

10/14/17^①

Bordemann & Wagemann

J. Lie Th 27 2017
555-567

[4] Fenn R, Rourke, C J Knot Th Ram 1 1992 no 4 343-406
~~forget it~~ (unavailable, not needed)

[7] Kinyon, M.K. J. Lie Th 17 2007 no 1 99-114
downloaded

[2] Covez, Simon AIF 63 2013 1-35
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Definition:

§ 2 (real) Leibniz algebra $\mathfrak{h}$ is vector space / $\mathbb{R}$

with linear map $[\cdot, \cdot]: \mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathfrak{h}$ (ignore $\otimes$ for now)

(= bilinear map $[\cdot, \cdot]: \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{h}$)

left Leibniz identity holds i.e. $y \rightarrow [x, y]$ is a derivation:

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

Two canonical subspaces:

$$\begin{aligned} Q(\mathfrak{h}) &= \text{linear combinations of brackets } [x, x] \\ &= \left\{ \sum_{r=1}^N \lambda_r [x_r, x_r] : N \geq 1, \lambda_1, \dots, \lambda_N \in \mathbb{R}, x_1, \dots, x_N \in \mathfrak{h} \right\} \end{aligned}$$

$$Z(\mathfrak{h}) = \{x \in \mathfrak{h} : [x, y] = 0 \forall y \in \mathfrak{h}\}$$

Remark

$\overline{\mathfrak{h}} := \mathfrak{h}/Q(\mathfrak{h})$ is a Lie algebra

$\mathfrak{h}/\mathfrak{g}(\mathfrak{h})$ is a Lie algebra

(we mentioned this in class)
I called $Q(\mathfrak{h})$, I

(see my correction on this point at the end of this note)
it is (not a) Lie algebra yet

Def 2.1

$(\mathfrak{h}, \rho, \mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \dot{\rho})$ is a

$\mathfrak{g}$ -augmented Lie algebra

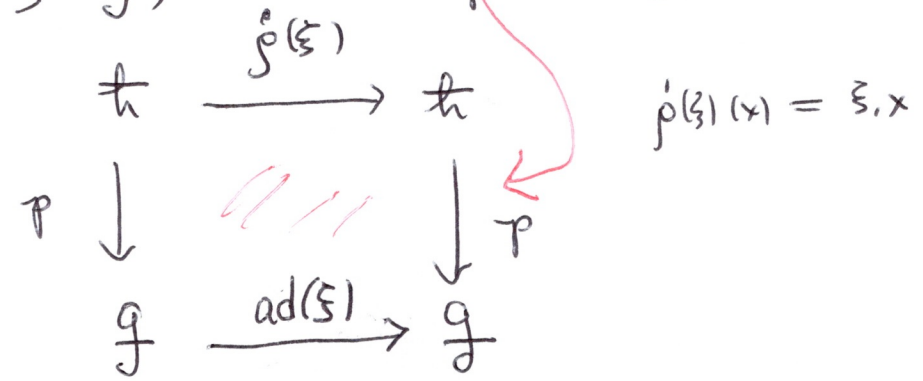
if

1. $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ is a real Lie algebra
2. $\mathfrak{h}$ is a left $\mathfrak{g}$ -module via the map $\dot{\rho}: \mathfrak{g} \times \mathfrak{h} \rightarrow \mathfrak{h}$, $\dot{\rho}_{\xi}(x) = \xi \cdot x$ $\xi \in \mathfrak{g}, x \in \mathfrak{h}$
3. $\rho: \mathfrak{h} \rightarrow \mathfrak{g}$ is a homomorphism of $\mathfrak{g}$ -modules

i.e. $\rho(\xi \cdot x) = [\xi, \rho(x)]_{\mathfrak{g}}$

Picture:

for $\xi \in \mathfrak{g}$, this diagram commutes:



