Seminar on Time-Frequency Analysis 2004-05

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Time-frequency analysis is a modern branch of harmonic analysis. It uses translations and modulations (multiplication by an exponential) for the analysis of functions and operators. It is a form of local Fourier analysis treating time and frequency simultaneously and symmetrically. The subject is motivated by applications in signal analysis and quantum mechanics.

An introduction to the subject is the book: Foundations of Time-Frequency Analysis by Karlheinz Grochenig 2001, [12].

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1 July 8, 2004—Uncertainty Principles—Bernard Russo

1.1 The classical uncertainty principle of Heisenberg-Pauli-Weyl in dimension $d = 1$

Theorem 1.1 (**[12, Theorem 2.2.1,page 26]**) *For $f \in L^2(\mathbf{R})$ and $a, b \in \mathbf{R}$,*

$$\int (x-a)^2 |f(x)|^2 \, dx \cdot \int (\omega - b)^2 |\hat{f}(\omega)|^2 \geq \frac{1}{16\pi^2} \|f\|_2^4.$$

Equality holds if and only if f is a multiple of $T_a M_b \varphi_c(x) = e^{2\pi i b(x-a)} e^{-\pi(x-a)^2/c}$ for some $c > 0$.

The proof follows from the following lemma applied to the self-adjoint operators $Xf(x) = xf(x)$ and $Pf(x) = \frac{1}{2\pi i} f'(x)$.

Lemma 1.2 (**[12, Lemma 2.2.2,page 27]**) *If A and B are (unbounded) self-adjoint operators, if $a, b \in \mathbf{R}$, and if $f \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$, then*

$$|([A, B]f|f)| \leq 2\|(B-b)f\| \|(A-a)f\|.$$

Equality holds if and only if $(A-a)f = ic(B-b)f$ for some $c \in \mathbf{R}$.

1.2 The uncertainty principle of Donoho and Stark

Theorem 1.3 (**[12, Theorem 2.3.1,page 30]**) *Suppose that $f \in L^2(\mathbf{R}^d)$ is ϵ_T-concentrated on $T \subset \mathbf{R}^d$ and$\hat{f}$ is ϵ_Ω-concentrated on $\Omega \subset \mathbf{R}^d$, that is*

$$\left(\int_{\mathbf{R}^d \setminus T} |f|^2 \right)^{1/2} \leq \epsilon_T \|f\|_2 \text{ and } \left(\int_{\mathbf{R}^d \setminus \Omega} |\hat{f}|^2 \right)^{1/2} \leq \epsilon_\Omega \|\hat{f}\|_2.$$

Then $|T||\Omega| \geq (1 - \epsilon_T - \epsilon_\Omega)^2$.

Corollary 1.4 *If $f \in L^2(\mathbf{R}^d)$ is supported on T and $\hat{f}$ is supported on Ω, then $|T||\Omega| \geq 1$.*

One can use the Poisson summation formula to prove the following qualitative uncertainty principle.

Theorem 1.5 (**[12, Theorem 2.3.3,page 32]**) *If $f \in L^1(\mathbf{R}^d)$ is supported on T, $\hat{f}$ is supported on Ω, and $|T||\Omega| < \infty$, then $f = 0$.*

1.3 Motivating remarks

- " f and $\hat{f}$ are two different, equivalent representations of the same object f. Each one makes visible and accessible rather different features of f"
- "It is impossible for a non-zero function and its Fourier transform to be simultaneously very small, that is, $f \sim g$ and $\hat{f} \sim \hat{g}$ cannot both hold with a high degree of accuracy if $f \neq g$"
- "If $\|f\|_2 = 1$, then f represents the state of a one-dimensional system, and the inequality $\|xf\|_2\|y\hat{f}\|_2 \geq (4\pi)^{-1}$ states that the product of the standard deviation of the position observable and the standard deviation of the momentum observable is at least $1/4\pi$."
- "In quantum mechanics, a particle's position and momentum cannot be determined simultaneously" (Heisenberg 1930)
- "In signal processing, there are limits on the extent to which the instantaneous frequency of a signal can be measured" (Gabor 1946)

1.4 Papers for further study

- Application of uncertainty principles to PDE, [7]
- Uncertainty principle as characterization of Hermite functions, [2]

2 July 19, 2004—Short Time Fourier Transform—Devin Greene

2.1 Short Time Fourier Transform

A function $f \in L^2(\mathbf{R}^d)$ can be thought of as a signal ($d = 1$,variable is time), image ($d = 2$), or quantum mechanical state ($d \geq 1$ is the number of particles). We want to analyze f in terms of frequencies. For this the Fourier transform $\hat{f}(\omega) = \int f(t)e^{-2\pi i\omega\cdot t}\,dt$ suffers from a lack of time localization. So, we use the Short time Fourier transform (STFT) with window g:

$$V_g f(x,\omega) = \int f(t)\overline{g(t-x)}e^{-2\pi i\omega\cdot t}\,dt.$$

A discontinuity of a window introduces artificial high frequencies. The best choice for a window is the Gaussian $e^{-\pi t^2/a}$, with variance $a > 0$. If $\phi_a(t) = e^{-\pi t^2/a}$, then $\hat{\phi}_a(\omega) = a^{d/2}\phi_{1/a}(\omega)$, and in particular $\hat{\phi}_1 = \phi_1$. Gaussians satisfy a semigroup property: $(a^{-d/2}\phi_a) * (b^{-d/2}\phi_b) = (a+b)^{-d/2}\phi_{a+b}$.

If a is small, the STFT with window ϕ_a has good time localization and bad frequency localization. If a is large, the STFT with window ϕ_a has bad time localization and good frequency localization.

An identity:

$$V_g f(x,\omega) = e^{-2\pi i\omega\cdot x}V_{\hat{g}}\hat{f}(\omega,-x). \tag{1}$$

Theorem 2.1 (**[12, Theorem 3.2.1,page 42]**) *For $f_i, f_2, g_1, g_2 \in L^2(\mathbf{R}^d)$,*

$$(V_{g_1}f_1|V_{g_2}f_2)_{L^2(\mathbf{R}^{2d})} = (g_2|g_1)(f_1|f_2)$$

Corollary 2.2 *$f \otimes \overline{g} \mapsto V_g f$ extends to an isometry of $L^2 \otimes \overline{L^2}$ (Hilbert-Schmidt operators) into $L^2(\mathbf{R}^{2d})$*

Properties of the STFT:

1. $V_g f$ is uniformly continuous and bounded.
2. $V_g f(x,\omega)$ is continuous in $g, f \in L^2$ for fixed (x,ω).
3. $V_g f \in L^2$ and $\|V_g f\|_2 = \|f\|_2\|g\|_2$. If $\|g\|_2 = 1$, then $f \mapsto V_g f$ is an isometry (energy is preserved).

4. Inversion formula (Immediate from Theorem 2.1): Let $f, g, \gamma \in L^2(\mathbf{R}^d)$. Then

$$f(t) = \frac{1}{(\gamma|g)} \int\int V_g f(x,\omega)\gamma(t-x)e^{2\pi i t\cdot\omega}\,dxd\omega$$

5. Suppose we start with some $L(x,\omega) \in L^2(\mathbf{R}^{2d})$. Then

$$f(t) := \frac{1}{(\gamma|g)} \int\int L(x,\omega)\gamma(t-x)e^{2\pi i t\cdot\omega}\,dxd\omega$$

is the signal whose STFT best approximates L.

2.2 Lieb's uncertainty principle

Theorem 2.3 (**[12, Theorem 3.3.2,page 50]**) *Let $f, g \in L^2(\mathbf{R}^d)$, $2 \le p < \infty$. Then*

$$\int\int |V_g f(x,\omega)|^p\,dxd\omega \le (\frac{2}{p})^d(\|f\|_2\|g\|_2)^p.$$

This implies the support of $V_g f(x,\omega)$ is "spread out."

2.3 Bargmann transform

For Gaussian windows, the STFT can be expressed analytically in terms of the complex variable $z = (x,\omega)$ via the Bargmann transform: if $g(t) = e^{-\pi t^2}$, then $V_g f(x,-\omega) = e^{\pi x\omega i}e^{-\pi|z|^2/2}Bf(z)$, where $Bf(z) = \int f(t)e^{-\pi t^2+2\pi zt-\pi z^2/2}\,dt$. Here $B : L^2(\mathbf{R}^d) \to \mathcal{F}^2(\mathbf{C}^d)$ (the Bargmann transform) is an isometry into the Fock space of entire functions with inner product $(F|G)_{\mathcal{F}^2(\mathbf{C}^d)} = \int F(z)\overline{G(z)}e^{-\pi|z|^2}\,d|z|$.

2.4 Papers for further study

- Hardy's theorem and STFT, [14]
- Time-frequency localization operators, [3]

3 August 5, 2004—Introduction to Gabor Frames—Devin Greene

It is natural and useful to try to replace the integrals in the inversion formula

$$f(t) = \frac{1}{(\gamma|g)} \int\int V_g f(x,\omega)\gamma(t-x)e^{2\pi i t\cdot\omega}\,dxd\omega$$

for the STFT by a discrete sum and ask if $\{V_g f(\alpha m, \beta n)\} \in \ell^2(\alpha\mathbf{Z}^d \times \beta\mathbf{Z}^d)$ and if

$$f = \sum_{k,n\in\mathbf{Z}^d} (f|T_{\alpha k}M_{\beta n}\gamma)T_{\alpha k}M_{\beta n}g$$

for some suitable windows $g, \gamma \in L^2(\mathbf{R}^d)$ and lattice parameters $\alpha, \beta > 0$. For example, if $d = 1, g = \chi_{[0,1)}, \alpha = \beta = 1$, then $V_g f(m,n) = \int_m^{m+1} f(t)e^{-2\pi i n t}\,dt$ is the n-th Fourier coefficient of $f\chi_{[m,m+1)}$. If $\{g(t-m)e^{2\pi i n t}\}$ is an orthonormal basis, the sequence $\{V_g f(m,n)\}$ determines f. To pursue this question requires a study of the sequence of vectors $g(t-\alpha m)e^{2\pi i\beta n t}$ and the inner products $(f|g(t-\alpha m)e^{2\pi i\beta n t})$, that is, frames.

