## 3

## Evolution Algebras

As a system of abstract algebra, evolution algebras are nonassociative algebras. There is no deep structure theorem for general nonassociative algebra. However, there are deep structure theorem and classification theorem for evolution algebras because we introduce concepts of dynamical systems to evolution algebras. In this chapter, we shall introduce the foundation of the evolution algebras. Section 1 contains basic definitions and properties. Section 2 introduces evolution operators and examines related algebras, including multiplication algebras and derived Lie algebras. Section 3 introduces a norm to an evolution algebra. In Section 4, we introduce the concepts of periodicity, algebraic persistency, and algebraic transiency. In the last section, we obtain the hierarchy of an evolution algebra. For illustration, there are examples in each section.

### 3.1 Definitions and Basic Properties

In this section, we establish the algebraic foundation for evolution algebras. We define evolution algebras by generators and defining relations. It is notable that the generator set of an evolution algebra can serve as a basis of the algebra. We study the basic algebraic properties of evolution algebras, for example, nonassociativity, non-power-associativity, and existence of unitary elements. We also study various algebraic concepts in evolution algebras, for example, evolution subalgebras and evolution ideals. In particular, we define occurrence relations among elements of an evolution algebra and the connectedness of an evolution algebra.

### 3.1.1 Departure point

We define algebras in terms of generators and defining relations. The method of generators and relations is similar to the axiomatic method, where the role of axioms is played by the relations.

Let us recall the formal definition of an algebra $A$ defined by the generators $x_{1}, x_{2}, \ldots, x_{v}$ and the defining relations

$$
f_{1}=0, f_{2}=0, \cdots, f_{r}=0
$$

(Both the set of generators and the set of relations, generally speaking, may be infinite. Since there is no principal difference between finite and infinite cases, we will only consider the finite cases for convenience.) We first consider a nonassociative and noncommutative free algebra $\Re$ with the set of generators $X=\left\{x_{1}, x_{2}, \cdots, x_{v}\right\}$ over a field $K$. It is necessary to point out that its elements are polynomials of noncommutative variables $x_{i}$ with coefficients from $K$ and the basis consists of bracketed words (bracketed monomials). By a bracketed word, we mean a monomial of variables $x_{1}, x_{2}, \cdots, x_{v}$ with brackets inserted so that the order of multiplications in the monomial is uniquely determined. In particular, all $f_{i}$ are elements of this free algebra $\Re$. Then we consider the ideal $I$ in $\Re$ generated by these elements (i.e., the smallest ideal contains these elements). The factor algebra $\Re / I$ is the algebra defined by the generators and the relations. We use notation

$$
\Re / I=\left\langle x_{1}, x_{2}, \cdots, x_{v} \mid f_{1}, f_{2}, \cdots, f_{r}\right\rangle
$$

for the algebra $A$ defined by the generators $x_{1}, x_{2}, \cdots, x_{v}$ and the defining relations $f_{1}=0, f_{2}=0, \cdots, f_{r}=0$.

Now let us define our evolution algebras.
Definition 1. Let $X=\left\{x_{1}, x_{2}, \cdots, x_{v}\right\}$ be the set of generators and $R=$ $\left\{f_{l}=x_{l}^{2}+\sum_{k=1}^{v} a_{l k} x_{k}, f_{i j}=x_{i} x_{j} \mid a_{l k} \in K, i \neq j, l, i, j=1,2, \cdots, v\right\}$ be the set of defining relations, where $K$ is a field, an evolution algebra is then defined by

$$
R(X)=\left\langle x_{1}, \cdots, x_{v} \mid x_{l}^{2}+\sum_{k=1}^{v} a_{l k} x_{k}, x_{i} x_{j}, i \neq j ; i, j, l \in \Lambda\right\rangle
$$

where $\Lambda$ is the index set, $\Lambda=\{1,2, \cdots, v\}$.
Remark 1. In many practical problems, the underlying field $K$ should be the real number field. We say an evolution algebra is real if the underlying field is the real number field $R$. We say an evolution algebra is nonnegative if it is real and any structural coefficient $a_{j k}$ in defining relations is nonnegative. An evolution algebra is called Markov evolution algebra if it is nonnegative and the summation of coefficients in each defining relation is $1, \sum_{k=1}^{v} a_{j k}=1$, for each $j$. We will study Markov evolution algebras in Chapter 4.

Remark 2. There are two types of trivial evolution algebras, zero evolution algebras and nonzero trivial evolution algebras. If the defining relations are given by $x_{i} \cdot x_{j}=0$ for all generators and any $x_{i}^{2}=0$, we say that the algebra generated by these generators is a zero evolution algebra. If the defining
relations are given by $x_{i} \cdot x_{j}=0$ for $i \neq j$ and $x_{i} \cdot x_{i}=k_{i} x_{i}$, where $k_{i} \in K$ is not a zero element, we say that the algebra generated by these generators is a nonzero trivial evolution algebra. To avoid triviality, we always assume that an evolution algebra is not a zero algebra.

To understand evolution algebras defined this way, we need to understand the properties of generators. To this end, we define a notion - the length of a bracketed word. Let $W\left(x_{1}, x_{2}, \cdots, x_{v}\right)$ be a bracketed word. We define the length of $W$, denoting it by $l(W)$, to be the sum of the number of occurrence of each generator $x_{i}$ in $W$. Thus, for the empty word $\phi, l(\phi)=0$, and for any generator $x_{i}, l\left(x_{i}\right)=1$. For example, $W=k\left(x_{1} x_{2}\right)\left(\left(x_{3} x_{1}\right) x_{2}\right)$, here $l(W)=5$, where $k \in K$. Using this notion, we can prove the following theorem.

Theorem 1. If the set of generators $X$ is finite, then the evolution algebra $R(X)$ is finite dimensional. Moreover, the set of generators $X$ can serve as a basis of the algebra $R(X)$.
Proof. We know that a general element of the evolution algebra $R(X)$ is a linear combination of reduced bracketed words. By a reduced bracketed word, we mean a bracketed word that is subject to the defining relations of $R(X)$. Therefore, if we can prove that any reduced word $W$ can be expressed as a linear combination of generators, we can conclude that $R(X)$ has the set of generators $X$ as its basis. Now we use induction to finish the proof.

If $l(w)=0$, then $w=\phi$, and if $l(w)=1$, then $w$ must be a certain generator $x_{i}$. Furthermore, if $l(w)=2, w$ has to be $x_{j}^{2}$ for some generator $x_{j}$, since $x_{i} x_{j}=0$ for two distinct generators. Since $x_{j}^{2}+\sum_{k=1}^{v} a_{j, k} x_{k}=0$, we have $w=x_{j}^{2}=\sum_{k=1}^{v}-a_{j, k} x_{k}$.

Now suppose that when $l(w)=n, w$ can be written as a linear combination of generators. Then let us look at the case of $l(w)=n+1$. Because $w$ here is a reduced bracketed word, the first multiplication in $w$ must be $x_{i} \cdot x_{i}$ for a certain generator $x_{i}$; otherwise $w=\phi$. Since $x_{i} \cdot x_{i}=\sum_{k=1}^{v}-a_{i, k} x_{k}$, after taking the first multiplication, $w$ will become a polynomial, each term of which has a length that is less than or equal to $n$. By induction, each term of the polynomial can be written as a linear combination of generators. Therefore, $w$ can also be written as a linear combination of generators. Hence, by induction, every reduced bracketed word can be written as a linear combination of generators. Thus, the generator set $X$ is a basis for $R(X)$.

We also need to prove that $X$ is a linear independent set. Suppose $\sum_{k} a_{k} x_{k}=0$, then multiply by $x_{k}$ on both sides of the equation, we have $a_{k} x_{k}^{2}=0$. Since $x_{k}^{2} \neq 0$, thus $a_{k}=0$, for every index $k$ (since $R(X)$ is not a zero algebra).

Actually, in the previous theorem, the restrictive condition of finiteness is not necessary, because any element of $R(X)$ is a finite linear combination of
reduced bracketed words and each reduced bracketed word has a finite length whether the number of generators is finite or infinite. Therefore, we have the following two equivalent definitions for evolution algebras.

Definition 2. Let $S=\left\{x_{1}, x_{2}, \ldots, x_{n}, \ldots\right\}$ be a countable set of letters, referred as the set of generators, $V_{S}$ be a vector space spanned by $S$ over a field $K$. We define a bilinear map $m$,

$$
m: \quad V_{S} \times V_{S} \longrightarrow V_{S}
$$

by

$$
\begin{aligned}
m\left(x_{i}, x_{j}\right) & =0, \quad \text { if } i \neq j \\
m\left(x_{i}, x_{i}\right) & =\sum_{k} a_{i, k} x_{k}, \quad \text { for any } i
\end{aligned}
$$

and bilinear extension onto $V_{S} \times V_{S}$. Then, we call the pair $\left(V_{S}, m\right)$ an evolution algebra.

Or, alternatively,
Definition 3. Let $(A, \cdot)$ be an algebra over a field $K$. If it admits a countable basis $x_{1}, x_{2}, \cdots, x_{n}, \cdots$, such that

$$
\begin{aligned}
x_{i} \cdot x_{j} & =0, \quad \text { if } i \neq j \\
x_{i} \cdot x_{i} & =\sum_{k} a_{i, k} x_{k}, \quad \text { for any } i
\end{aligned}
$$

we then call this algebra an evolution algebra. We call the basis a natural basis.
Now, let us discuss several basic properties of evolution algebras. They are corollaries of the definition of an evolution algebra.