③

Definition of a morphism of $\mathfrak{g}$ -augmented Lie algs

$$(\mathfrak{h}, \mathfrak{p}, \mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \dot{\rho}) \quad \text{and} \quad (\mathfrak{h}', \mathfrak{p}', \mathfrak{g}', [\cdot, \cdot]_{\mathfrak{g}'}, \dot{\rho}')$$

is (ϕ, Φ) where

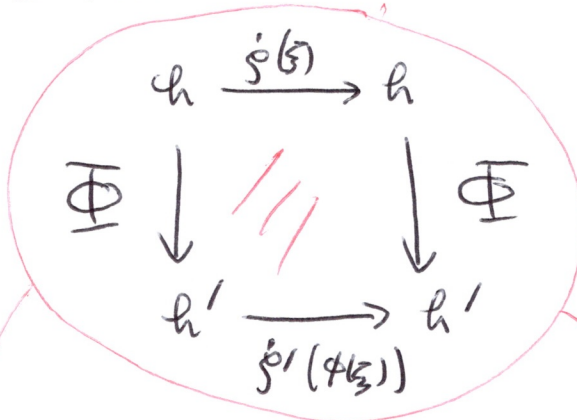
- ① $\phi: \mathfrak{g} \rightarrow \mathfrak{g}'$ is a Lie alg. homo
- $\Phi: \mathfrak{h} \rightarrow \mathfrak{h}'$ is a Lie-module homo over ϕ
(i.e. $\Phi(\xi, x) = \phi(\xi) \cdot \Phi(x)$)
 $x \in \mathfrak{h}, \xi \in \mathfrak{g}$

and

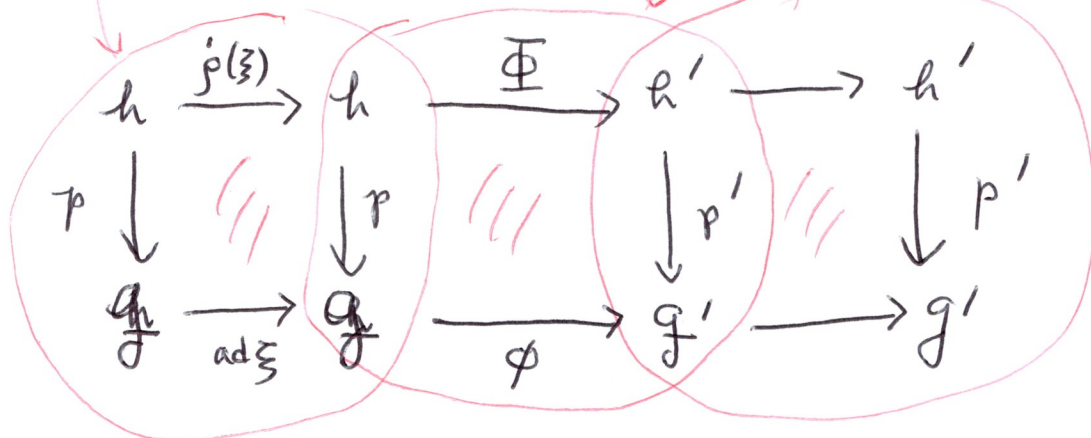
- ② $\mathfrak{p}' \circ \Phi = \phi \circ \mathfrak{p}$

Picture for ① $\Phi(\xi, x) = \phi(\xi) \cdot \Phi(x)$

(the diagrams commute)



Picture for ② $\mathfrak{p}' \circ \Phi = \phi \circ \mathfrak{p}$



Proposition 2.1 Let $(\mathfrak{h}, p, q, [\cdot, \cdot]_g, \dot{p})$ be an augmented Leibniz algebra. Define

a bracket on $\mathfrak{h}$: $[x, y]_{\mathfrak{h}} = p(x) \cdot y$ ($x, y \in \mathfrak{h}$)

Then

1. $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ is a Leibniz algebra on which q acts by derivations: Need to prove