A *frame* in a separable Hilbert space H is a family $\{e_j : j \in J\} \subset H$ satisfying

$$A\|f\|^2 \le \sum_{j\in J} |(f|e_j)|^2 \le B\|f\|^2$$

for some $A, B > 0$ and all $f \in H$. The frame is *tight* if A and B can be chosen to be equal.

Examples of frames:

- an orthonormal basis
- the image of an orthonormal basis under a co-isometry, that is, U^*e_j where U is an isometry (into) and e_j is an ONB. This is a tight frame and shows that a tight frame with $A = B = 1$ need not be an ONB.
- With $H = L^2[0,1]$ and $\beta > 0$, consider $\{e^{2\pi ij\beta t} : j \in \mathbf{Z}\}$. If $\beta = 1$, this is an ONB. If $\beta > 1$ the closed span of this set consists of $1/\beta$-periodic functions, so you can find $f \neq 0$ such that $\sum |(f|e^{2\pi ij\beta t})|^2 = 0$. If $\beta < 1$, you get a tight frame with $A = B = 1/\beta$.
- A non-tight frame is given by $\{Te_j\}$, where T is an invertible operator on H and e_j is an ONB: $\|T^{-1}\|^{-2}\|f\|^2 \leq \sum |(f|Te_j)|^2 \leq \|T\|^2\|f\|^2$. (Note: it is enough that T^* has closed range)

Some operators associated to a frame $\{e_j\}_{j\in J}$.

- *coefficient* (or *analysis*) operator: $C : H \to \ell^2(J)$, $f \mapsto \{(f|e_j)\}$. C is bounded above and below by B and A respectively, and hence C has closed range.
- *reconstruction* (or *synthesis*) operator: $D = C^*$, $Da = \sum a_j e_j$ for $a = \{a_j\} \in \ell^2(J)$.
- *frame* operator: $S = DC = C^*C$ is positive and invertible, and $Sf = \sum (f|e_j)e_j$
- Properties of the frame operator: $f = S^{-1}Sf = \sum (f|e_j)S^{-1}e_j$ and $\gamma_j := S^{-1}e_j$ is a frame called the *dual frame*, with bounds B^{-1} and A^{-1}. Also, $f = \sum (f|e_j)\gamma_j = \sum (f|\gamma_j)e_j$. Finally, if $\{e_j\}$ is tight, then $S = A\,Id = B\,\mathrm{Id}$ and $\gamma_j =$ constant e_j.

4 August 12, 2004—Some trace class pseudodifferential operators arising in time-frequency analysis— Bernard Russo

Let's recall the definitions of *translation operator*: $T_x f(t) = f(t-x)$, *modulation operator*: $M_\omega f(t) = e^{2\pi i\omega t}f(t)$, and the Short time Fourier transform: $V_g f(x,\omega) = (f|M_\omega T_x g) = \int_{\mathbf{R}^d} f(t)\overline{g}(t-x)e^{-2\pi i\omega t}\,dt$. The *time-frequency localization operator*, with windows φ_1, φ_2 and symbol a is defined by

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbf{R}^{2d}} a(x,\omega)V_{\varphi_1}f(x,\omega)M_\omega T_x\varphi_2(t)\,dxdt.$$

Conditions for $A_a^{\varphi_1,\varphi_2}$ to be bounded or belong to a Schatten class on $L^2(\mathbf{R}^d)$ are given in [3], which is a paper we wish to study. In this talk, we prove a result needed in [3], namely Theorem 4.2 below.

4.1 Basics of pseudo-differential operators

The object of PDE is to solve equations like $Af(x) = \sum_{|\alpha|\leq N}\sigma_\alpha(x)D^\alpha f(x) = g(x)$. By Fourier inversion ($D^\alpha f(x) = \int_{\mathbf{R}^d}\hat{f}(\omega)(2\pi i\omega)^\alpha e^{2\pi ix\cdot\omega}\,d\omega$), this becomes $Af(x) = \int_{\mathbf{R}^d}[\sum_{|\alpha|\leq N}\sigma_\alpha(x)(2\pi i\omega)^\alpha]\hat{f}(\omega)e^{2\pi ix\cdot\omega}\,d\omega$. This motivates the definition which follows.

Let σ be a tempered distribution on $\mathbf{R}^{2d}$. The operator $K_\sigma f(x) = \int \sigma(x,\omega)\hat{f}(\omega)e^{2\pi ix\cdot\omega}d\omega$ is the *pseudo-differential operator* with *Kohn-Nirenberg* symbol σ. The Kohn-Nirenberg correspondence is the map $\sigma \mapsto K_\sigma$. Some special cases are:

- If $\sigma(x,\omega) = m(x)$, then K_σ is the multiplication operator $f \mapsto mf$
- If $\sigma(x,\omega) = \mu(\omega)$, then K_σ is the Fourier multiplier $f \mapsto \mathcal{F}^{-1}(\mu\hat{f})$. If, in addition $\mu = \hat{h}$, then K_σ is the convolution operator $f \mapsto h * f$, or if $\mu(\omega) = \sum_{|\alpha|\leq N}\sigma_\alpha(2\pi i\omega)^\alpha$, then K_σ is the differential operator with constant coefficients $f \mapsto \sum_{|\alpha|\leq N}\sigma_\alpha D^\alpha f$.
- If $\sigma(x,\omega) = m(x)\mu(\omega)$ and $\mu = \hat{h}$, then K_σ is the product-convolution operator $f \mapsto m\cdot(h*f)$.

It is important to note that K_σ can be expressed as

- an integral operator: $K_\sigma f(x) = \int_{\mathbf{R}^d} k(x,y)f(y)\,dy$ where $k(x,y) = \sigma(x,\cdot)\check{}(y-x) = \mathcal{F}_2\sigma(x,y-x)$
- a convolution operator: $K_\sigma f(x) = f * h_x(x)$, where $h_x(y) = \mathcal{F}^{-1}\sigma(x,y) = \sigma(x,\cdot)\check{}(y)$
- a superposition of time-frequency shifts: $K_\sigma f = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \hat{\sigma}(\eta,u) M_\eta T_{-u} f\,dud\eta$.

In Weyl's approach to quantization, he came up with the following variation of the last representation. The *Weyl transform* is the correspondence $\sigma \mapsto L_\sigma$, where σ is a tempered distribution and L_σ is the ψ-do $L_\sigma f = \int\int \hat{\sigma}(\xi,u) e^{-\pi i \xi\cdot u} T_{-u} M_\xi\,dud\xi$. σ is the *Weyl symbol* and $\hat{\sigma}$ is the *spreading function*.

4.2 Uncertainty principle for STFT

The following is an uncertainty principle for the STFT, which we just state here. The weighted L^p-spaces L^p_a are defined by $\|f\|_{p,a} = (\int_{\mathbf{R}^d} |f(x)|^p (1+|x|)^{ap}\,dx)^{1/p}$.

Theorem 4.1 (**[11, Theorem 1]**) *If*

$$\left(\frac{a}{d} - \frac{1}{p'}\right)\left(\frac{b}{d} - \frac{1}{q'}\right) > \max\left(\frac{1}{pq}, \frac{1}{2p}, \frac{1}{2q}, \frac{1}{4}\right),$$

then there is a constant $C = C(a,b,p,q,g)$ such that for all $f \in L^p_a \cap \mathcal{F}^{-1}L^q_b$,

$$\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} |V_g f(x,y)|\,dxdy \le C(\|f\|_{p,a} + \|\hat{f}\|_{q,b}).$$

The converse is also true.

Because $\|f\|_2^2\|g\|_2^2 \le \int\int |V_g|^2 \le \|V_g f\|_\infty \|V_g f\|_1 \le \|f\|_2\|g\|_2\|V_g f\|_1$, Theorem 4.1 implies the uncertainty principle $\|f\|_2 \le C(\|f\|_{p,a} + \|\hat{f}\|_{q,b})$. It also implies the embedding of $L^p_a \cap \mathcal{F}^{-1}L^q_b$ in the space $S_0 = \{f \in L^1(\mathbf{R}^d) : V_g f \in L^1(\mathbf{R}^{2d})\}$.

4.3 Some trace class ψdo's arising in time-frequency analysis

Theorem 4.2 (**[11, Theorem 3]**) *Let $\sigma \in S_0$. Then the ψdo L_σ (in Weyl's sense) is a trace class operator.*

The proof depends on the following (amazing) computational lemma, the inversion formula for the STFT and the elementary identity (1). It is so elegant that I cannot resist sketching it here. First, let me state the lemma.