Corollary 1. 1) Evolution algebras are not associative, in general.
2) Evolution algebras are commutative, flexible.
3) Evolution algebras are not power-associative, in general.
4) The direct sum of evolution algebras is also an evolution algebra.
5) The Kronecker product of evolution algebras is an evolution algebra.

Proof. We always work with a generator set $\left\{e_{1}, e_{2}, \cdots, e_{n}, \cdots\right\}$, and consider evolution algebras to be nontrivial.

1) Generally, for some index $i, e_{i} \cdot e_{i}=\sum_{j} a_{i j} e_{j}$, there is $j \neq i$, such that $a_{i j} \neq 0$. Therefore, we have $\left(e_{i} \cdot e_{i}\right) \cdot e_{j} \neq 0$. But $e_{i} \cdot\left(e_{i} \cdot e_{j}\right)=e_{i} \cdot 0=0$. That is, $\left(e_{i} \cdot e_{i}\right) \cdot e_{j} \neq e_{i} \cdot\left(e_{i} \cdot e_{j}\right)$.
2) For any two elements $x$ and $y$ in an evolution algebra, $x=\sum_{i} a_{i} e_{i}$ and $y=\sum_{i} b_{i} e_{i}$, we have

$$
x \cdot y=\sum_{i} a_{i} e_{i} \cdot \sum_{j} b_{j} e_{j}=\sum_{i, j} a_{i} b_{j} e_{i} \cdot e_{j}=\sum_{i} a_{i} b_{i} e_{i}^{2}=y \cdot x .
$$

Therefore, any evolution algebra is commutative. Recall that an algebra is flexible if it satisfies $x(y x)=(x y) x$. It is easy to see that a commutative algebra is flexible. Therefore, any evolution algebra is flexible.
3) Take $e_{i}$, we look at

$$
\begin{aligned}
\left(e_{i} \cdot e_{i}\right) \cdot\left(e_{i} \cdot e_{i}\right) & =\sum_{k} a_{i k} e_{k} \cdot \sum_{l} a_{i l} e_{l}=\sum_{k} a_{i k}^{2} e_{k}^{2} \\
\left(\left(e_{i} \cdot e_{i}\right) \cdot e_{i}\right) \cdot e_{i} & =\left(\left(\sum_{k} a_{i k} e_{k}\right) \cdot e_{i}\right) e_{i} \\
& =\left(a_{i i} e_{i}^{2}\right) \cdot e_{i}=\left(a_{i i} \sum_{k} a_{i k} e_{k}\right) \cdot e_{i} \\
& =a_{i i}^{2} e_{i}^{2}
\end{aligned}
$$

generally,

$$
\left(e_{i} \cdot e_{i}\right) \cdot\left(e_{i} \cdot e_{i}\right) \neq\left(\left(e_{i} \cdot e_{i}\right) \cdot e_{i}\right) \cdot e_{i} .
$$

Thus, an evolution algebra is not necessarily power-associative.
4) Consider two evolution algebras $A_{1}, A_{2}$ with generator sets $\left\{e_{i} \mid\right.$ $\left.i \in \Lambda_{1}\right\}$ and $\left\{\eta_{j} \mid j \in \Lambda_{2}\right\}$, respectively. Then, $A_{1} \oplus A_{2}$ has a generator set $\left\{e_{i}, \eta_{j} \mid i \in \Lambda_{1}, j \in \Lambda_{2}\right\}$, once we identify $e_{i}$ with $\left(e_{i}, 0\right)$, $\eta_{j}$ with $\left(0, \eta_{j}\right)$. Actually, this generator set is a natural basis for $A_{1} \oplus A_{2}$. We can verify this as follows:

$$
\begin{aligned}
e_{i} \cdot e_{i} & =\sum_{k} a_{i k} e_{k} \\
e_{i} \cdot e_{j} & =0, \quad \text { if } i \neq j \\
\eta_{i} \cdot \eta_{i} & =\sum_{k} b_{i k} \eta_{k} \\
\eta_{i} \cdot \eta_{j} & =0, \quad \text { if } i \neq j \\
e_{i} \cdot \eta_{j} & =\left(e_{i}, 0\right) \cdot\left(0, \eta_{j}\right)=0 .
\end{aligned}
$$

Therefore $A_{1} \oplus A_{2}$ is an evolution algebra. It is clear that the dimension of $A_{1} \oplus A_{2}$ is the sum of the dimension of $A_{1}$ and that of $A_{2}$. The proof is similar when the number of summands of the direct sum is bigger than 2 .
5) First consider two evolution algebras $A_{1}$ and $A_{2}$ with generator sets $\left\{e_{i} \mid i \in \Lambda_{1}\right\}$ and $\left\{\eta_{j} \mid j \in \Lambda_{2}\right\}$, respectively. On the tensor product of two vector spaces $A_{1}$ and $A_{2}, A_{1} \otimes_{K} A_{2}$, we define a multiplication in the usual way. That is, for $x_{1} \otimes x_{2}$ and $y_{1} \otimes y_{2}$, we define $\left(x_{1} \otimes x_{2}\right) \cdot\left(y_{1} \otimes y_{2}\right)=$ $x_{1} y_{1} \otimes x_{2} y_{2}$. Then, we have the Kronecker product of these two algebras. This Kronecker product is also an evolution algebra, because the generator set of
the Kronecker product is $\left\{e_{i} \otimes \eta_{j} \mid i \in \Lambda_{1}, j \in \Lambda_{2}\right\}$, and the defining relations are given by

$$
\begin{aligned}
& \left(e_{i} \otimes \eta_{j}\right) \cdot\left(e_{i} \otimes \eta_{j}\right) \neq 0 \\
& \left(e_{i} \otimes \eta_{j}\right) \cdot\left(e_{k} \otimes e_{l}\right)=0, \text { if } i \neq k \text { or } j \neq l .
\end{aligned}
$$

As to its dimension, we have $\operatorname{dim}\left(A_{1} \otimes A_{2}\right)=\operatorname{dim}\left(A_{1}\right) \operatorname{dim}\left(A_{2}\right)$. The proof is similar when the number of factors of Kronecker product is greater than 2.

### 3.1.2 Existence of unity elements

For an evolution algebra $A$, we can use a standard construction to obtain an algebra $A_{1}$ that does contain a unity element, such that $A_{1}$ has (an isomorphic copy of) $A$ as an ideal and $A_{1} / A$ has dimension 1 over $K$. We take $A_{1}$ to be the set of all ordered pairs $(k, x)$ with $k \in K$ and $x \in A$; addition and multiplication are defined by

$$
(k, x)+(c, y)=(k+c, x+y)
$$

and

$$
(k, x) \cdot(c, y)=(k c, k y+c x+x y)
$$

where $k, c \in K, x, y \in A$. Then $A_{1}$ is an algebra over $K$ with unitary element $(1,0)$, where 1 is the unity element of the field $K$ and 0 is the empty element of $A$. The set $A^{\prime}$ of all pairs $(0, x)$ in $A_{1}$ with $x$ in $A$ is an ideal of $A_{1}$ which is isomorphic to $A$. For commutative Jordan algebras and alternative algebras, we know that by adjoining a unity element to them we obtain the same type of nonassociative algebras. However, in the case of evolution algebras, $A_{1}$ is no longer an evolution algebra generally. Although the subset $\left\{(1,0),\left(0, e_{i}\right): i \in \Lambda\right\}$ of $A_{1}$ is a basis, and so is a generator set of algebra $A_{1}$, this subset does not satisfy the condition of generator set of an evolution algebra. The following proposition characterizes an evolution algebra with a unity element.

Proposition 1. An evolution algebra has a unitary element if and only if it is a nonzero trivial evolution algebra.

Proof. Let an evolution algebra $A$ has a generator set $\left\{e_{i} \mid i \in \Lambda\right\}$, and $\mu=$ $\sum_{i} a_{i} e_{i}$ be a unity element. We then have $\mu e_{i}=e_{i}$ for each $i \in \Lambda$. That is,

$$
e_{i}=\left(\sum_{j} a_{j} e_{j}\right) e_{i}=a_{i} e_{i}^{2}=a_{i} \sum_{j} a_{i j} e_{j}
$$

We have to have $a_{i} a_{i i}=1$ and $a_{i j}=0$ if $i \neq j$. That means $A$ must be a nonzero trivial evolution algebra, and the unity element is given by $\mu=$ $\sum_{i} \frac{1}{a_{i i}} e_{i}$. On the other hand, if $A$ is a nonzero trivial evolution algebra, it is easy to check that there is a unity element, which is given by $\mu$.

### 3.1.3 Basic definitions

We need some more basic definitions: evolution subalgebras, evolution ideals, principal powers, plenary powers, and simple evolution algebras. Now, let's define them.