- $[x, [y, z]_{\mathfrak{h}}]_{\mathfrak{h}} = [[x, y]_{\mathfrak{h}}, z]_{\mathfrak{h}} + [y, [x, z]_{\mathfrak{h}}]_{\mathfrak{h}}$
- $\dot{p}_{\xi}([x, y]_{\mathfrak{h}}) = [x, \dot{p}_{\xi}(y)]_{\mathfrak{h}} + [\dot{p}_{\xi}(x), y]_{\mathfrak{h}}$

proof of first bullet: $[x, [y, z]_{\mathfrak{h}}]_{\mathfrak{h}} = p(x) \cdot [y, z]_{\mathfrak{h}} = p(x) \cdot (p(y) \cdot z)$

$$= p(x) \cdot (p(y) \cdot z) + \underbrace{0}_{\text{II}}$$

$$= \underbrace{p(x) \cdot (p(y) \cdot z)}_{\text{I}} - \underbrace{p(y) \cdot (p(x) \cdot z)}_{\text{II}} + \underbrace{p(y) \cdot (p(x) \cdot z)}_{\text{III}}$$

$$= \underbrace{[p(x), p(y)]_g \cdot z}_{\text{I}} + \underbrace{[y, [x, z]_{\mathfrak{h}}]_{\mathfrak{h}}}_{\text{II}}$$

$$= p(p(x) \cdot y) \cdot z + \text{II}$$

$$= [p(x) \cdot y, z]_{\mathfrak{h}} + \text{II}$$

$$\stackrel{\text{def of } [\cdot, \cdot]_{\mathfrak{h}}}{=} [[x, y]_{\mathfrak{h}}, z]_{\mathfrak{h}} + \text{II}$$

3. in definition 2.1

def of $[\cdot, \cdot]_{\mathfrak{h}}$

proving the first bullet.

Proof of
second bullet:

rewrite the second bullet as

$$\xi \circ (p(x) \circ y) \stackrel{?}{=} p(x) \circ (\xi \circ y) + p(\xi \circ x) \circ y$$

(by 3. of definition 2.1)

$$= [\xi, p(x)]_g \circ y$$

This holds by the definition of $\hbar$ being a g -module

i.e.

$$[\xi, p(x)]_g \circ y = \xi \circ (p(x) \circ y) - p(x) \circ (\xi \circ y)$$

This proves the second bullet.

Continue here with the statement of Proposition 2.1

There is more to the first assertion

~~2.~~

1. (continued) If (Φ, ϕ) is a morphism of augmented Leibniz algebras, then $\bar{\Phi}$ is a morphism of Leibniz algebras.

[This proof and the proof of ~~the~~ assertions 2 & 3 are left as easy (but tedious) exercises]

CORRECTION !

10/15/17 ①

to my notes on pp 269-272

page ③ of NOTES

let L be a right Leibniz algebra

i.e. $[\cdot, z]$ is a derivation

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]$$

let $I = \text{span} \{ [x, x] : x \in L \}$ (the linear span is enough)

claim I is a left ideal and a right ideal

(i.e. two sided so that L/I is a Leibniz alg
actually a Lie algebra)

Recall from the notes for October 13 (Notes for pp 269-272)

~~that~~ that in a right Leibniz algebra

$$[x, [y, z]] = - [x, [z, y]]$$

Therefore if $x, y \in L$ then

$$[y, [x, x]] = - [y, [x, x]]$$

so $[y, [x, x]] = 0$ & I is a left ideal.

To see that it is a right ideal consider

$$[[y, y] + x, [y, y] + x] = [x, x] + [[y, y], [y, y]] + [x, [y, y]]$$

+ $[[y, y], x]$, so $= 0$ (left ideal)

(there is a misprint in Rakhimov's survey on Gol'ton of p. 229)

$$[[y, y], x] = [[y, y] + x, [y, y] + x] - [x, x] - [[y, y], [y, y]] \quad \therefore \text{right ideal} \quad \square$$