Lemma 4.3 (**[11, Lemma 8]**) *Let Φ and ϕ be the Gaussians $\Phi(p,q) = e^{-\pi(p^2+q^2)/2}$, $p,q \in \mathbf{R}^d$ and $\phi(t) = e^{-\pi t^2}$, $t \in \mathbf{R}^d$. Write $x = (x_1,x_2), y = (y_1,y_2) \in \mathbf{R}^d \times \mathbf{R}^d$. Let $\Pi_{x,y}$ be the ψdo (in Weyl's sense) corresponding to the symbol $\mathcal{F}^{-1}(M_y T_x \Phi)$. Then $\Pi_{x,y}$ is a rank-one operator, explicitly, $\Pi_{x,y} = \lambda f \otimes g$, where $\lambda = e^{2\pi i x_2\cdot y_2}$, $g = M_{-y_1-x_2/2}T_{-y_2+x_1/2}\phi$ and $f = M_{-y_1+x_2/2}T_{-y_2-x_1/2}\phi$.*

It follows from this lemma that $\|\Pi_{x,y}\|_{\mathcal{S}_1} = \|\phi\|_2^2$, where $\mathcal{S}_1$ denotes the space of trace class operators on $L^2(\mathbf{R}^d)$. By the inversion formula with $\gamma = g = \Phi$, $\hat{\sigma} = \int\int V_\Phi\hat{\sigma}(x,y) M_y T_x \Phi\,dxdy$. Therefore

$$\begin{aligned} L_\sigma f(t) &= \int\int \hat{\sigma}(p,q) e^{i\pi p\cdot q} e^{2\pi i q\cdot t} f(t+p)\,dpdq \\ &= \int\int \left(\int\int V_\Phi\hat{\sigma}(x,y) M_y T_x \Phi(p,q)\,dxdy\right) e^{i\pi p\cdot q} e^{2\pi i q\cdot t} f(t+p)\,dpdq \\ &= \int\int V_\Phi\hat{\sigma}(x,y) \left(\int\int M_y T_x \Phi(p,q) e^{i\pi p\cdot q} e^{2\pi i q\cdot t} f(t+p)\,dpdq\right) dxdy \\ &= \int\int V_\Phi\hat{\sigma}(x,y) \Pi_{x,y} f(t)\,dxdy. \end{aligned} \tag{2}$$

Thus $L_\sigma = \int\int V_\Phi\hat{\sigma}(x,y)\Pi_{x,y}\,dxdy$. By (1), $S_\Phi\hat{\sigma} \in L^1(\mathbf{R}^{2d})$, so by (2), $L_\sigma \in \mathcal{S}_1$, and $\|L_\sigma\|_{\mathcal{S}_1} \le \|S_\Phi\hat{\sigma}\|_{L^1}\|\|\phi\|_2^2$.

4.4 Papers for further study

Besides [11], the following two papers are needed in [3].

- Hilbert-Schmidt case, [1]
- Interpolation, [9]

5 August 18, 2004—The dual of a Gabor frame—Devin Greene

5.1 Motivation and basic operators

Let f be a signal, and g a window (concentrated at 0). We think of $|V_g f(x,\omega)|$ as the density of the signal at time x and frequency ω. As illustrated in the handouts, a Gaussian window is better than the indicator function of $[-1/2,1/2]$.

Recall that $f \mapsto V_g f$ is "analysis", and the inversion formula gives rise to "synthesis":$V_g f \mapsto f$. Lattice anyone? If we consider $f \mapsto \{V_g f(m\alpha, n\beta)\}_{m,n\in\mathbf{Z}^d}$ as analysis, can we perform synthesis, that is, is it true that $f(t) = \sum_{m,n} V_g f(m\alpha,n\beta)\gamma(t-m\alpha)e^{2\pi i n\beta t}$? Note that $V_g f(m\alpha,n\beta) = (f|M_{n\beta}T_{m\alpha}g)$ so the answer depends on whether $\{M_{n\beta}T_{m\alpha}g\}_{m,n\in\mathbf{Z}^d}$ is a frame, that is,

$$A\|f\|_2^2 \leq \sum_{m,n} |(f|M_{n\beta}T_{m\alpha}g)|^2 \leq B\|f\|_2^2.$$

In this case, the analysis, synthesis and frame operators, $C_g^{\alpha,\beta}$, $D_g^{\alpha,\beta}$, and $S_g^{\alpha,\beta}$, are given by

- $C_g f = \{V_g f(m\alpha,n\beta)\}_{m,n\in\mathbf{Z}^d}$
- $D_g a = \sum a_{n,m} g(t-m\alpha)e^{2\pi i n\beta t}$
- $S_g f = \sum_{n,m} (f|M_{n\beta}T_{m\alpha}g)M_{n\beta}T_{m\alpha}g$

Theorem 5.1 (**[12, Prop. 5.2.1,page 94]**) *If $\mathcal{G}(g,\alpha,\beta) := \{M_{n\beta}T_{m\alpha}g\}_{m,n\in\mathbf{Z}^d}$ is a frame and $\gamma := S_g^{-1}g$ (where S_g is the corresponding frame operator), then $\mathcal{G}(\gamma,\alpha,\beta)$ is the dual frame.*

Proof. The dual frame is $\{S^{-1}M_{n\beta}T_{m\alpha}g\}$. The proposed dual frame is $\{M_{n\beta}T_{m\alpha}S^{-1}g\}$. The theorem requires $M_{n\beta}T_{m\alpha}S^{-1}g = S^{-1}M_{n\beta}T_{m\alpha}g$. But $S_g\gamma = \sum(\gamma|M_{n\beta}T_{m\alpha}g)M_{n\beta}T_{m\alpha}g = g$ and $S_g M_{n\beta}T_{m\alpha}\gamma = \sum_{n',m'}(M_{n\beta}T_{m\alpha}\gamma|M_{n'\beta}T_{m'\alpha}g)M_{n'\beta}T_{m'\alpha}g = M_{n\beta}T_{m\alpha}g$ by translation. □

If $\mathcal{G}(g,\alpha,\beta)$ is a frame, it is called a *Gabor frame.* We shall establish sufficient conditions for

1. $\sum_{m,n} |(f|M_{n\beta}T_{m\alpha}g)|^2 \leq B\|f\|_2^2$
2. $A\|f\|_2^2 \leq \sum_{m,n} |(f|M_{n\beta}T_{m\alpha}g)|^2$

For this, we need the *Wiener space* $W = \{g \in L^\infty(\mathbf{R}^d) : \|g\|_W = \sum_n \sup_{0\leq x\leq 1} |g(x+n)| < \infty\}$.

Lemma 5.2 *If $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame and $G_n(t) := \sum_m g(t-m\alpha)\overline{g}(t-m\alpha-n/\beta)$, then $S_g f(t) = \beta^{-1}\sum_n G_n(t)f(t-n/\beta)$.*

5.2 Papers for further study

A potentially fruitful topic implicit in today's talk is the general theory of "atomic decompositions," see [12, page 95]. A good starting point is the paper [8]. This will be the subject of a future talk in the seminar. The setup is an integrable, irreducible, unitary representation π of a locally compact group G on a Hilbert space H, a fixed function $g \in H$ (think wavelet) and a dense set $\{x_i\}_{i\in I}$ in G such that for each member f of some collection of functions on G we have numbers $\{\lambda_i\}$ such that $f = \sum_{i\in I}\lambda_i\pi(x_i)g$.

6 August 26, 2004—Sufficient conditions for Gabor frames—Devin Greene

6.1 Upper boundedness of the (presumptive) frame operator

We shall now consider the frame operator to be dependent on two windows as follows: $D_{g,\gamma}^{(\alpha,\beta)} := D_\gamma^{(\alpha,\beta)} C_g^{(\alpha,\beta)}$. Lemma 5.2 remains true in this generality, that is, $S_{g,\gamma}f(t) = \beta^{-1}\sum_n G_n(t)f(t-n/\beta)$, with $G_n(t) = \sum_m \gamma(t-m\alpha)\overline{g}(t-m\alpha-n/\beta)$. For later use, we introduce the notation $G_{k,n}(t) = \sum_m \gamma(t-m\alpha-k/\beta)\overline{g}(t-m\alpha-n/\beta)$.

Theorem 6.1 (**[12, Cor. 6.2.3,page 109]**) $(S_{g,g}f|f) \le (1+1/\alpha)(1+1/\beta)\|g\|_W^2\|f\|_2^2$.

The following generalization of Theorem 6.1 is [12, Theorem 6.3.2,p. 113]: for $1 \le p \le \infty$,

$$\|S_{g,\gamma}\|_{L^p\to L^p} \le 2^d(1+1/\alpha)^d(1+1/\beta)^d\|g\|_W\|\gamma\|_W.$$

6.2 Conditions for Gabor frames

By Theorem 6.2 below, sufficient conditions on g,α,β to generate a tight Gabor frame are

1. $\sum_m |g(t-m\alpha)|^2 = \text{constant}$
2. $\operatorname{supp} g$ is compact and $\beta > 2\operatorname{diam}\operatorname{supp} g$

In this case, $G_{k,n}(t) = \delta_{kn}\cdot\text{constant}$ and $(S_{g,g}f|f) = \text{constant}\cdot Id$. Examples are (1) $b_0 = \chi_{[1/2,1/2]}$, $\alpha = 1$, $\beta < 1$ (for $\beta = 1$, the frame is the orthonormal basis $\{\chi_{[1/2,1/2]}(t-m)e^{2\pi int} : m,n \in \mathbf{Z}\}$); (2) $b_1 := b_0 * b_0$ and $b_n := b_{n-1} * b_0$ leads to the theory of B-splines.

Theorem 6.2 (**[12, Theorem 6.5.1,page 121]**) *If $g \in W$, $\sum_m |g(t-m\alpha)|^2 \ge a > 0$ for all t, then there exists $\beta_0 > 0$ such that for all $\beta \le \beta_0$, $\mathcal{G}(g,\alpha,\beta)$ is a frame.*

The following two results are some answers to the general question "For a given g, for which values of α and β is $\mathcal{G}(g,\alpha,\beta)$ a frame?"