Definition 4. 1) Let $A$ be an evolution algebra, and $A_{1}$ be a subspace of $A$. If $A_{1}$ has a natural basis $\left\{e_{i} \mid i \in \Lambda_{1}\right\}$, which can be extended to a natural basis $\left\{e_{j} \mid j \in \Lambda\right\}$ of $A$, we call $A_{1}$ an evolution subalgebra, where $\Lambda_{1}$ and $\Lambda$ are index sets and $\Lambda_{1}$ is a subset of $\Lambda$.
2) Let $A$ be an evolution algebra, and $I$ be an evolution subalgebra of $A$. If $A I \subseteq I$, we call $I$ an evolution ideal.
3) Let $A$ and $B$ be evolution algebras, we say a linear homomorphism $f$ from $A$ to $B$ is an evolution homomorphism, if $f$ is an algebraic map and for a natural basis $\left\{e_{i} \mid i \in \Lambda\right\}$ of $A,\left\{f\left(e_{i}\right) \mid i \in \Lambda\right\}$ spans an evolution subalgebra of $B$. Furthermore, if an evolution homomorphism is one to one and onto, it is an evolution isomorphism.
4) Let $A$ be a commutative algebra, we define principal powers of $a \in A$ as follows:

$$
\begin{aligned}
a^{2}= & a \cdot a \\
a^{3}= & a^{2} \cdot a \\
& \cdots \cdots \\
a^{n}= & a^{n-1} \cdot a
\end{aligned}
$$

and plenary powers of $a \in A$ as follows:

$$
\begin{aligned}
a^{[1]}= & a^{(2)}=a \cdot a \\
a^{[2]}= & a^{\left(2^{2}\right)}=a^{(2)} \cdot a^{(2)} \\
a^{[3]}= & a^{\left(2^{3}\right)}=a^{(4)} \cdot a^{(4)} \\
& \cdots \cdots \cdots \\
a^{[n]}= & a^{\left(2^{n}\right)}=a^{\left(2^{n-1}\right)} \cdot a^{\left(2^{n-1}\right)} .
\end{aligned}
$$

For convenience, we denote $a^{[0]}=a$.
Then, we have a property

$$
\left(a^{[n]}\right)^{[m]}=a^{[n+m]}
$$

where $n$ and $m$ are positive integers. The proof of this property can be obtained by counting the number of a that contains in the mth plenary power of $a^{[n]}$, therefore

$$
\left(a^{[n]}\right)^{[m]}=\left(a^{\left(2^{n}\right)}\right)^{\left(2^{m}\right)}=a^{\left(2^{n} 2^{m}\right)}=a^{\left(2^{n+m}\right)}=a^{[n+m]}
$$

5) We say an evolution algebra $E$ is connected if $E$ can not be decomposed into a direct sum of two proper evolution subalgebras.
6) An evolution algebra $E$ is simple if it has no proper evolution ideal.
7) An evolution algebra $E$ is irreducible if it has no proper subalgebra.

Natural bases of evolution algebras play a privileged role among all other bases, since the generators represent alleles in genetics and states generally in other problems. Importantly, natural bases are privileged for mathematical reasons, too. The following example illustrates this point.

Example 1. Let $E$ be an evolution algebra with basis $e_{1}, e_{2}, e_{3}$ and multiplication defined by $e_{1} e_{1}=e_{1}+e_{2}, e_{2} e_{2}=-e_{1}-e_{2}, e_{3} e_{3}=-e_{2}+e_{3}$. Let $u_{1}=e_{1}+e_{2}, u_{2}=e_{1}+e_{3}$. Then $\left(\alpha u_{1}+\beta u_{2}\right)\left(\gamma u_{1}+\delta u_{2}\right)=\alpha \gamma u_{1}^{2}+(\alpha \delta+$ $\beta \gamma) u_{1} u_{2}+\beta \delta u_{2}^{2}=(\alpha \delta+\beta \gamma) u_{1}+\beta \delta u_{2}$. Hence, $F=K u_{1}+K u_{2}$ is a subalgebra of $E$. However, $F$ is not an evolution subalgebra.

Let $v_{1}, v_{2}$ be a basis of $F$. Then $v_{1}=\alpha u_{1}+\beta u_{2}, v_{2}=\gamma u_{1}+\delta u_{2}$ for some $\alpha, \beta, \gamma, \delta \in K$ such that $D=\alpha \delta-\beta \gamma \neq 0$. By the above calculation, $v_{1} v_{2}=$ $(\alpha \delta+\beta \gamma) u_{1}+\beta \delta u_{2}$. Assume that $v_{1} v_{2}=0$. Then $\beta \delta=0$ and $\alpha \delta+\beta \gamma=0$. If $\beta=0$, we have $\alpha \delta=0$. Then, $D=0$, a contradiction. If $\delta=0$, we reach the same contradiction. Hence $v_{1} v_{2} \neq 0$, and $F$ is not an evolution subalgebra.

We have just seen that evolution algebras are not closed under subalgebras. This is one reason we define these new notions, such as evolution subalgebras. We shall see the relations between these concepts in next subsection.

### 3.1.4 Ideals of an evolution algebra

Classically, an ideal $I$ in an algebra $A$ is first a subalgebra, and then it satisfies $A I \subseteq I$ and $I A \subseteq I$. In the setting of evolution algebras, an evolution ideal is first an evolution subalgebra. However, the conditions for evolution subalgebras seem enough for evolution ideals. We have the following property.

Proposition 2. Any evolution subalgebra is an evolution ideal.
Proof. Let $E_{1}$ be an evolution subalgebra of $E$, then $E_{1}$ has a generator set $\left\{e_{i} \mid i \in \Lambda_{1}\right\}$ that can be extended to a generator set of $E,\left\{e_{i} \mid i \in \Lambda\right\}$, where $\Lambda_{1}$ is a subset of $\Lambda$. For $x \in E_{1}$, and $y \in E$, we write $x=\sum_{i \in \Lambda_{1}} x_{i} e_{i}$ and $y=$ $\sum_{i \in \Lambda} y_{i} e_{i}$, where $x_{i}, y_{i} \in K$, we then have the product $x y=\sum_{i \in \Lambda_{1}} x_{i} y_{i} e_{i}^{2} \in$ $E_{1}$. Therefore, $E_{1} E \subseteq E_{1}$. Since $E$ is a commutative algebra, $E_{1}$ is a two-sided ideal.

This property makes the concept of evolution ideals superfluous. We will use the notion, evolution ideals, as an equivalent concept of evolution subalgebras. As we know, a simple algebra does not have a proper ideal. And an evolution algebra is irreducible if it does not have a proper subalgebra. So, from the above proposition, an irreducible evolution algebra is a simple
evolution algebra, and a simple evolution algebra is an irreducible evolution algebra. They are, actually, the same concepts in evolution algebras. As in general algebra theory, if an evolution algebra can be written as a direct sum of evolution subalgebras, we call it a semisimple evolution algebra. Then we have the following corollary.

Corollary 2. 1) A semisimple evolution algebra is not connected.
2) A simple evolution algebra is connected.

### 3.1.5 Quotients of an evolution algebra

To study structures of evolution algebras, particularly, hierarchies of evolution algebras, quotients of evolution algebras should be studied. Let $E_{1}$ be an evolution ideal of an evolution algebra $E$, then the quotient algebra $\bar{E}=E / E_{1}$ consists of all cosets $\bar{x}=x+E_{1}$ with the induced operations $k \bar{x}=\overline{k x}$, $\bar{x}+\bar{y}=\overline{x+y}, \bar{x} \cdot \bar{y}=\overline{x y}$. We can easily verify that $\bar{E}$ is an evolution algebra. The canonical map $\pi: x \mapsto \bar{x}$ of $E$ onto $\bar{E}$ is an evolution homomorphism with the kernel $E_{1}$.

Lemma 1. Let $\eta_{1}, \eta_{2}, \cdots, \eta_{m}$ be elements of an evolution algebra $E$ with dimension $n$, and satisfies $\eta_{i} \eta_{j}=0$ when $i \neq j$. If some of these elements form a basis of $E$, then there are $(m-n)$ zeroes in this sequence.

Proof. Suppose $\eta_{1}, \eta_{2}, \cdots, \eta_{n}$ form a natural basis of $E$. Then, $\eta_{n+k}, 1 \leq k \leq$ $(m-n)$, can be expressed as a linear combination of $\eta_{i}, 1 \leq i \leq n$. That is, $\eta_{n+k}=\sum_{i=1}^{n} a_{i} \eta_{i}$. Multiplying by $\eta_{i}$ on both sides of this equation, we have $\eta_{n+k} \eta_{i}=a_{i} \eta_{i}^{2}=0$; then, $a_{i}=0$, for each $i, 1 \leq i \leq n$. Therefore, $\eta_{n+k}=0$, where $1 \leq k \leq m-n$.

Theorem 2. Let $E_{1}$ and $E_{2}$ be evolution algebras, and $f: E_{1} \longrightarrow E_{2}$ be an evolution algebraic homomorphism. Then, $K=\operatorname{kernel}(f)$ is an evolution subalgebra of $E_{1}$, and $E_{1} / K$ is isomorphic to $E_{2}$ if $f$ is surjective. Or, $E_{1} / K$ is isomorphic to $f\left(E_{1}\right)$.

Proof. Let $e_{1}, e_{2}, \cdots, e_{m}$ be a natural basis of $E_{1}$, by the definition of evolution algebra homomorphism, $f\left(e_{1}\right), f\left(e_{2}\right), \cdots, f\left(e_{m}\right)$ span an evolution subalgebra of $E_{2}$; denote this subalgebra by $B$. When $\operatorname{dim}(B)=m$, it is easy to see that $K=\operatorname{kernel}(f)=0 . K$ is the zero subalgebra. When $\operatorname{dim}(B)=n<m$, we will prove $\operatorname{dim}(K)=m-n$. For $i \neq j, f\left(e_{i}\right) f\left(e_{j}\right)=f\left(e_{i} e_{j}\right)=0$, and some of $f\left(e_{i}\right) \mathrm{s}$ form a natural basis of the image of $E_{1}$, which is an evolution subalgebra of $E_{2}$. By the Lemma 1, there are $m-n$ zeroes; let's say $f\left(e_{n+1}\right)=0, \cdots$, $f\left(e_{m}\right)=0$. That means, $e_{n+1}, \cdots, e_{m} \in K$. Actually, they span an evolution subalgebra, which is the kernel $K$ of $f$ with dimension $m-n$.

