the entire equation can be multiplied throughout by -1. The result is the following catalogue of normal forms:

Theorem 13.1. *By an invertible linear coordinate change, every conic can be put in one of the following normal forms:*

1. $X^2 + Y^2 + P = 0$
2. $X^2 - Y^2 + P = 0$
3. $X^2 + Y + Q = 0$
4. $X^2 + Q = 0$ □

In case (1) we get an ellipse (indeed a circle) if $P < 0$, a point if $P = 0$, and the empty set if $P > 0$. In case (2) we get a (rectangular) hyperbola if $P \neq 0$ and two distinct intersecting lines if $P = 0$. Case (3) is a parabola. Case (4) is a pair of parallel lines if $Q < 0$, a 'double' line if $Q = 0$, and empty if $Q > 0$.

Transforming back into the original (x, y) coordinates, circles transform into ellipses, rectangular hyperbolas transform into general hyperbolas, parabolas transform into parabolas, lines transform into lines, and points transform into points.

Even the conics, then, exhibit a rich set of possibilities when viewed as curves in the real plane $\mathbf{R}^2$. The situation simplifies somewhat if we consider the same equations, but in *complex* variables; it simplifies even more if we work in projective space. In complex coordinates, the map $Y \mapsto iY$ sends Y^2 to $-Y^2$, a scaling that cannot be performed over the reals. This coordinate transformation sends normal form (1) to normal form (2) and thereby abolishes the distinction between hyperbolas and ellipses.

13.2 Projective Space

We now show that in projective space, all the different types of conic section other than the double line and the point can be transformed into each other. (This is the case even in *real* projective space.) First, we recall the basic notions of projective geometry. For further details, see Coxeter [16], Loney [44], or Roe [62].

Definition 13.2. The *real projective plane* $\mathbf{RP}^2$ is the set of lines L through the origin in $\mathbf{R}^3$. Each such line is referred to as a *projective point*. Each plane through the origin in $\mathbf{R}^3$ is called a *projective line*. A projective point

13.1 Review of Conics

The simplest real plane curves are straight lines, which can be defined as the set of solutions $(x, y) \in \mathbf{R}^2$ to a *linear* (or degree 1) polynomial equation

$$Ax + By + C = 0 \tag{13.1}$$

where $A, B, C \in \mathbf{R}$ are constants and $AB \neq 0$.

Next in order of complexity come the *conic sections* or *conics*, defined by a general quadratic (or degree 2) polynomial equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{13.2}$$

where $A, B, C, D, E, F \in \mathbf{R}$ are constants and $ABC \neq 0$.

It is well known (and can be found in most elementary texts on coordinate geometry or linear algebra) that conic sections can be classified into seven different types: ellipse, hyperbola, parabola, two distinct lines, one 'double' line, a point, or empty. A good way to see this is to transform Equation (13.2) into a simpler form, usually known as a *normal form*, by a change of coordinates. In fact, a general invertible linear change of coordinates

$$\begin{aligned} X &= ax + by \\ Y &= cx + dy \end{aligned}$$

(with $ad - bc \neq 0$ for invertibility) transforms (13.2) into one or other of the forms

$$\begin{aligned} \epsilon_1 X^2 + \epsilon_2 Y^2 + P &= 0 \\ X^2 + Y + Q &= 0 \end{aligned}$$

where $P, Q \in \mathbf{R}$ and $\epsilon_1, \epsilon_2 = 0, 1$, or -1.

The usual proof of this (see for example Loney [44] page 323, Anton [1] page 359, or Roe [62] page 251) begins by rotating coordinates orthogonally to diagonalise the quadratic form $Ax^2 + Bxy + Cy^2$, which changes (13.2) to the slightly simpler form

$$\lambda_1 x'^2 + \lambda_2 y'^2 + \alpha x' + \beta y' + \gamma = 0.$$

If $\lambda_1 \neq 0$ then the term $\alpha x'$ can be eliminated by 'completing the square', and similarly if $\lambda_2 \neq 0$ then the term $\beta y'$ can be eliminated. The coefficients of x'^2 and y'^2 can be scaled to $0, 1$, or -1 by multiplying them by a nonzero constant; furthermore, x' and y' can be interchanged if necessary. Finally,