Theorem 6.3 (**[12, Cor. 7.5.1,page 138]**) *If $\mathcal{G}(g,\alpha,\beta)$ is a frame, then $\alpha\beta \le 1$.*

Theorem 6.4 (**[12, Theorem 7.5.3,page 140]**) *If g is a Gaussian, then $\mathcal{G}(g,\alpha,\beta)$ is a frame if and only if $\alpha\beta < 1$.*

7 October 28, 2003 (that's right, 2003)—Wiener's lemma for twisted convolution; Introduction to Time-frequency analysis—Bernard Russo

7.1 Wiener's lemma for twisted convolution

This subsection is based on the paper [13] (which was brought to my attention by Ingrid Daubechies during her visit to UCI in August 2003).

1. BANACH ALGEBRA PROOF OF WIENER'S THEOREM

Complex commutative Banach algebra with identity: Associative and commutative algebra B over $\mathbf{C}$ with unit e and complete norm $\|\cdot\|$ with $\|e\| = 1$, $\|xy\| \le \|x\|\|y\|$

spectrum: For $x \in B$, $\sigma(x) = \{\lambda \in \mathbf{C} : \lambda e - x \text{ is invertible in } B\}$

THEOREM: $\sigma(x) \ne \emptyset$ and is compact.

1:1 correspondence: $\{\text{maximal ideals } M\} \leftrightarrow \{\ne 0 \text{ homos } \gamma : B \to \mathbf{C}\}$, $\mathcal{M}_B \leftrightarrow \Gamma(B)$ given by $M = \ker\gamma$

$\Gamma(B) \subset B^*$ is weak*-closed subset of unit ball, so is compact in the weak*-topology.

The Gelfand transform $B \ni x \mapsto \hat{x} \in C(\Gamma(B))$ is an algebra homomorphism ($\hat{x}(\gamma) = \gamma(x)$)

THEOREM: $x \in B$ is invertible $\Leftrightarrow \gamma(x) \neq 0\ \forall \gamma \in \Gamma(B)$, that is, $\hat{x}$ is invertible in $C(\Gamma(B))$
EXAMPLES $C(\Omega),\ L^1(\mathbf{T}),\ A(\mathbf{T}) = \{f \in L^1(\mathbf{T}) : \hat{f} \in \ell^1(\mathbf{Z})\},\ \|f\|_A = \|\hat{f}\|_{\ell^1(\mathbf{Z})}$
$\Gamma(A(\mathbf{T})) = \mathbf{T}$ as topological spaces. If $\gamma \leftrightarrow e^{it}$, then $\gamma(f) = f(e^{it})$
WIENER'S THEOREM: Let $f \in A(\mathbf{T})$. If $f(e^{it}) \neq 0, \forall t \in \mathbf{R}$, then $\frac{1}{f} \in A(\mathbf{T})$
$C(\Omega)$ and $L^1(\mathbf{T})$ are (commutative) **Banach *-algebras**.

2. EQUIVALENT FORM OF WIENER'S THEOREM
WIENER'S THEOREM: $f \in L^1(\mathbf{T}),\ \hat{f} \in \ell^1(\mathbf{Z}),\ f(e^{it}) \neq 0\ \forall t \Rightarrow \frac{1}{f} \in L^1(\mathbf{T})$ and $\widehat{\frac{1}{f}} \in \ell^1(\mathbf{Z})$
CONVOLUTION THEOREM: $a \in \ell^1(\mathbf{Z}),\ T_a c := a * c$ invertible on $\ell^2(\mathbf{Z}) \Rightarrow a$ invertible in $\ell^1(\mathbf{Z})$
This implies that T_a is invertible on $\ell^p(\mathbf{Z}^d)$, $1 \leq p \leq \infty$

3. TWISTED CONVOLUTION
Convolution: For $a, b \in \ell^1(\mathbf{Z}^d)$, $(a * b)_m = \sum_{k \in \mathbf{Z}^d} a_k b_{m-k}$
$\ell^1(\mathbf{Z}^d)$ is a commutative Banach *-algebra with involution $(a_k)^* = (\overline{a_{-k}})$
Change of notation: For $a, b \in \ell^1(\mathbf{Z}^{2d})$, $(a * b)_{m,n} = \sum_{k,l \in \mathbf{Z}^d} a_{kl} b_{m-k,n-l}$
Twisted convolution: For $\theta > 0$, $a, b \in \ell^1(\mathbf{Z}^{2d})$, $(a \natural_\theta b)_{m,n} = \sum_{k,l \in \mathbf{Z}^d} a_{kl} b_{m-k,n-l} e^{2\pi i \theta (m-k)\cdot l}$

PROPERTIES:

- $\ell^1 \natural_\theta \ell^p \subset \ell^p\ (1 \leq p \leq \infty)$
- $(\ell^1(\mathbf{Z}^{2d}), \natural_\theta, ^*)$ is a Banach *-algebra with $(a_{kl})^* = (\overline{a_{-k,-l}} e^{2\pi i \theta k \cdot l})$
- $\theta \in \mathbf{Z} \Rightarrow \natural_\theta = *$
- $\theta - \theta' \in \mathbf{Z} \Rightarrow \natural_\theta = \natural_{\theta'}$
- $\natural_\theta$ is commutative $\Leftrightarrow \theta \in \mathbf{Z}$

Theorem 7.1 (**[13, Theorem 1.1]**) *Fix $\theta > 0$. If $a \in \ell^1(\mathbf{Z}^{2d})$ and $T_a c := a \natural_\theta c$ is invertible on $\ell^2(\mathbf{Z}^{2d})$, then a is invertible in $\ell^1(\mathbf{Z}^{2d})$*

REMARKS:

- This implies that T_a is invertible on all $\ell^p(\mathbf{Z}^{2d})$ $(1 \leq p \leq \infty)$
- If $\theta \in \mathbf{Q}$, you can use commutative methods to prove Theorem 1 (This is done in the book by Gröchenig)
- If $\theta \notin \mathbf{Q}$, commutative methods break down.
- Noncommutative methods used:
  - relation of twisted convolution to ordinary convolution on a Heisenberg group
  - the group algebra $L^1(G)$ of a nilpotent group is a *symmetric* Banach *-algebra.

4. ROTATION C*-ALGEBRAS
For $\alpha > 0, \beta > 0$, the C*-algebra $C^*(\alpha, \beta)$ generated by $\{T_{\alpha k} M_{\beta l} : k, l \in \mathbf{Z}^d\} \subset B(L^2(\mathbf{R}^d))$ is a representation of the **rotation algebra**.

REMARKS:

- If $\alpha\beta \in \mathbf{Q}$, the structure of $C^*(\alpha, \beta)$ is well-known and not deep.
- If $\alpha\beta \notin \mathbf{Q}$, then $C^*(\alpha, \beta)$ has no nontrivial two-sided closed ideals (i.e. is a simple C*-algebra), but its detailed structure is complex.
- Since the Gabor frame operator $S = S_{g,\alpha,\beta}$ commutes with all time-frequency shifts $T_{\alpha k} M_{\beta l}$, it lies in the commutant of $C^*(\alpha, \beta)$, which is known to be generated by $\{T_{k/\beta} M_{l/\alpha} : k, l \in \mathbf{Z}^d\}$. Thus (Janssen 1995) $Sf = (\alpha\beta)^{-d} \sum_{k,l \in \mathbf{Z}^d} \langle g, T_{k/\beta} M_{l/\alpha} g \rangle T_{k/\beta} M_{l/\alpha} f$.

"For the better understanding of the deeper properties of S, it is natural to study the (Banach) subalgebra of $C^*(\alpha,\beta)$ consisting of absolutely convergent series of time-frequency shifts." Given $\alpha > 0, \beta > 0$ and a sub-additive weight v, the operator algebra $\mathcal{A}_v(\alpha,\beta)$ in $B(L^2(\mathbf{R}^d))$ is defined to be

$$\mathcal{A}_v(\alpha,\beta) = \{A = \sum_{k,l\in\mathbf{Z}^d} a_{kl} T_{\alpha k} M_{\beta l}, : a = (a_{kl}) \in \ell^1_v(\mathbf{Z}^{2d})\}$$

REMARKS:

- $(\ell^1_v(\mathbf{Z}^{2d}), \natural_\theta, *)$ is a Banach *-algebra with the norm $\|a\|_{1,v} = \|a\|_{\ell^1_v(\mathbf{Z}^{2d})} = \sum_{r\in\mathbf{Z}^{2d}} |a_r| v(r)$
- $\mathcal{A}_v(\alpha,\beta)$ is a Banach algebra in the norm $\|A\|_{\mathcal{A}_v} = \|a\|_{1,v}$
- $\mathcal{A}_v(\alpha,\beta)$ is a dense subalgebra of $C^*(\alpha,\beta)$
- If $\alpha\beta = \theta$, and $\pi(a) = A \in \mathcal{A}_v(\alpha,\beta)$, then $\pi(a\natural_\theta b) = \pi(a)\pi(b)$

"Once the relation between the Gabor frame operator and twisted convolution is understood, then Wiener's lemma for twisted convolution may be applied to the problem of inverting the Gabor frame operator." The following theorem is equivalent to Theorem 7.1 and implies Theorem 7.3.

Theorem 7.2 ([**13, Theorem 1.4**]) *If $A \in \mathcal{A}_v(\alpha,\beta)$ is invertible in $C^*(\alpha,\beta)$, then it is invertible in $\mathcal{A}_v(\alpha,\beta)$.*

From [12, p.288]: "So far, operator algebras have found only marginal use in time-frequency analysis (DAUBECHIES, Landau, Landau, Janssen, Gabardo, Han, LARSON). A full exploration of the interplay between time-frequency analysis and operator algebras is still missing, but this connection holds great promise for the future."