Set a map

$$
\bar{f}: E_{1} / K \longrightarrow f\left(E_{1}\right)
$$

by

$$
x+K \longmapsto f(x) .
$$

It is not hard to see that $\bar{f}$ is an isomorphic.
We may conclude that an evolution algebra can be homomorphic and can only be homomorphic to its quotients. We will study the automorphism group of an evolution algebra in the next section.

### 3.1.6 Occurrence relations

When an element in a basis is viewed as an allele in genetics, or a state in stochastic processes, we are most interested in the following questions: when does the allele $e_{i}$ give rise to the allele $e_{j}$ ? when does a state appear in the next step of the process? To address this question, we introduce a notion, occurrence relations.

Let $E$ be an evolution algebra with the generator set $\left\{e_{1}, e_{2}, \cdots, e_{v}\right\}$. We say $e_{i}$ occurs in $x \in E$, if the coefficient $\alpha_{i} \in K$ is nonzero in $x=\sum_{j=1}^{v} \alpha_{j} e_{j}$. When $e_{i}$ occurs in $x$, we write $e_{i} \prec x$.

It is not hard to see that if $e_{i} \prec e_{i}^{[n]}$, then $\left\langle e_{i}\right\rangle \subseteq\left\langle e_{i}\right\rangle$, where $\langle x\rangle$ means the evolution subalgebra generated by $x$.

When we work on nonnegative evolution algebras, we can obtain a type of partial order among elements.

Lemma 2. Let $E$ be a nonnegative evolution algebra. Then for every $x, y \in$ $E^{+}$, and $n \geq 0$, there is $z \in E^{+}$, such that $(x+y)^{[n]}=x^{[n]}+z$, where $E^{+}=\sum \alpha_{i} e_{i} ; \alpha_{i} \geq 0$.

Proof. We prove the lemma by induction on n . We have $(x+y)^{[0]}=x^{[0]}+y$, and it suffices to set $z=y$. Also, $(x+y)^{[1]}=x^{[1]}+2 x y+y^{2}$. Since $E^{+}$is closed under addition, multiplication, and multiplication by positive scalars, $z=2 x y+y^{2}$ belongs to $E^{+}$.

Assume the claim is true for $n>1$. In particular, give $x, y \in E^{+}$, let $w \in E^{+}$such that $(x+y)^{[n]}=x^{[n]}+w$. Then $(x+y)^{[n+1]}=\left(x^{[n]}+w\right)^{[1]}=$ $\left(x^{[n]}\right)^{[1]}+z=x^{[n+1]}+z$ for some $z \in E^{+}$.

Proposition 3. Let $E$ be a nonnegative evolution algebra. When $e_{i} \prec e_{j}^{[n]}$ and $e_{j} \prec e_{k}^{[m]}$, then $e_{i} \prec e_{k}^{[n+m]}$

Proof. We have $e_{k}^{[m]}=\alpha_{j} e_{j}+y$ for some $\alpha_{j} \neq 0$ and $y \in E$, such that $e_{j}$ does not occur in $y$. We also have $\alpha_{j}>0$ and $y \in E^{+}$. By Lemma 2, $e_{k}^{[n+m]}=\left(e_{k}^{[m]}\right)^{[n]}=\left(\alpha_{j} e_{j}+y\right)^{[n]}=\left(\alpha_{j} e_{j}\right)^{[n]}+z=\alpha_{j}^{\left(2^{n}\right)} e_{j}^{[n]}+z$ for some $z \in E^{+}$. Now, $e_{j}^{[n]}=\beta_{i} e_{i}+v$ for some $\beta_{i}>0$ and $v \in E$ that $e_{i}$ does not occur in $v$. We therefore conclude that $e_{i} \prec e_{k}^{[n+m]}$.

We can have a type of partial order relation among the generators of an evolution algebra $E$. Let $e_{i}$ and $e_{j}$ be any two generators of $E$, if $e_{i}$ occurs in a plenary power of $e_{j}$, for example, $e_{i}$ occurs in $e_{j}^{[n]}$, we then set $e_{i}<e_{j}$, or just $e_{i} \prec e_{j}^{[n]}$. This relation is a partial order in the following sense.
(1) $e_{i} \prec e_{i}^{[0]}$, for any generator of $E$.
(2) If $e_{i} \prec e_{j}^{[n]}$ and $e_{j} \prec e_{i}^{[m]}$, then we say that $e_{i}$ and $e_{j}$ intercommunicate. Generally, $e_{i}$ and $e_{j}$ are not necessarily the same, but the evolution subalgebra generated by $e_{i}$ and the one by $e_{j}$ are the same.
(3) If $e_{i} \prec e_{j}^{[n]}$ and $e_{j} \prec e_{k}^{[m]}$, then $e_{i} \prec e_{k}^{[n+m]}$. This is Proposition 3 .

### 3.1.7 Several interesting identities

At the end of this section, let us give several interesting formulae, they are identities.

Proposition 4. 1) Let $\left\{e_{i} \mid i \in \Lambda\right\}$ be a natural basis of an evolution algebra $A$, then $\left\{e_{i}^{2} \mid i \in \Lambda\right\}$ generates a subalgebra $A$.
2) Let $\left\{e_{i} \mid i \in \Lambda\right\}$ be a natural basis of an evolution algebra $A$, then we have the following identities:

$$
\begin{array}{rlrl}
e_{i}^{m} & =a_{i i}^{m-2} e_{i}^{2}, & \forall i \in \Lambda, & \forall m \geq 2 \\
e_{i}^{2} \cdot e_{j} & =a_{i j} e_{j}^{2}, \quad \forall i, j \in \Lambda, & & \\
\left(e_{i}^{m}\right)^{2} & =a_{i i}^{2 m-4} e_{i}^{(4)}, \quad \forall i \in \Lambda, & \forall m \geq 2 \\
e_{i}^{4} \cdot e_{i}^{4} & =a_{i i}^{4} e_{i}^{(2)}, \quad \forall i, j \in \Lambda, & &
\end{array}
$$

where $a_{i j}$ 's are structural constants of $A$.
3) Let $\left\{e_{i} \mid i \in \Lambda\right\}$ be a natural basis of an evolution algebra, then, for any finite subset $\Lambda_{0}$ of the index set $\Lambda$, we have

$$
\left(\sum_{j \in \Lambda_{0}} e_{j}\right)^{2}=\sum_{j \in \Lambda_{0}} e_{j}^{2}
$$

Proof. 1) Since $\left\{e_{i} \mid i \in \Lambda\right\}$ be a generator set, so

$$
\begin{aligned}
e_{i}^{2} & =\sum_{k} a_{i k} e_{k}, \\
e_{i}^{2} \cdot e_{i}^{2} & =\sum_{k} a_{i k} e_{k} \cdot \sum_{l} a_{i l} e_{l}=\sum_{l, k} a_{i k} a_{i l} e_{k} \cdot e_{l}=\sum_{k} a_{i k}^{2} e_{k}^{2}, \\
e_{i}^{2} \cdot e_{j}^{2} & =\sum_{k} a_{i k} e_{k} \cdot \sum_{l} a_{j l} e_{l}=\sum_{l, k} a_{i k} a_{j k} e_{k}^{2} .
\end{aligned}
$$

Thus, any product of linear combinations of $e_{i}^{2}$ can still be written as a linear combination of $e_{i}^{2}$. This means that $\left\{e_{i}^{2} \mid i \in \Lambda\right\}$ generates a subalgebra of $A$.
2) Since

$$
\begin{aligned}
& e_{i}^{2}=\sum_{k} a_{i k} e_{k} \\
& e_{i}^{3}=e_{i}^{2} \cdot e_{i}=\left(\sum_{k} a_{i k} e_{k}\right) \cdot e_{i}=a_{i i} e_{i}^{2}
\end{aligned}
$$

If $e_{i}^{m-1}=a_{i i}^{m-3} e_{i}^{2}$, for any integer $m>2$, then

$$
e_{i}^{m}=e_{i}^{m-1} \cdot e_{i}=a_{i i}^{m-3} e_{i}^{2} \cdot e_{i}=a_{i i}^{m-3}\left(\sum_{k} a_{i k} e_{k}\right) \cdot e_{i}=a_{i i}^{m-2} e_{i}^{2}
$$

By induction, we got the first formula.
As to the second formula, we have

$$
e_{i}^{2} \cdot e_{j}=\left(\sum_{k} a_{i k} e_{k}\right) \cdot e_{j}=a_{i j} e_{j}^{2}
$$

As to the third formula, we see

$$
\left(e_{i}^{m}\right)^{2}=e_{i}^{m} \cdot e_{i}^{m}=a_{i i}^{2 m-4} e_{i}^{2} \cdot e_{i}^{2}=a_{i i}^{2 m-4} e_{i}^{(4)}
$$

Taking $m=4$, we have

$$
e_{i}^{4} \cdot e_{i}^{4}=a_{i i}^{4} e_{i}^{2} \cdot e_{i}^{2}=a_{i i}^{4} e_{i}^{(4)}
$$

3) By directly computing, we have

$$
\left(\sum_{j \in \Lambda_{0}} e_{j}\right)^{2}=\sum_{j \in \Lambda_{0}} e_{j} \cdot \sum_{i \in \Lambda_{0}} e_{i}=\sum_{i, j \in \Lambda_{0}} e_{i} \cdot e_{j}=\sum_{j \in \Lambda_{0}} e_{j}^{2}
$$

### 3.2 Evolution Operators and Multiplication Algebras

Traditionally, in the study of nonassociative algebras, one usually studies the associative multiplication algebra of a nonassociative algebra and its derived Lie algebra to try to understand the nonassociative algebra. In this section, we also study the multiplication algebra of an evolution algebra and conclude that any evolution algebra is centroidal. We characterize the automorphism group of an evolution algebra and its derived Lie algebra. Moreover, from the viewpoint of dynamics, we introduce the evolution operator for an evolution algebra. This evolution operator will reveal the dynamic information of an evolution algebra. Because we work with a generator set of an evolution algebra, it is also necessary for us to study the change of generator set, or transformations of natural bases.

