow[dr, phantom, "/\!/"] & \mathfrak{h}' \arrow[r] \arrow[d, "p'"] \arrow[dr, phantom, "/\!/"] & \mathfrak{h}' \arrow[d, "p'"] \\
\mathfrak{g} \arrow[r, "\mathrm{ad}\,\xi"'] & \mathfrak{g} \arrow[r, "\phi"'] & \mathfrak{g}' \arrow[r] & \mathfrak{g}'
\end{tikzcd}$$

Proposition 2.1 Let $(\mathfrak{h}, p, \mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \dot{\rho})$ be an augmented Leibniz algebra. Define a bracket on $\mathfrak{h}$: $[x,y]_{\mathfrak{h}} = p(x)\circ y$. $(x, y \in \mathfrak{h})$

Then

1. $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ is a Leibniz algebra on which $\mathfrak{g}$ acts by derivations: Need to prove

- $$[x,[y,z]_{\mathfrak{h}}]_{\mathfrak{h}} = [[x,y]_{\mathfrak{h}}, z]_{\mathfrak{h}} + [y,[x,z]_{\mathfrak{h}}]_{\mathfrak{h}}$$
- $$\dot{\rho}_{\xi}\left([x,y]_{\mathfrak{h}}\right) = [x, \dot{\rho}_{\xi}(y)]_{\mathfrak{h}} + [\dot{\rho}_{\xi}(x), y]_{\mathfrak{h}}$$

proof of first bullet: $[x,[y,z]_{\mathfrak{h}}]_{\mathfrak{h}} = p(x)\circ[y,z]_{\mathfrak{h}} = p(x)\circ(p(y)\circ z)$

$$= p(x)\circ(p(y)\circ z) + 0$$

$$= \underbrace{p(x)\circ(p(y)\circ z) - p(y)\circ(p(x)\circ z)} + \underbrace{p(y)\circ(p(x)\circ z)}$$

$$= \underbrace{[p(x),p(y)]_{\mathfrak{g}}}\circ z + [y,[x,z]_{\mathfrak{h}}]_{\mathfrak{h}}$$

3. in definition 2.1

$$= p(p(x)\circ y)\circ z + \text{''}$$

def of $[\cdot,\cdot]_{\mathfrak{h}}$

$$= [p(x)\circ y, z]_{\mathfrak{h}} + \text{''}$$

$$= [[x,y]_{\mathfrak{h}}, z]_{\mathfrak{h}} + \text{''}$$

proving the first bullet.

proof of second bullet:

rewrite the second bullet as

$$\xi \cdot (p(x) \cdot y) \overset{?}{=} p(x) \cdot (\xi \cdot y) + p(\xi \cdot x) \cdot y$$

by 3. of definition 2.1
$$\left(= [\xi, p(x)]_{\mathfrak{g}} \cdot y \right)$$

This holds by the definition of $\mathfrak{h}$ being a $\mathfrak{g}$-module

i.e.

$$[\xi, p(x)]_{\mathfrak{g}} \cdot y = \xi \cdot (p(x) \cdot y) - p(x) \cdot (\xi \cdot y)$$

This proves the second bullet.

Continue here with the statement of Proposition 2.1

There is more to the first assertion

1. (continued) If (Φ, ϕ) is a morphism of augmented Leibniz algebras, then Φ is a morphism of Leibniz algebras.

[This proof and the proofs of ~~stat~~ assertions 2 & 3 are left as easy (but tedious) exercises]

10/15/17 ①

CORRECTION !

to my notes on pp 269–272 | page ③ of NOTES

let L be a right leibniz algebra

i.e. $[\bullet, z]$ is a derivation

$$[[x,y],z] = [[x,z],y] + [x,[y,z]]$$

let $I = \text{span}\{[x,x] : x \in L\}$ (the linear span is enough)

claim I is a left ideal and a right ideal
(i.e. two sided so that L/I is a Leibniz alg actually a Lie algebra)

Recall from the notes for October 13 (Notes for pp 269–272) that in a right leibniz algebra

$$[x,[y,z]] = -[x,[z,y]]$$

Therefore if $x, y \in L$ then

$$[y,[x,x]] = -[y,[x,x]]$$

so $[y,[x,x]] = 0$ & I is a left ideal.

To see that it is a right ideal consider

$$[[y,y]+x,[y,y]+x] = [x,x] + [[y,y],[y,y]] + \underbrace{[x,[y,y]]}_{\substack{=0 \\ \text{(left ideal)}}}$$

$$+ [[y,y],x] \text{ , so}$$

(there is a misprint in Rakhmov's survey on bottom of p. 229)

$$[[y,y],x] = [[y,y]+x,[y,y]+x] - [x,x] - [[y,y],[y,y]]$$

$\therefore$ right ideal $\boxtimes$