7.2 Introduction to time-frequency analysis

1. FROM THE PREFACE OF [12]

Time-frequency analysis is a modern branch of harmonic analysis. It uses the structure of translations and modulations for the analysis of functions and operators. It is a form of local Fourier analysis treating time and frequency simultaneously and symmetrically. It is motivated by applications in signal analysis and quantum mechanics.

applied side interactions are with: signal analysis, communication theory, image processing
physics interactions are with: phase space analysis, coherent state theory
mathematics interactions are with: Fourier analysis, complex analysis, noncommutative harmonic analysis (Heisenberg group), representation theory, pde, ψdo, operator algebras, numerical analysis

Chapters 1-8: Core material about Time-frequency analysis for $L^2(\mathbf{R}^d)$ using only Hilbert space and analysis

Uncertainty principles in harmonic analysis, Short-time Fourier transform, Bargmann transform, Wigner distribution, Gabor frames, Zak transform.

Chapters 11-14: Quantitative theory of Time-frequency analysis (as opposed to the pure L^2-theory)

Modulation spaces, Gabor analysis, Window design, Wiener's lemma, use of operator algebras, Banach frames

Chapters 9-10: Time-frequency analysis from an abstract viewpoint (Heisenberg group, representation theory); Time-frequency analysis and wavelets (wavelet frames and bases, multiresolution analyses)

2. TIME-FREQUENCY ANALYSIS, SHORT TIME FOURIER TRANSFORM

The goal of **time-frequency analysis** is to understand the properties of functions or distributions simultaneously in time and frequency. In physics, this may be called **phase-space analysis**. The main tool is the *short-time Fourier transform*, which is also called **radar ambiguity function, coherent state transform, cross Wigner distribution**

translation operator on $\mathbf{R}^d$: $T_x f(t) = f(t-x)$
modulation operator on $\mathbf{R}^d$: $M_\omega f(t) = e^{2\pi i\omega\cdot t} f(t)$
short-time Fourier transform (STFT): (of f with respect to a fixed **window** g)
$V_g f(x,\omega) = \int_{\mathbf{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i\omega\cdot t} = \langle f, M_\omega T_x g\rangle$

This can be manipulated into: $V_g f(x,\omega) = \langle \hat{f}, T_\omega M_x \hat{g}\rangle = e^{-2\pi i x\cdot\omega} V_{\hat{g}} \hat{f}(\omega, -x)$

"For g and $\hat{g}$ well localized (e.g. $g \in \mathcal{S}(\mathbf{R}^d)$), $V_g f(x,\omega)$ measures the magnitude of f in a neighborhood of x and of $\hat{f}$ in a neighborhood of ω."

"For practical and numerical purposes one prefers a discrete version of the STFT and aims for **Gabor** expansions of the form $f = \sum_{k,l\in\mathbf{Z}^d} \langle f, T_{\alpha k} M_{\beta l} g\rangle T_{\alpha k} M_{\beta l}\gamma$, where the coefficient $c_{kl} = \langle f, T_{\alpha k} M_{\beta l} g\rangle$ describes the combined time-frequency behavior of f at a point $(\alpha k, \beta l)$ in the time-frequency plane $\mathbf{R}^{2d}$."

(g,γ) is a **pair of dual windows** and $\alpha > 0, \beta > 0$.

To construct Gabor expansions, start with g, α, β and study the associated **Gabor frame operator** $S = S_{g,\alpha,\beta}$

$$Sf = \sum_{k,l\in\mathbf{Z}^d} \langle f, T_{\alpha k} M_{\beta l}\, g\rangle T_{\alpha k} M_{\beta l}\, g$$

REMARKS:

- S is a positive operator; $\langle Sf, f\rangle \geq 0$
- If S is invertible, then $\gamma := S^{-1}g$ enables a Gabor expansion $f = \sum_{k,l\in\mathbf{Z}^d} \langle f, T_{\alpha k} M_{\beta l} g\rangle T_{\alpha k} M_{\beta l}\gamma$, and the following norms are equivalent: $\|f\|_2$, $\left(\sum_{k,l\in Z^d} |\langle f, T_{\alpha k} M_{\beta l} g\rangle|^2\right)^{1/2}$, $\left(\sum_{k,l\in Z^d} |\langle f, T_{\alpha k} M_{\beta l}\gamma\rangle|^2\right)^{1/2}$
- Thus, $\{T_{\alpha k} M_{\beta l} g : k,l \in \mathbf{Z}^d\}$ and $\{T_{\alpha k} M_{\beta l}\gamma : k,l \in \mathbf{Z}^d\}$ are **frames** for $L^2(\mathbf{R}^d)$.
- Invertibility of S is well understood on $L^2(\mathbf{R}^d)$.

"For genuine time-frequency analysis, the pure L^2-theory is insufficient. With $\chi := \chi_{[0,1]^d}$, $\{T_k M_l : k,l \in \mathbf{Z}^d\}$ is an ONB for $L^2(\mathbf{R}^d)$, but $\hat{\chi}$ decays slowly. The coefficients $c_{kl} = \langle f, T_k M_l \chi\rangle$ do not furnish any frequency location. It is not possible to distinguish a smooth function from a rough function by looking at the coefficients in the Gabor expansion."

"For a better description of a function in both time *and* frequency, we need a pair (γ, g) of dual windows enabling a Gabor expansion and possessing good decay and smoothness properties. If we use $\gamma = S^{-1}g$, these properties of (γ, g) are equivalent to the invertibility of S on other function spaces." **EXAMPLE:** (Janssen 1995) If $S_{g,\alpha,\beta}$ is invertible on $L^2(\mathbf{R}^d)$ and if $g \in \mathcal{S}(\mathbf{R}^d)$, then $\gamma = S^{-1}g \in \mathcal{S}(\mathbf{R}^d)$.

To state the main result on window design, fix a **symmetric weight function** $v : \mathbf{R}^{2d} \to [1,\infty)$ and let $\phi(x) = e^{-\pi x^2}$ on $\mathbf{R}^d$. The **modulation space** M_v^1 is $\{f : \mathbf{R}^{2d} \to \mathbf{C} : \int_{\mathbf{R}^d} |V_\phi f(z)| v(z)\, dz < \infty\}$

Theorem 7.3 (**[13, Theorem 1.2]**) *Assume that v is a sub-exponential weight function on $\mathbf{R}^{2d}$. If $g \in M_v^1$ and $S_{g,\alpha,\beta}$ is invertible on $L^2(\mathbf{R}^d)$, then $S_{g,\alpha,\beta}$ is invertible on M_v^1, and thus $\gamma := S^{-1}g \in M_v^1$.*

REMARKS:

- structure is similar to that of Theorem 7.1; invertibility on a Hilbert space and smoothness of the symbol imply invertibility on other function spaces
- For $\alpha\beta \in \mathbf{Q}$, this is "classical" (i.e. commutative)
- For $\alpha\beta \notin \mathbf{Q}$, this is new and deep (i.e. noncommutative)

8 October 29, 2004—Review; Preview; The Gabor von Neumann algebra and Rieffel's incompleteness theorem—Bernard Russo

8.1 Review

During the course of the six meetings in the summer of 2004 (sections 1-6, and one seminar talk in the fall of 2003 (section 7), we have covered the following topics:

- **Uncertainty principles**—From Chapter 2 of [12]—Seminar talk by B. Russo, July 8, 2004 (section 1)
- **Short time Fourier transform**—From Chapter 3 of [12]—Seminar talk by D. Greene, July 19, 2004 (section 2)
- **Gabor frames**—From Chapters 5, 6, and 7 of [12]—Seminar talks by D. Greene, August 5,18 and 26, 2004 (sections 3,5, and 6)
- **Time-frequency localization operators**—From [3] and chapter 14 of [12]—Seminar talk by B. Russo, August 12, 2004 (section 4)
- **Wiener's Lemma for twisted convolution; Introduction to time-frequency analysis**—Seminar talk by B. Russo, Oct. 28, 2003 (section 7).

8.2 Preview

During the past two months (September and October 2004), I have been surfing the literature and have come up with five rich areas for further study. These topics will be explored from their original sources in future seminar talks. Here I just indicate a sample reference for each topic.

- **Density of Gabor frames**—One of the many references for this topic, both from [12] and beyond, is [4], which is the source for the rest of this talk.
- **Uncertainty principles**—The survey paper [10] is a very rich source of results and ideas for future work.
- **Atomic decompositions**—See subsection 5.2 for some information on this topic.
- **Time-frequency localization operators**—This differs from the work [3] cited in subsection 8.1. There, the theme was the interplay between the function spaces containing symbols and spaces of operators containing the operators. Here, the theme is to estimate the number of eigenvalues of the Berezin-Toeplitz localization operator lying in a subinterval of [0,1]. Not suprisingly, this was started by Daubechies and company in 1988. The latest paper seems to be [5].
- **Continuous wavelet transforms**—Here the theme is the following. Given a locally compact group G and a representation π on a Hilbert space H, can you find vectors $\phi \in H$ such that for all $f \in H$,

  $$f = \int_G (f|\pi(x)\phi)\pi(x)\phi\, dx?$$

  I'll mention two references here: [16], and [6]

8.3 The Gabor-von Neumann algebra, its trace and commutant

Gabor time-frequency lattices are sequences of functions $g_{m\alpha,n\beta}(t) = e^{-2\pi i\alpha mt}g(t-n\beta)$ generated by a fixed function $g(t)$. The behavior of the lattice $\{(m\alpha, n\beta)\}$ can be connected to that of the dual lattice $\{(m/\beta, n/\alpha)\}$. The general problems is to find coefficients $c_{m,n}(f)$ such that for all $f \in L^2(\mathbf{R})$, $f(x) = \sum_{m,n} c_{m,n} g_{m\alpha,n\beta}(x)$. Daubechies and others have studied this in the context of frames since 1990. In Theorem 8.3 below, following [4], we shall prove that $\alpha\beta \leq 1$ is necessary for the $g_{m\alpha,n\beta}$ to span L^2. Rieffel proved this using the coupling constant of von Neumann algebras and Lie theory. In [4], only elementary von Neumann algebra theory is used, and the existence of the coupling constant is derived as a result (see subsection 9.2).