### 3.2.1 Evolution operators

Definition 5. Let $E$ be an evolution algebra with a generator set $\left\{e_{i} \mid i \in \Lambda\right\}$. We define a $K$-linear map $L$ to be

$$
\begin{aligned}
L: & E \longrightarrow E \\
& e_{i} \mapsto e_{i}^{2} \forall i \in \Lambda
\end{aligned}
$$

then linear extension onto $E$.
Consider $L$ as a linear transformation, ignoring the algebraic structure of $E$, then under a natural basis (the generator set), we can have the matrix representation of the evolution operator $L$. Since

$$
L\left(e_{i}\right)=e_{i}^{2}=\sum_{k} a_{k i} e_{k} \quad \forall i \in \Lambda
$$

then we have

$$
\left(\begin{array}{ccccc}
a_{11} & a_{12} & \cdots & a_{1 n} & \cdots \\
a_{21} & a_{22} & \cdots & a_{2 n} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n 1} & a_{n 2} & \cdots & a_{n n} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right)
$$

If $E$ is a finite dimensional algebra, this matrix will be of finite size. An evolution operator, not being an algebraic map though, can reveal dynamical properties of the evolution algebra, as we will see later on.

Alternatively, by using a formal notation $\theta=\sum_{i \in \Lambda} e_{i}$, no matter whether $\Lambda$ is finite or infinite, we can define $L$ as follows:

$$
L(x)=\theta \cdot x=\left(\sum_{i \in \Lambda} e_{i}\right) \cdot x
$$

for any $x \in E$. According to the distributive law of product to addition in algebra $E, L$ is a linear map. Because

$$
L\left(e_{i}\right)=\left(\sum_{i \in \Lambda} e_{i}\right) \cdot e_{i}=e_{i}^{2}, \quad \forall i \in \Lambda
$$

this definition for an evolution operator is the same as the previous one. We do not feel uncomfortable about the notation $\theta=\sum_{i \in \Lambda} e_{i}$, when $\Lambda$ is infinite, since the product $\left(\sum_{i \in \Lambda} e_{i}\right) \cdot x$ is always finite. We may call this $\theta$ a universal element.

Now, we state a theorem that will be used to get the equilibrium state or a fixed point of the evolution of an evolution algebra.

Theorem 3. If $E_{0}$ is an evolution subalgebra of an evolution algebra $E$, then the evolution operator $L$ of $E$ leaves $E_{0}$ invariant.

Proof. Let $\left\{e_{i} \mid i \in \Lambda_{0}\right\}$ be a natural basis of $E_{0}$, and $\left\{e_{i} \mid i \in \Lambda\right\}$ be its extension to a natural basis of $E$, where $\Lambda_{0} \subset \Lambda$. Given $x \in E_{0}$, then $x=\sum_{i \in \Lambda_{0}} c_{i} e_{i}$, and the action of the evolution operator is

$$
L(x)=\sum_{i \in \Lambda_{0}} c_{i} e_{i}^{2}=\sum_{i \in \Lambda_{0}, \quad k \in \Lambda_{0}} c_{i} a_{k i} e_{k}
$$

since $E_{0}$ is a subalgebra. Therefore, $L(x) \in E_{0}$, then $L\left(E_{0}\right) \subset E_{0}$. Furthermore, $L^{n}\left(E_{0}\right) \subset E_{0}$, for any positive integer $n$.

### 3.2.2 Changes of generator sets (Transformations of natural bases)

Let $\left\{e_{i} \mid i \in \Lambda\right\}$ and $\left\{\eta_{j} \mid j \in \Lambda\right\}$ be two generator sets (natural bases) for an evolution algebra $E$. Suppose the transformation between them is given by $e_{i}=\sum_{k} a_{k i} \eta_{k}$ or $\eta_{i}=\sum_{k} b_{k i} e_{k}$. And suppose the defining relations are $e_{i} \cdot e_{j}=0$ if $i \neq j, e_{i}^{2}=\sum_{k} p_{k i} e_{k}$, and $\eta_{i} \cdot \eta_{j}=0$ if $i \neq j, \eta_{i}^{2}=\sum_{k} q_{k i} \eta_{k}$, $i, j \in \Lambda$, respectively. Then, we have

$$
\begin{aligned}
e_{i} \cdot e_{j} & =\left(\sum_{k} a_{k i} \eta_{k}\right) \cdot\left(\sum_{k} a_{k j} \eta_{k}\right) \\
& =\sum_{k} a_{k i} a_{k j} \eta_{k}^{2}=\sum_{v, k} a_{k i} a_{k j} q_{v k} \eta_{v} \\
& =\sum_{v} \sum_{k} q_{v k} a_{k i} a_{k j} \eta_{v}=0 .
\end{aligned}
$$

Since each component coefficient of zero vector must be 0 , we get $\sum_{k} q_{v k} a_{k i} a_{k j}=$ 0 for $v \in \Lambda$ and $i \neq j$. Similarly, from

$$
\begin{aligned}
e_{i} \cdot e_{i} & =\left(\sum_{k} a_{k i} \eta_{k}\right)^{2}=\sum_{k} a_{k i}^{2} \eta_{k}^{2} \\
& =\sum_{v, k} a_{k i}^{2} q_{v k} \eta_{v}=\sum_{v, k, u} a_{k i}^{2} q_{v k} b_{u v} e_{u} \\
& =\sum_{u} p_{u i} e_{u}
\end{aligned}
$$

we get $p_{u i}=\sum_{v, k} b_{u v} q_{v k} a_{k i}^{2}$. Thus, summarizing all these information together, we have

$$
\begin{aligned}
A^{-1} Q A^{(2)} & =P \\
Q(A * A) & =0
\end{aligned}
$$

where $A=\left(a_{i j}\right), Q=\left(q_{i j}\right), P=\left(p_{i j}\right), A^{(2)}=\left(a_{i j}^{2}\right)$ and "*" of two matrices is defined as follows.

Let $A=\left(a_{i j}\right)$ and $B=\left(b_{i j}\right)$ be two $n \times n$ matrices, then $A * B=\left(c_{i j}^{k}\right)$ is a matrix with size $n \times \frac{n(n-1)}{2}$, where $c_{i j}^{k}=a_{k i} \cdot b_{k j}$ for pairs $(i, j)$ with $i<j$, the rows are indexed by $k$ and the columns indexed by pairs $(i, j)$ with the lexicographical order.

We can also use $B$ to describe the above condition

$$
\begin{aligned}
B^{-1} P B^{(2)} & =Q \\
P(B * B) & =0
\end{aligned}
$$

where $B A=A B=I$.

### 3.2.3 "Rigidness" of generator sets of an evolution algebra

By "rigidness," we mean that an evolution operator is specified by a generator set. Let us illustrate this point in the following way. Given a generator set $\left\{e_{i} \mid i \in \Lambda\right\}$, we have an evolution operator, denoted by $L_{e}$. When the generator set is changed to $\left\{\eta_{j} \mid j \in \Lambda\right\}$, we also have an evolution operator, denoted by $L_{\eta}$. Since a generator set is also a natural basis in evolution algebras, it might be expected that $L_{e}$ and $L_{\eta}$, as linear maps, should be the same. However, they are different, unless additional conditions are imposed. Therefore, an evolution operator is not just a linear map. It is a map related to a specific generator set. This property is very useful to study the dynamic behavior of an algebra, because a multiplication in an algebra is viewed as a dynamical step. In the following lemma, we describe an additional condition about transformations of natural bases that guarantee $L_{e}$ and $L_{\eta}$ will be the same linear map.

Lemma 3. $L_{e}$ and $L_{\eta}$ are the same invertible linear map if and only if the generator sets $\left\{e_{i} \mid i \in \Lambda\right\}$ and $\left\{\eta_{j} \mid j \in \Lambda\right\}$ are the same, or if one can be obtained from the other by a permutation.