Given $g \in L^2$, and $\alpha, \beta > 0$, the coefficient operator $T_{g;\alpha,\beta}$ is defined formally by $T_{g;\alpha,\beta} f = \{(f|g_{m\alpha,n\beta})\}_{m,n\in\mathbf{Z}}$. $T_{g;\alpha,\beta}$ is an "unbounded" operator from $L^2(\mathbf{R})$ to $\ell^2(\mathbf{Z}^2)$, which is densely defined and closable. In this talk, for simplicity, we shall assume it is always bounded, in which case, $T^*_{g;\alpha,\beta} : \ell^2(\mathbf{Z}^2) \to L^2(\mathbf{R})$ is given by $T^*_{g;\alpha,\beta} c = \sum_{m,n} c_{m,n} g_{m\alpha,n\beta}$.

Let $W(p,q)$ denote the unitary operator $W(p,q)f(x) = e^{-2\pi ipx} f(x-q)$. Then $W(j\alpha, k\beta)$ commutes with $W(m/\beta, n/\alpha)$, and using the identity $W(a,b) = W(a,0)W(0,b) = e^{-2\pi iab}W(0,b)W(a,0)$, we have

$$W(j_1\alpha, k_1\beta)W(j_2\alpha, k_2\beta) = e^{2\pi ik_1j_2\alpha\beta}W((j_1+j_2)\alpha, (k_1+k_2)\beta)$$

and in particular $W(j\alpha, k\beta)W(-j\alpha, -k\beta) = e^{-2\pi ijk\alpha\beta}I$, so $W(j\alpha, k\beta)^{-1} = e^{2\pi ijk\alpha\beta}W(-j\alpha, -k\beta)$.

If $\mathcal{T}$ denotes the collection of unitary operators $\{W(j\alpha, k\beta) : j,k \in \mathbf{Z}\}$, then $\operatorname{span}\mathcal{T}$ is a subalgebra of $B(H)$ $(H = L^2(\mathbf{R}))$, which contains I and is closed under the adjoint operation. Hence, the closure $\mathcal{A}$ of $\operatorname{span}\mathcal{T}$ in the weak operator topology is a von Neumann algebra.

Recall that for any subset $\mathcal{S}$ of $B(H)$, its *commutant* is defined to be $\mathcal{S}' := \{T \in B(H) : TS = ST \forall S \in \mathcal{S}\}$, so that for instance $\mathcal{S} \subset \mathcal{S}''$ and $\mathcal{S}' = \mathcal{S}'''$. Then $\mathcal{T}' = \mathcal{A}'$, and von Neumann's celebrated double commutant theorem states that $\mathcal{A} = \mathcal{A}''$ and coincides with the strong operator closure of $\operatorname{span}\mathcal{T}$. Notice that, in this notation, $W(m/\beta, n/\alpha) \in \mathcal{A}'$. The following proposition is proved in [4, Appendix 6.1].

Proposition 8.1 *The von Neumann algebra $\mathcal{A}'$ is generated by the family of unitary operators $\{W(m/\beta, n/\alpha) : m,n \in \mathbf{Z}\}$.*

Proof. (Sketch) Since the operator $W(j\alpha, 0)$, which is multiplication by $e^{-2\pi ij\alpha t}$, belongs to $\mathcal{A}$, every multiplication operator $f \mapsto hf$ by an essentially bounded function h of period $1/\alpha$ belongs to $\mathcal{A}$. Every $T \in \mathcal{A}'$ commutes with such operators, and it follows that $Tf(t) = \sum m_k(t) f(t - k/\alpha)$, where $\operatorname{ess\,sup}_t \sum_k |m_k(t)|^2 < \infty$ and m_k has period β.

Let $\mathcal{C}$ denote the von Neumann algebra generated by the $W(j/\beta, k/\alpha)$ and $S \in \mathcal{C}'$. By the previous paragraph, $Sf(t) = \sum_j n_j(t) f(t - j\beta)$, where n_j is of period $1/\alpha$ and $\operatorname{ess\,sup}_t \sum_j |n_j(t)|^2 < \infty$.

Using these forms of S and T one shows that $STf = TSf$ for every $f \in L^2$ supported on an interval smaller than $\min\{\beta, 1/\alpha\}$; and then $ST = TS$. Thus $\mathcal{C}' \subset \mathcal{A}'' = \mathcal{A}$, and since $\mathcal{C} \subset \mathcal{A}'$, we have $\mathcal{C}' \supset \mathcal{A}'' = \mathcal{A}$, so $\mathcal{C}' = \mathcal{A}$. Then $\mathcal{A}' = \mathcal{C}'' = \mathcal{C}$. □

Every Gabor-von Neumann algebra $\mathcal{A}$ has a trace, that is, a linear functional $\operatorname{tr}_{\mathcal{A}}$ satisfying $\operatorname{tr}_{\mathcal{A}}(I) = 1$, $\operatorname{tr}_{\mathcal{A}}(XY) = \operatorname{tr}_{\mathcal{A}}(YX)$, for $X, Y \in \mathcal{A}$ and $\operatorname{tr}_{\mathcal{A}}(X^*X) > 0$ unless $X = 0$. This trace can be given explicitly, for

$$T = \sum_{|j|,|k|\le N} c_{jk} W(j\alpha, k\beta) = c_{00}I + \cdots$$

by $\operatorname{tr}_{\mathcal{A}}(T) = c_{00}$. It must be shown that this is well-defined and extends to all of $\mathcal{A}$.

Using the commutation formula $W(a,b) = W(a,0)W(0,b) = e^{-2\pi iab}W(0,b)W(a,0)$, one shows that the trace defined above is unique provided $\alpha\beta$ is irrational (see [4, page 457]).

There is a very nice formula for calculating the trace, which among other things, facilitates the extension to $\mathcal{A}$. Let χ_i, $1 \le i \le K$ denote the characteristic functions of disjoint contiguous intervals, none longer than $\min(1/\alpha, \beta)$, that together make up the interval $[0, 1/\alpha]$. Then for any $T \in \mathcal{A}$,

$$\operatorname{tr}_{\mathcal{A}}(T) = \alpha \sum_{i=1}^{K} (T\chi_i|\chi_i). \tag{3}$$

(*Proof.* It is sufficient to check this for $T = W(j\alpha, k\beta)$. Since the intervals are no longer than β, any two translates of one of them by multiples of β are disjoint. Thus the only non-zero contribution to (3) comes from $k = 0$, whereupon the sum becomes $\alpha \int_0^{1/\alpha} e^{2\pi i\alpha jt}\,dt$, which is zero, unless $j = 0$.)

Proposition 8.2 *For any $f, g \in L^2$, $T^*_{f;\alpha,\beta}T_{g;\alpha,\beta}$ commutes with $W(j\alpha, k\beta)$ for all $j,k \in \mathbf{Z}$. That is, $T^*_{f;\alpha,\beta}T_{g;\alpha,\beta} \in \mathcal{A}'$.*

Proof. It suffices to verify that

$$\sum_{m,n}(W(j\alpha,k\beta)h|W(m\alpha,n\beta)g)W(m\alpha,n\beta)f=\sum_{m,n}(h|W(m\alpha,n\beta)g)W(j\alpha,k\beta)W(m\alpha,n\beta)f.$$

8.4 The Daubechies-Landau-Landau proof of Rieffel's incompleteness theorem

Theorem 8.3 (**[4, Theorem 6.1]**) *If for some $g\in L^2(\mathbf{R})$, the functions $\{W(j\alpha,k\beta)g\}_{j,k\in\mathbf{Z}}$ span L^2, then $\alpha\beta\le 1$.*

Proof. Let T denote $T_{g;\alpha,\beta}$ and set $p_\epsilon=(\epsilon I+T^*T)^{-1}g$. Then $(g|p_\epsilon)=\epsilon\|p_\epsilon\|^2+\|Tp_\epsilon\|^2$ so that $(g|p_\epsilon)\ge\|Tp_\epsilon\|^2$. But $(g|p_\epsilon)=(T^*e_{00}|p_\epsilon)$ so that $(g|p_\epsilon)\le\|Tp_\epsilon\|$ and therefore $(g|p_\epsilon)\le 1$.

Assume for the moment that

$$T_{\varphi;\alpha,\beta}S=T_{S^*\varphi;\alpha,\beta}\text{ for all }S\in\mathcal{A}'\text{ and }\varphi\in L^2,\tag{4}$$

and

$$\mathrm{tr}_{\mathcal{A}'}(T^*_{f;\alpha,\beta}T_{g;\alpha,\beta})=\frac{1}{\alpha\beta}(f|g)\text{ for all }f,g\in L^2(\mathbf{R}).\tag{5}$$

From (4), $T^*T(\epsilon I+T^*T)^{-1}=T^*T_{p_\epsilon;\alpha,\beta}$ so from (5),

$$\alpha\beta\,\mathrm{tr}_{\mathcal{A}'}(T^*T(\epsilon I+T^*T)^{-1})=(g|p_\epsilon)\le 1.$$

As $\epsilon\to 0$, $T^*T(\epsilon I+T^*T)^{-1}$ approaches the projection P on the range of T^*T. By assumption, the range of T^* is dense, and therefore so is the range of T^*T, so $P=I$ and $\alpha\beta\le 1$.

We now prove (4) and (5). The first one is a one-liner: if $u\in L^2$, then

$$T_{\varphi;\alpha,\beta}Su=\{(Su|W(m\alpha,n\beta)\varphi)\}=\{(u|S^*W(m\alpha,n\beta)\varphi)\}=\{(u|W(m\alpha,n\beta)S^*\varphi)\}=T_{S^*\varphi;\alpha,\beta}u.$$

Since $T^*_{f;\alpha,\beta}T_{g;\alpha,\beta}\in\mathcal{A}'$, we can use (3) to compute its trace, namely

$$\mathrm{tr}_{\mathcal{A}'}(T^*_{f;\alpha,\beta}T_{g;\alpha,\beta})=\frac{1}{\beta}\sum_j(T^*_{f;\alpha,\beta}T_{g;\alpha,\beta}\chi_j|\chi_j)=\frac{1}{\beta}\sum_j(T_{g;\alpha,\beta}\chi_j|T_{f;\alpha,\beta}\chi_j),$$

with χ_j the characteristic functions of intervals I_j smaller than $1/\alpha$ that decompose $[0,\beta]$. For each k, look at the component $c_{m,n}=(\chi_j|W(m\alpha,n\beta)f)$ of $T_{f;\alpha,\beta}\chi_j$:

$$c_{m,n}=\int\chi_j(t)e^{2\pi im\alpha t}\overline{f(t-n\beta)}\,dt=\int\chi_j(t+n\beta)\overline{f(t)}e2\pi im\alpha(t+n\beta)\,dt=e^{2\pi im\alpha n\beta}\widehat{[\overline{f(t)}\chi_j(t+n\beta)]}(m\alpha).$$

Thus, the $c_{m,n}$ are the Fourier *coefficients* in the basis $\{e^{-2\pi i\alpha jt}\}$ of the function $\overline{f(t)}\chi_j(t+n\beta)$, a function supported on an interval no longer than $1/\alpha$, and similarly for g. Then by Parseval,

$$(T_g\chi_j|T_f\chi_j)=\sum_n\sum_m e^{2\pi im\alpha n\beta}\widehat{[\overline{g(t)}\chi_j(t+n\beta)]}(m\alpha)\overline{e^{-2\pi im\alpha n\beta}\widehat{[\overline{f(t)}\chi_j(t+n\beta)]}(m\alpha)}$$

$$=\frac{1}{\alpha}\sum_n(\overline{g(t)}\chi_j(t+n\beta)|\overline{f(t)}\chi_j(t+n\beta))_{L^2(I_j)}.$$

Summing over j, we have

$$\mathrm{tr}_{\mathcal{A}'}(T^*_{f;\alpha,\beta}T_{g;\alpha,\beta})=\frac{1}{\beta}\sum_j(T_{g;\alpha,\beta}\chi_j|T_{f;\alpha,\beta}\chi_j)=\frac{1}{\alpha\beta}\sum_n(\overline{g(t)}\chi_{[0,\beta]}(t+n\beta)|\overline{f(t)})=\frac{1}{\alpha\beta}(f|g)_{L^2}.$$

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9.1 Two more applications of von Neumann algebras

We first give two more applications of von Neumann algebras to Gabor frames. Theorem 9.2 subsumes the proof of the Wexler-Raz identity (cf. [12, Theorem 7.3.1,page 133]). This fact and the significance of Theorem 9.1 for Gabor frames will be explained in due course. For now, Theorems 9.1 and 9.2 will be used to prove the existence of the coupling constant in subsection 9.2 below.

Although the statements of Theorems 9.1 and 9.2 do not use the GNS (Gelfand-Naimark-Segal) construction arising from the trace $\mathrm{tr}_{\mathcal{A}}$ (or $\mathrm{tr}_{\mathcal{A}'}$), their proofs both do. The GNS construction is defined as follows. Let $L^2(\mathcal{A})$ be the Hilbert space which is the completion of $\mathcal{A}$ in the norm coming from the inner product $[A,B] := \mathrm{tr}_{\mathcal{A}}(AB^*)$, and define a *-isomorphism π of $\mathcal{A}$ into $B(L^2(\mathcal{A}))$ by $\pi(A)B = AB$ for $A, B \in \mathcal{A}$. It is easy to see that the generators $W(j\alpha, k\beta)$ form an orthonormal basis for $L^2(\mathcal{A})$.

Theorem 9.1 (**[4, Theorem 4.3]**) *Let $g \in L^2$, $\alpha > 0, \beta > 0$. Then*

(a) *$T_{g;\alpha,\beta}$ is bounded $L^2(\mathbf{R}) \to \ell^2(\mathbf{Z}^2)$ if and only if $T_{g;1/\beta,1/\alpha}$ is.*

(b) *$T^*_{g;\alpha,\beta}T_{g;\alpha,\beta}$ is invertible if and only if $T_{g;1/\beta,1/\alpha}T^*_{g;1/\beta,1/\alpha}$ is.*

Proof. Let $V : L^2(\mathcal{A}') \to \ell^2(\mathbf{Z}^2)$ be the "coefficient operator" for the Hilbert space $L^2(\mathcal{A}')$, that is, the extension of $\mathcal{A}' \ni T \mapsto \{[T, W(j/\beta, k/\alpha)]\}_{j,k\in\mathbf{Z}} \in \ell^2(\mathbf{Z}^2)$. Let $U : \ell^2(\mathbf{Z}^2) \to \ell^2(\mathbf{Z}^2)$ be the conjugate linear isometry defined by $U\{\gamma_{j,k}\} = \{\overline{\gamma}_{-j,-k}e^{2\pi ijk/\alpha\beta}\}$. Then

- $V(T^*_{f;\alpha,\beta}T_{g;\alpha,\beta}) = (\alpha\beta)^{-1}T_{g;1/\beta,1/\alpha}f$ for $f, g \in L^2$ such that $T^*_{g;\alpha,\beta}T_{g;\alpha,\beta}$ is bounded.
  (By Proposition 8.2, $T^*_{f;\alpha,\beta}T_{g;\alpha,\beta} \in \mathcal{A}' \subset L^2(\mathcal{A}')$.)

  $\Big[$ Proof: $\mathrm{tr}_{\mathcal{A}'}(T^*_{f;\alpha,\beta}T_{g;\alpha,\beta}W(j/\beta,k/\alpha)^*) = \mathrm{tr}_{\mathcal{A}'}(T^*_{f;\alpha,\beta}T_{W(j/\beta,k/\alpha)g;\alpha,\beta})$
  $= (\alpha\beta)^{-1}(f|W(j/\beta,k/\alpha)g) = (\alpha\beta)^{-1}T_{g;1/\beta,1/\alpha}f\Big]$

- $V(X^*) = UV(X)$ for $X \in \mathcal{A}' \subset L^2(\mathcal{A}')$.
  $\big[$ Proof: If $X = \sum \gamma_{j,k}W(j/\beta,k/\alpha) \in \mathcal{A}' \subset L^2(\mathcal{A}')$, then
  $X^* = \sum \overline{\gamma}_{j,k}e^{2\pi ijk/\alpha\beta}W(-j/\beta,-k/\alpha) = \sum \overline{\gamma}_{-j,-k}e^{2\pi ijk/\alpha\beta}W(j/\beta,k/\alpha)\big]$

- $U(T_{p;1/\beta,1/\alpha}q) = T_{q;1/\beta,1/\alpha}p$ for $p, q \in L^2$.
  $\big[$ Proof: $U(T_{p;1/\beta,1/\alpha}q) = U\{(q|W(j/\beta,k/\alpha)p)\}_{j,k\in\mathbf{Z}} =$
  $\{\overline{(q|e^{-2\pi ijk/\alpha\beta}W(-j/\beta,-k/\alpha))}\} = \{(p|W(j/\beta,k/\alpha)q)\} = T_{q;1/\beta,1/\alpha}p\Big]$

- $X^*g = T^*_{g;1/\beta,1/\alpha}V(X^*)$ for $g \in L^2$ and $X \in \mathcal{A}' \subset L^2(\mathcal{A}')$.
  $\big[$ Proof: If $X = W(j/\beta,k/\alpha)$ then $V(X) = e_{j,k} \in \ell^2(\mathbf{Z}^2)$ and
  $T^*_{g;1/\beta,1/\alpha}V(X) = T^*_{g;1/\beta,1/\alpha}e_{j,k} = g_{j/\beta,k/\alpha} = W(j/\beta,k/\alpha)g = Xg\Big]$

- $V(T^*_{f;\alpha,\beta}T_{g;\alpha,\beta}X) = \frac{1}{\alpha\beta}UT_{f;1/\beta,1/\alpha}T^*_{g;1/\beta,1/\alpha}UV(X)$
  $\Big[$ Proof: If $X \in \mathrm{sp}\,\{W(j/\beta,k/\alpha)\}$ then $V(T^*_{f;\alpha,\beta}T_{g;\alpha,\beta}X) = V(T^*_{f;\alpha,\beta}T_{X^*g;\alpha,\beta}) =$
  $(\alpha\beta)^{-1}T_{X^*g;1/\beta,1/\alpha}f = (\alpha\beta)^{-1}UT_{f;1/\beta,1/\alpha}X^*g = (\alpha\beta)^{-1}UT_{f;1/\beta,1/\alpha}T^*_{g,1/\beta,1/\alpha}V(X^*)$
  $= (\alpha\beta)^{-1}UT_{f;1/\beta,1/\alpha}T^*_{g,1/\beta,1/\alpha}UV(X)\Big]$

This last equation extends by continuity to all $X \in L^2(\mathcal{A}')$ resulting in

$$UV\pi(T^*_{f;\alpha,\beta}T_{g;\alpha,\beta})(UV)^{-1} = \frac{1}{\alpha\beta}T_{f;1/\beta,1/\alpha}T^*_{g;1/\beta,1/\alpha} \quad (\text{ on } \ell^2(\mathbf{Z}^2)).$$

The map $T \mapsto UV\pi(T)(UV)^{-1}$ is a conjugate linear algebra isomorphism and thus preserves the spectrum of all elements, which spectrum can be computed in any C*-algebra containing the element. Thus (with $f = g$), $\sigma(\alpha\beta T^*_{g;\alpha,\beta}T_{g;\alpha,\beta}) = \sigma(T_{g;1/\beta,1/\alpha}T^*_{g;1/\beta,1/\alpha})$ proving both (a) and (b). □

The Wexler-Raz theorem (cf. [12, Theorem 7.3.1,page 133]) states that if $g, h \in L^2$ satisfy the bi-orthogonality relations $(h|g_{m/\beta,n/\alpha}) = \alpha\beta\delta_{m,0}\delta_{n,0}$, then for all $f \in L^2$, $f(x) = \sum (f|h_{m\alpha,n\beta})g_{m\alpha,n\beta}$. We shall see later that this result is subsumed by the following theorem.