Proof. Here we use the same notations as those used in the previous subsection. The matrix representation of $L_{\eta}$ is $Q$ under the generator set $\left\{\eta_{j} \mid j \in \Lambda\right\}$, and

$$
\begin{aligned}
L_{\eta}\left(e_{1}, e_{2}, \cdots, e_{n}\right) & =L_{\eta}\left(\eta_{1}, \eta_{2}, \cdots, \eta_{n}\right) A \\
& =\left(\eta_{1}, \eta_{2}, \cdots, \eta_{n}\right) Q A \\
& =\left(e_{1}, e_{2}, \cdots, e_{n}\right) A^{-1} Q A
\end{aligned}
$$

Thus, the matrix representation of $L_{\eta}$ is $A^{-1} Q A$ under the generator set $\left\{e_{i} \mid i \in \Lambda\right\}$. But as we know, the matrix representation of $L_{e}$ is $P$ under the natural basis $\left\{e_{i} \mid i \in \Lambda\right\}$. Therefore, $P=A^{-1} Q A$, if $L_{\eta}$ and $L_{e}$ can be taken as the same linear maps. From the previous subsection, we know $A^{-1} Q A^{(2)}=P$,
so we have $A^{-1} Q A=A^{-1} Q A^{(2)}$. Since $L_{\eta}$ is invertible, we then have $A=A^{(2)}$. Similarly, we have $B=B^{(2)}$. Since $a_{i j}=a_{i j}^{2}, a_{i j}$ must be 1 or 0 and $b_{i j}$ must also be 1 or 0 , then we can prove $A$ can only be a permutation matrix as follows:

$$
\left(\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1 n} \\
a_{21} & a_{22} & \cdots & a_{2 n} \\
\cdots & \cdots & \cdots & \cdots \\
a_{n 1} & a_{n 2} & \cdots & a_{n n}
\end{array}\right)\left(\begin{array}{cccc}
b_{11} & b_{12} & \cdots & b_{1 n} \\
b_{21} & b_{22} & \cdots & b_{2 n} \\
\cdots & \cdots & \cdots & \cdots \\
b_{n 1} & b_{n 2} & \cdots & b_{n n}
\end{array}\right)=\left(\begin{array}{ccc}
1 & \cdots & 0 \\
\cdots & \ddots & \cdots \\
0 & \cdots & 1
\end{array}\right)
$$

Without loss of generality, suppose that $a_{11} \neq 0, a_{12} \neq 0$, and $a_{1 k}=0$ for $k \geq$ 3. Then we have $a_{11} b_{11}+a_{12} b_{21}=1$. Thus, we have either $b_{11} \neq 0$ or $b_{21} \neq 0$. But only one of these two entries can be nonzero, otherwise $a_{11} b_{11}+a_{12} b_{21}=2$. Now, suppose $b_{21} \neq 0$, and $b_{11}=0$, then $a_{11} b_{12}+a_{12} b_{22}=0$, then we must have $b_{12}=0$; and by $a_{11} b_{13}+a_{12} b_{23}=0$, we have $b_{13}=0$; inductively, $b_{1 j}=0, j=2,3, \cdots$. This means $b_{11}=b_{12}=\cdots=b_{1 n}=0$. This contradicts the nonsingularity of $B$. If we suppose $b_{11} \neq 0$, and $b_{21}=0$, similarly we get $b_{21}=b_{22}=\cdots=b_{2 n}=\cdots=0$. That is a contradiction. Therefore, every row of $A$ can only have one entry that is not zero. Similarly, we can prove that every column of $A$ can only have one entry that is nonzero. Therefore, $A$ is a permutation matrix.

### 3.2.4 The automorphism group of an evolution algebra

Given an evolution algebra $E$, it is important to know how many generator sets $E$ can have. To study this problem, we need to study the automorphism group of an evolution algebra.

Proposition 5. Let $g$ be an automorphism of an evolution algebra $E$ with a generator set $\left\{e_{i} \mid i \in \Lambda\right\}$, then $G^{-1} P G^{(2)}=P$ and $P(G * G)=0$, where $G$ and $P$ are the matrix representations of $g$ and $L$ respectively.

Proof. Write $g\left(e_{i}\right)=\sum_{k} g_{k i} e_{k}$ and $G=\left(g_{i j}\right)$. For $i \neq j$, we have

$$
\begin{aligned}
g\left(e_{i} \cdot e_{j}\right) & =0 \\
& =g\left(e_{i}\right) g\left(e_{j}\right) \\
& =\sum_{k} g_{k i} e_{k} \cdot \sum_{k} g_{k j} e_{k} \\
& =\sum_{k} g_{k i} g_{k j} e_{k}^{2} \\
& =\sum_{k, v} p_{v k} g_{k i} g_{k j} e_{v} .
\end{aligned}
$$

So we have $\sum_{k} p_{v k} g_{k i} g_{k j}=0$, for each $v$. That is $P(G * G)=0$. For $i=i$,

$$
\begin{aligned}
g\left(e_{i} \cdot e_{i}\right) & =g\left(e_{i}\right) g\left(e_{i}\right) \\
& =\sum_{k} g_{k i}^{2} e_{k}^{2} \\
& =\sum_{k, j} g_{k i}^{2} p_{j k} e_{j} \\
& =\sum_{k, j} p_{k i} g_{j k} e_{j} .
\end{aligned}
$$

Thus, we have $\sum_{k} p_{j k} g_{k i}^{2}=\sum_{k} g_{j k} p_{k i}$. That is $P G^{(2)}=G P$, thus $G^{-1} P G^{(2)}=P$.

Therefore, we can characterize the automorphism group of $E$ as

$$
\text { Auto }(E)=\left\{G \mid G^{-1} P G^{(2)}=P, \text { and } P(G * G)=0\right\}
$$

We can use the automorphism group to give a description of the collection of all generator sets. We write it as a corollary.
Corollary 3. Let $B=e_{i}: i \in \Lambda$ be a generator set of an evolution algebra $E$. Then the family $g(B): g \in \operatorname{Auto}(E)$ is the collection of all different generator sets of $E$.

### 3.2.5 The multiplication algebra of an evolution algebra

Let $E$ be an algebra, denote $L_{a}$ and $R_{a}$ as the operators of the left and right multiplication by the element $a$ respectively:

$$
\begin{aligned}
& L_{a}: x \mapsto a \cdot x \\
& R_{a}: x \mapsto x \cdot a .
\end{aligned}
$$

The subalgebra of the full matrix algebra $\operatorname{Hom}(E, E)$ of the endomorphisms of the linear space $E$, generated by all the operators $L_{a}, a \in E$, is called the operator algebra of left multiplication of the algebra $E$, denoted by $L(E)$. The operator algebra of right multiplication $R(E)$ of the algebra $E$ is defined analogously. The subalgebra of $\operatorname{Hom}(E, E)$ generated by all the operators $L_{a}, R_{a}, a \in E$ is called the multiplication algebra of the algebra $E$, denoted by $M(E)$, which is actually the enveloping algebra of all operators $L_{a}, R_{a}$, $a \in E$.

Corollary 4. If $E$ is an evolution algebra, $L(E)=R(E)=M(E)$ is an associative algebra with a unit.
Proof. Since $E$ is commutative, it is obvious.
Corollary 5. If $E$ is an evolution algebra with a natural basis $\left\{e_{i} \mid i \in \Lambda\right\}$, then $\left\{L_{i} \mid i \in \Lambda\right\}$ spans a linear space, denoted by $\operatorname{span}(L, E)$, which is the set of all the operators of left (right) multiplication, where $L_{i}=L_{e_{i}}$. The vector space $\operatorname{span}(L, E)$ and $E$ have the same dimension. Generally, we also have $\operatorname{dim}(E)<\operatorname{dim}(L(E))$ if $\operatorname{dim} E^{2} \neq 1$.

Proof. For any operator of left multiplication $L_{x}$, we can write $x=\sum_{i} a_{i} e_{i}$ uniquely, then by the linearity of multiplication in $E, L_{x}=\sum_{i} a_{i} L_{i}$. If

$$
L_{x}=L_{y}
$$

for $y=\sum_{i} b_{i} e_{i}$, then

$$
L_{x}\left(e_{k}\right)=L_{y}\left(e_{k}\right), \text { and, }\left(\sum_{i} a_{i} e_{i}\right) \cdot e_{k}=\left(\sum_{i} b_{i} e_{i}\right) \cdot e_{k}
$$

Thus,

$$
\begin{aligned}
a_{k} e_{k}^{2} & =b_{k} e_{k}^{2}, \\
\left(a_{k}-b_{k}\right) e_{k}^{2} & =0 \\
\left(a_{k}-b_{k}\right) \sum_{i} p_{k i} e_{i} & =0
\end{aligned}
$$

Since $E$ is a nontrivial algebra, there is $j, p_{k j} \neq 0$, and $\left(a_{k}-b_{k}\right) p_{k j} e_{j}=0$, thus $a_{k}-b_{k}=0$ for each $k$. Therefore $x=y$. This means that $x \mapsto L_{x}$ is an injection. So the linear space that is spanned by all operators of left multiplication can be spanned by the set $\left\{L_{i} \mid i \in \Lambda\right\}$. Moreover the set $\left\{L_{i} \mid i \in \Lambda\right\}$ is a basis for $\operatorname{span}(L, E)$. However, since the algebra $E$ is not associative, $x \mapsto L_{x}$ is not an algebraic map from $E$ to $L(E)$. Generally, $\left\{L_{i} \mid i \in \Lambda\right\}$ is not a basis for $L(E)$. Since $\operatorname{dim} E^{2}>1$, there are different generators $e_{i}$ and $e_{j}$ whose square vectors $e_{i}^{2}$ and $e_{j}^{2}$ are not parallel to each other. For the sake of simplicity, we denote them as $e_{1}$ and $e_{2}$. We claim that $L_{2} \circ L_{1}$ can not be represented by a linear combination of $L_{i}, i \in \Lambda$. Suppose $L_{2} \circ L_{1}=\sum_{i} a_{i} L_{i}$, then

$$
\begin{aligned}
& L_{2} \circ L_{1}\left(e_{k}\right)=\left(\sum_{i} a_{i} L_{i}\right)\left(e_{k}\right) \\
& k \neq 1, \quad 0=a_{k} e_{k}^{2}, \quad a_{k}=0 \\
& k=1, \quad L_{2}\left(e_{1}^{2}\right)=a_{1} e_{1}^{2}, \quad p_{12} e_{2}^{2}=a_{1} e_{1}^{2} ;
\end{aligned}
$$

so

$$
p_{12} p_{2 k}=a_{1} p_{1 k}, \forall k
$$

If $a_{1}$ was not zero, $p_{1 k}=\frac{p_{12}}{a_{1}} p_{2 k}, \forall k$, but it is not possible since $e_{1}^{2}$ and $e_{2}^{2}$ are not parallel. Therefore, $L(E)$ can not be spanned by $\left\{L_{i} \mid i \in \Lambda\right\}$.