Theorem 9.2 (**[4, Theorem 3.1]**) *Let $f, g, h \in L^2$, $\alpha > 0, \beta > 0$. If $T_{f;\alpha,\beta}$, $T_{g;\alpha,\beta}$ and $T_{h;1/\beta,1/\alpha}$ are bounded, then*

$$T^*_{f;\alpha,\beta}T_{g;\alpha,\beta}h = \frac{1}{\alpha\beta}T^*_{h;1/\beta,1/\alpha}T_{g;1/\beta,1/\alpha}f. \tag{6}$$

Proof. The expansion of $T^*_{f;\alpha,\beta}T_{g;\alpha,\beta} \in \mathcal{A}'$ with respect to the orthonormal basis $\{W(j/\beta, k/\alpha)\}$ of $L^2(\mathcal{A}')$ is given by $T^*_{f;\alpha,\beta}T_{g;\alpha,\beta} = \sum \gamma_{j,k}W(j/\beta, k/\alpha)$ where $\{\gamma_{j,k}\} = (\alpha\beta)^{-1}T_{g;1/\beta,1/\alpha}f$. Apply this to $h \in L^2$. On the left side, you get $T^*_{f;\alpha,\beta}T_{g;\alpha,\beta}h$. On the right side, you get $\sum \gamma_{j,k}W(j/\beta, k/\alpha)h = T^*_{h;1/\beta,1/\alpha}(\{\gamma_{j,k}\}) = T^*_{h;1/\beta,1/\alpha}(\alpha\beta)^{-1}T_{g;1/\beta,1/\alpha}f$, as required. Wait a minute, there is an overlooked point here. The formula $T^*_{f;\alpha,\beta}T_{g;\alpha,\beta} = \sum \gamma_{j,k}W(j/\beta, k/\alpha)$ is convergence in the norm of $L^2(\mathcal{A}')$ whereas (6) is an equality in the norm of $L^2(\mathbf{R})$. So it must be shown that the convergence of $\sum \gamma_{jk}W(j/\beta, k/\alpha)$ to $T_{f;\alpha,\beta}$, $T_{g;\alpha,\beta}$ in the norm of $L^2(\mathcal{A}')$ implies that, for $h \in L^2$, $\sum \gamma_{jk}W(j/\beta, k/\alpha)h$ converges to $T^*_{f;\alpha,\beta}T_{g;\alpha,\beta}h$ in $L^2(\mathbf{R})$. This requires an approximation argument which is carried out in [4, Appendix 6.3,pp. 468-469].

9.2 Existence of the coupling constant via Gabor frames

Last time we gave the Daubechies-Landau-Landau proof of Rieffel's incompleteness theorem using elementary von Neumann algebra theory but avoiding Rieffel's intractable coupling constant argument. Now we give the proof, (from the same paper and based on time-frequency analysis ideas) of the existence of the coupling constant for the von Neumann algebra generated by the basic time-frequency operators. The significance of the coupling constant will be mentioned.

Theorem 9.3 (**[4, Theorem 6.2]**) *Let $g \in L^2$, $\alpha > 0, \beta > 0$, and let $\mathcal{A}$ be the von Neumann generated by $\{W(j\alpha, k\beta) : j, k \in \mathbf{Z}\}$. Let Y be the closure of the subspace $\mathcal{A}g$ and let P be the projection onto Y. Similarly, let Y' be the closure of the subspace $\mathcal{A}'g$ and P' the projection onto Y'. Then $P \in \mathcal{A}'$, $P' \in \mathcal{A}$, and*

$$\frac{tr_{\mathcal{A}'}(P)}{tr_{\mathcal{A}}(P')} = \frac{1}{\alpha\beta} \quad (\text{ independent of } g).$$

Proof. For all $T \in \mathcal{A}$, $T(Y) \subset Y$ and $T^*(Y) \subset Y$, so $PTP = TP$ and $PT^*P = T^*P$. Then $PT = (T^*P)^* = (PT^*P)^* = PTP = TP$, so $P \in \mathcal{A}'$. Similarly, $P' \in \mathcal{A}$.

For any Hilbert space operator T, the ranges of T^* and T^*T have the same orthogonal complement. We apply this to $T_{g;\alpha,\beta}$ which is assumed to be bounded and let S denote $T^*_{g;\alpha,\beta}T_{g;\alpha,\beta}$ so that $S \in \mathcal{A}'$. Now Y is the closure of the range of $T^*_{g;\alpha,\beta}$ and hence of S, and by functional calculus, $P = \lim_{k\to\infty}(I - (I - S/\|S\|)^k)$ (strong operator topology). Therefore,

$$\mathrm{tr}_{\mathcal{A}'}(P) = \lim_{k\to\infty}\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}\mathrm{tr}_{\mathcal{A}'}(S^j)/\|S\|^j.$$

Similarly, Y' is the closure of the range of $T^*_{g;1/\beta,1/\alpha}$ and hence of $Q := T^*_{g;1/\beta,1/\alpha}T_{g;1/\beta,1/\alpha}$, $Q \in (\mathcal{A}')' = \mathcal{A}$, so

$$\mathrm{tr}_{\mathcal{A}}(P') = \lim_{k\to\infty}\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}\mathrm{tr}_{\mathcal{A}}(Q^j)/\|Q\|^j.$$

By Theorem 9.2, with $f = g = h$, $Sg = (\alpha\beta)^{-1}Qg$, and therefore

$$S^k g = S^{k-1}Sg = (\alpha\beta)^{-1}S^{k-1}Qg = (\alpha\beta)^{-1}QS^{k-1}g = \cdots = (\alpha\beta)^{-k}Q^k g \text{ and } \mathrm{tr}_{\mathcal{A}'}(S^k) =$$

$$\mathrm{tr}_{\mathcal{A}'}(SS^{k-1}) = \mathrm{tr}_{\mathcal{A}'}(T^*_{g;\alpha,\beta}T_{g;\alpha,\beta}S^{k-1}) = \mathrm{tr}_{\mathcal{A}'}(T^*_{g;\alpha,\beta}T_{S^{k-1}g;\alpha,\beta}) = (\alpha\beta)^{-1}(g|S^{k-1}g) = (\alpha\beta)^{-k}(g|Q^{k-1}g).$$

By the same argument applied to Q but stopping one step earlier:

$$Q^k g = Q^{k-1}Qg = \alpha\beta Q^{k-1}Sg = \alpha\beta SQ^{k-1}g \text{ and } \mathrm{tr}_{\mathcal{A}}(Q^k) =$$

$$\mathrm{tr}_{\mathcal{A}}(QQ^{k-1}) = \mathrm{tr}_{\mathcal{A}}(T^*_{g;1/\beta,1/\alpha}T_{g;1/\beta,1/\alpha}Q^{k-1}) = \mathrm{tr}_{\mathcal{A}}(T^*_{g;1/\beta,1/\alpha}T_{Q^{k-1}g;1/\beta,1/\alpha}) = \alpha\beta(g|Q^{k-1}g).$$

From the proof of Theorem 9.1, $\|S\| = \|T^*_{g;\alpha,\beta}T_{g;\alpha,\beta}\| = (\alpha\beta)^{-1}\|T^*_{g;1/\beta,1/\alpha}T_{g;1/\beta,1/\alpha}\| = (\alpha\beta)^{-1}\|Q\|$. So

$$\frac{\mathrm{tr}_{\mathcal{A}'}(S^j)}{\|S\|^j} = \frac{(\alpha\beta)^{-j}(g|Q^{j-1}g)}{(\alpha\beta)^{-1}\|Q\|^j} = \frac{\mathrm{tr}_{\mathcal{A}}(Q^k)}{\alpha\beta\|Q\|^j},$$

so that

$$\frac{\mathrm{tr}_{\mathcal{A}'}(P)}{\mathrm{tr}_{\mathcal{A}}(P')} = \lim_{k\to\infty} \frac{\sum_{j=1}^k (-1)^{j+1}\binom{k}{j}\frac{\mathrm{tr}_{\mathcal{A}'}(S^j)}{\|S\|^j}}{\sum_{j=1}^k (-1)^{j+1}\binom{k}{j}\frac{\mathrm{tr}_{\mathcal{A}}(Q^j)}{\|Q\|^j}} = \lim_{k\to\infty}(\alpha\beta)^{-1} = (\alpha\beta)^{-1}.$$

This completes the proof for the case that $T_{g;\alpha,\beta}$ is bounded. The general case follows by an approximation argument. □

The significance of this result is the following property of the coupling constant in von Neumann algebra theory (cf. [15, Exercise 9.6.30]).

Corollary 9.4 *Suppose that ϕ is a *-isomorphism of the Gabor von Neumann algebra corresponding to the lattice with parameters α, β onto the Gabor von Neumann algebra corresponding to the lattice with parameters α', β'. Then there exists a unitary operator U on $L^2(\mathbf{R})$ such that $\phi(T) = UTU^{-1}$ if and only if $\alpha\beta = \alpha'\beta'$.*

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