### 3.2.6 The derived Lie algebra of an evolution algebra

As for any algebra, the subspace $\operatorname{Der}(E)$ of derivations of an evolution $E$ is a Lie algebra. Here, let us characterize an element that belongs to the $\operatorname{Der}(E)$. Let $\left\{e_{i} \mid i \in \Lambda\right\}$ be a generator set of $E, D \in \operatorname{Der}(E)$, and suppose
$D\left(e_{i}\right)=\sum_{k} d_{k i} e_{k}$ for $i \in \Lambda$. By the definition of derivation $D(x y)=D(x) y+$ $x D(y)$, we have

$$
\begin{aligned}
D\left(e_{i} e_{j}\right) & =D\left(e_{i}\right) e_{j}+e_{i} D\left(e_{j}\right) \\
& =\left(\sum_{k} d_{k i} e_{k}\right) e_{j}+e_{i}\left(\sum_{k} d_{k j} e_{k}\right) \\
& =d_{j i} e_{j}^{2}+d_{i j} e_{i}^{2} \\
& =d_{j i} \sum_{k} p_{k j} e_{k}+d_{i j} \sum_{k} p_{k i} e_{k} \\
& =\sum_{k}\left(d_{j i} p_{k j}+d_{i j} p_{k i}\right) e_{k} \\
& =0,
\end{aligned}
$$

so, for $i \neq j, p_{k j} d_{j i}+p_{k i} d_{i j}=0, i \in \Lambda$. We also have

$$
\begin{aligned}
D\left(e_{i}^{2}\right) & =D\left(\sum_{k} p_{k i} e_{k}\right) \\
& =\sum_{k} p_{k i} D\left(e_{k}\right) \\
& =\sum_{j, k} p_{k i} d_{j k} e_{j} \\
& =2 \sum_{j} d_{i i} p_{j i} e_{j},
\end{aligned}
$$

so, we get for any $i, j \in \Lambda, 2 p_{j i} d_{i i}=\sum_{k} p_{k i} d_{j k}$. Therefore, we have

$$
\begin{aligned}
\operatorname{Der}(E) & =\left\{D \in \operatorname{End}(E) \mid p_{k j} d_{j i}+p_{k i} d_{i j}=0, \text { for } i \neq j ; 2 p_{j i} d_{i i}\right. \\
& \left.=\sum_{k} p_{k i} d_{j k}\right\} .
\end{aligned}
$$

### 3.2.7 The centroid of an evolution algebra

We recall that the centroid $\Gamma(E)$ of an algebra $E$ is the set of all linear transformations $T \in \operatorname{Hom}(E, E)$ that commute with all left and right multiplication operators

$$
T L_{x}=L_{x} T, \quad T R_{y}=R_{y} T, \text { for all } x, y \in E
$$

Or, the centroid centralizes the multiplication algebra $M(E)$. That is

$$
\Gamma(E)=\operatorname{Cent}_{H o m(E, E)}(M(E)) .
$$

Theorem 4. Any evolution algebra is centroidal.
Proof. Let $T$ be an element of the centroid $\Gamma(E)$. Suppose $T\left(e_{i}\right)=\sum_{k} t_{k i} e_{k}$, for $i \neq j$, we have

$$
\begin{aligned}
T L_{e_{j}}\left(e_{i}\right) & =T\left(e_{j} e_{i}\right)=0 \\
& =L_{e_{j}} T\left(e_{i}\right) \\
& =e_{j}\left(\sum_{k} t_{k i} e_{k}\right) \\
& =t_{j i} e_{j}^{2}=t_{j i} \sum_{k} p_{k j} e_{k}
\end{aligned}
$$

thus, $t_{i j}=0$. Then, look at

$$
\begin{aligned}
T L_{e_{i}}\left(e_{i}\right) & =T\left(e_{i}^{2}\right)=T\left(\sum_{k} p_{k i} e_{k}\right) \\
& =\sum_{k} p_{k i} T\left(e_{k}\right)=\sum_{k, j} p_{k i} t_{j k} e_{j} \\
& =\sum_{j, k} t_{j k} p_{k i} e_{j}
\end{aligned}
$$

and

$$
\begin{aligned}
L_{e_{i}} T\left(e_{i}\right) & =e_{i} \sum_{k} t_{k i} e_{k} \\
& =t_{i i} e_{i}^{2}=t_{i i} \sum_{k} p_{k i} e_{k}
\end{aligned}
$$

comparing them, we can have

$$
\begin{aligned}
& t_{i i} p_{j i}=\sum_{k} t_{j k} p_{k i}, \text { for } j \in \Lambda \\
& t_{i i} p_{j i}=t_{j j} p_{j i}, \text { for } j \in \Lambda
\end{aligned}
$$

Thus, we must have $t_{i i}=t_{j j}$. Therefore,

$$
T\left(e_{i}\right)=k(T) e_{i},
$$

where $k(T)$ is a scalar in the ground field $K$. That is $T$ is a scalar multiplication. So, we can conclude that $\Gamma(E) \cong K, E$ is centroidal.

### 3.3 Nonassociative Banach Algebras

To describe the evolution flow quantitatively in an evolution algebra, it is necessary to introduce a norm. As we will see, under this norm, an evolution algebra becomes a Banach algebra. We will define a norm for an evolution algebra first and then prove that any finite dimensional evolution algebra is a Banach algebra.

### 3.3.1 Definition of a norm over an evolution algebra

Let $E$ be an evolution algebra with a generator set $\left\{e_{i} \mid i \in \Lambda\right\}$. Define a function $N$ from $E$ to the underlying field $K$ as follows,

$$
\begin{gathered}
N: E \longrightarrow K \\
N(x)=\sum_{i}\left|a_{i}\right|,
\end{gathered}
$$

where $x \in E$ and $x=\sum_{i} a_{i} e_{i}$. We can verify that $N$ is a norm as follows:

- Nonnegativity

$$
N(x)=\sum_{i}\left|a_{i}\right| \geq 0
$$

Furthermore, if $N(x)=0$, then $N(x)=\sum_{i}\left|a_{i}\right|=0$. Thus $\left|a_{i}\right|=0$, which means that $a_{i}$ must be 0 . That is, $x=0$. Therefore $N(x)=0$ if and only if $x=0$.

- Linearity $N(a x)=|a| N(x), a \in K$, since $N(a x)=\sum_{i}\left|a a_{i}\right|=|a| \sum_{i}\left|a_{i}\right|=$ $|a| N(x)$.
- Triangle inequality $N(x+y) \leq N(x)+N(y)$. For $x=\sum_{i} a_{i} e_{i}$ and $y=$ $\sum_{i} b_{i} e_{i}$, we have

$$
\begin{aligned}
N(x+y) & =N\left(\sum_{i}\left(a_{i}+b_{i}\right) e_{i}\right) \\
& =\sum_{i}\left|a_{i}+b_{i}\right| \\
& \leq \sum_{i}\left(\left|a_{i}\right|+\left|b_{i}\right|\right) \\
& =\sum_{i}\left|a_{i}\right|+\sum_{i}\left|b_{i}\right| \\
& =N(x)+N(y) .
\end{aligned}
$$

Thus, an evolution algebra is a normed algebra. We denote $N(x)=\|x\|$.
Proposition 6. Any evolution operator $L$ is a bounded linear operator.
Proof. For $x \in E, x=\sum_{i} a_{i} e_{i}$ under a natural basis $\left\{e_{i} \mid i \in \Lambda\right\}$ of an evolution algebra $E$, we have

$$
\begin{aligned}
& L(x)=\sum_{i} a_{i} L\left(e_{i}\right)=\sum_{i} a_{i} e_{i}^{2}=\sum_{i j} a_{i} p_{j i} e_{j} . \\
& N(L(x))=\sum_{j}\left|\sum_{i} a_{i} p_{j i}\right| \leq \sum_{j} \sum_{i}\left|a_{i} p_{j i}\right| \\
& \leq \sum_{i}\left|a_{i}\right| \sum_{j}\left|p_{j i}\right| \leq \sum_{i}\left|a_{i}\right| c_{i} \\
& \leq c N(x)
\end{aligned}
$$

where $c_{i}=\sum_{j}\left|p_{j i}\right|$, and $c=\max \left\{c_{i} \mid i \in \Lambda\right\}$. Therefore, $T$ is bounded.

Corollary 6. Each element of $S P(E, E)$ is a bounded linear operator, where $S P(L, E)$ is the linear space of all the operators of left multiplication of $E$.

Proof. We know $S P(L, E)=\operatorname{Span}\left(L_{i}: i \in \Lambda\right)$ over $K$. We have $L_{i}(x)=a_{i} e_{i}^{2}$, if $x=\sum_{i} a_{i} e_{i}$. Then we can see

$$
\begin{aligned}
N\left(L_{i}(x)\right) & =N\left(a_{i} \sum_{j} p_{j i} e_{j}\right) \\
& \leq c_{i}\left|a_{i}\right| \leq c \sum_{i}\left|a_{i}\right|=c N(x),
\end{aligned}
$$

so $L_{i}$ is bounded.
Now, $\forall \theta \in S p(L, E)$, write $\theta=\sum_{i} \beta_{i} L_{i}, \beta_{i} \in K$. For any $x=\sum a_{i} e_{i}$, we have

$$
\theta(x)=\sum_{i} \beta_{i} L_{i}(x)=\sum_{i} \beta_{i} a_{i} e_{i}^{2}=\sum_{i j} \beta_{i} a_{i} p_{j i} e_{j}
$$

then

$$
\begin{aligned}
N(\theta(x)) & =\sum_{i j}\left|\beta_{i} a_{i} p_{j i}\right| \leq c \sum_{i}\left|\beta_{i} a_{i}\right| \\
& \leq c \sum_{i}\left|\beta_{i}\right| \cdot \sum_{i}\left|a_{i}\right| \\
& \leq c b N(x)
\end{aligned}
$$

where $b=\sum_{i}\left|\beta_{i}\right|$ is a constant for a given operator $\theta$. Therefore $\theta$ is bounded.

### 3.3.2 An evolution algebra as a Banach space

In Functional Analysis, there is a theorem that a linear operator is bounded if and only if it is a continuous operator. From Proposition 6 and Corollary 6, evolution operators and left multiplication operators are all bounded. Therefore, they are continuous under the topology induced by the metric $\rho(x, y)=$ $N(x-y)$, for $x, y \in E$.

Theorem 5. Let $E$ be an evolution algebra with finite dimension $n$, then it is complete as a normed linear space. That is, $E$ is a Banach space.

Proof. Let $x^{m}=\sum_{i=1}^{n} a_{i}^{m} e_{i}, m=1,2, \cdots$, be a sequence in $E$, then we have

$$
\begin{aligned}
\rho\left(a_{i}^{m} e_{i}, a_{i}^{k} e_{i}\right) & =N\left(a_{i}^{m} e_{i}-a_{i}^{k} e_{i}\right) \\
& =\left|a_{i}^{m}-a_{i}^{k}\right| \leq \sum_{i=1}^{n}\left|a_{i}^{m}-a_{i}^{k}\right| \\
& =\rho\left(x^{m}, x^{k}\right) \leq n \cdot \max _{1 \leq i \leq n}\left|a_{i}^{m}-a_{i}^{k}\right| .
\end{aligned}
$$

When $x^{m}$ is a Cauchy sequence, then, for any $\varepsilon>0$, there is an integer $m_{0}$, and for any integers $m, k>m_{0}$, we have $\rho\left(x^{m}, x^{k}\right)<\varepsilon$. So, we have $\left|a_{i}^{m}-a_{i}^{k}\right|<\varepsilon / n$. By the Cauchy principle in Real Analysis, there is a number $b_{i}$, such that $\left|a_{i}^{m}-b_{i}\right|<\varepsilon / n$. That is, the coordinate sequence $a_{i}^{m}$ converges to $b_{i}, i=1,2, \cdots, n$. If we denote $x^{0}=\sum_{i=1}^{n} b_{i} e_{i}$, then

$$
\rho\left(x^{m}, x^{0}\right)=\sum_{i=1}^{n}\left|a_{i}^{m}-a_{i}^{k}\right| \leq \varepsilon .
$$

This means that $x^{m}$ converges to $x^{0}$. Therefore, $E$ is a complete normed linear space, i.e. $E$ is a Banach space.

Corollary 7. For a finite dimensional evolution algebra $E$, it is a nonassociative Banach algebra.

Proof. It is an immediate consequence of Theorem 5.
Theorem 6. Let $E$ be a finite dimensional evolution algebra, and $B L(E \rightarrow$ $E)$ be the set of all bounded linear operators over $E$, then the subspace $L(E)$ of $B L(E \rightarrow E)$, all left multiplication operators of $E$, is a Banach subalgebra of $B L(E \rightarrow E)$.

Proof. In Functional Analysis, there is a theorem that when $X$ is Banach space, $\Re(X \longrightarrow X)$, the space of all bounded linear operators from $X$ to $X$, is a Banach algebra. Because $E$ is a Banach algebra, $B L(E \rightarrow E)$ is also a Banach algebra. Since each element of $L(E)$ is bounded and the composite of two elements of $L(E)$ is also bounded, then the operator algebra of left multiplication is a subalgebra of $B L(E \rightarrow E)$,

But we know, generally, $L(E)$ is not a Banach subalgebra of $B L(E \rightarrow E)$.

### 3.4 Periodicity and Algebraic Persistency

In this section, we introduce a periodicity for each generator of an evolution algebra. It turns out all generators of a nonnegative simple evolution algebra have the same periodicity. We also introduce an algebraic persistency and an algebraic transiency for each generator of an evolution algebra. They are basic concepts in the study of evolution in algebras.

### 3.4.1 Periodicity of a generator in an evolution algebra

Definition 6. Let $e_{j}$ be a generator of an evolution algebra $E$, the period d of $e_{j}$ is defined to be the greatest common divisor of the set $\left\{\log _{2} m \mid e_{j}<\left(e_{j}^{(m)}\right)\right\}$, where power $e_{j}^{(m)}$ is some $k$ th plenary power, $2^{k}=m$. That is

$$
d=g . c . d .\left\{\log _{2} m \mid e_{j}<\left(e_{j}^{(m)}\right)\right\} .
$$

If d is 1 , we say $e_{j}$ is aperiodic; if the set $\left\{\log _{2} m \mid e_{j}<\left(e_{j}^{(m)}\right)\right\}$ is empty, we define $d=\infty$.

To understand this definition, we give a proposition that states relations between evolution operators and plenary powers of an element.

Proposition 7. Generator $e_{j}$ has the period $d$ if and only if $d$ is the greatest common divisor of the set $\left\{n \mid \rho_{i} L^{n}\left(e_{i}\right) \neq 0\right\}$. That is

$$
d=g . c . d .\left\{n \mid \rho_{i} L^{n}\left(e_{i}\right) \neq 0\right\},
$$

where $\rho_{i}$ is a projection map of $E$, which maps every element of $E$ to its $e_{i}$ component.

Proof. We introduce a notion - plenary powers of a matrix. Let

$$
\begin{aligned}
& \left(e_{1}, e_{2}, \cdots \cdots, e_{n}\right) \cdot\left(e_{1}, e_{2}, \cdots \cdots, e_{n}\right) \\
= & \left(e_{1}^{2}, e_{2}^{2}, \cdots \cdots, e_{n}^{2}\right)=\left(e_{1}, e_{2}, \cdots \cdots, e_{n}\right) B,
\end{aligned}
$$

where $B=\left(p_{i j}\right)$ is the structural constant matrix of $E$.
Look at

$$
\begin{aligned}
& \left(e_{1}^{2}, e_{2}^{2}, \cdots \cdots, e_{n}^{2}\right) \cdot\left(e_{1}^{2}, e_{2}^{2}, \cdots \cdots, e_{n}^{2}\right) \\
= & \left(e_{1}, e_{2}, \cdots \cdots, e_{n}\right) B \cdot\left(e_{1}, e_{2}, \cdots \cdots, e_{n}\right) B \\
= & \left(e_{1}^{(4)}, e_{2}^{(4)}, \cdots \cdots, e_{n}^{(4)}\right) \\
= & \left(\sum_{k} p_{k 1} e_{k}, \sum_{k} p_{k 2} e_{k}, \cdots \cdots, \sum_{k} p_{k n} e_{k}\right) \\
& \cdot\left(\sum_{k} p_{k 1} e_{k}, \sum_{k} p_{k 2} e_{k}, \cdots \cdots, \sum_{k} p_{k n} e_{k}\right) \\
= & \left(\sum_{k} p_{k 1}^{2} e_{k}^{2}, \sum_{k} p_{k 2}^{2} e_{k}^{2}, \cdots \cdots, \sum_{k} p_{k n}^{2} e_{k}^{2}\right) \\
= & \left(e_{1}^{2}, e_{2}^{2}, \cdots \cdots, e_{n}^{2}\right)\left(\begin{array}{cc:}
p_{11}^{2} & p_{21}^{2} \\
p_{12}^{2} & p_{22}^{2} \\
\vdots & p_{n 1}^{2} \\
\vdots & p_{n 2}^{2} \\
p_{1 n}^{2} p_{2 n}^{2} & \vdots \\
\vdots
\end{array}\right) \\
= & \left(e_{1}, e_{2}^{2}, \cdots \cdots, e_{n}\right) B B^{(2)} .
\end{aligned}
$$

We also compute

$$
\begin{aligned}
& \left(e_{1}^{(4)}, e_{2}^{(4)}, \cdots \cdots, e_{n}^{(4)}\right) \\
& \cdot\left(e_{1}^{(4)}, e_{2}^{(4)}, \cdots \cdots, \cdot, e_{n}^{(4)}\right) \\
= & \left(e_{1}^{(8)}, e_{2}^{(8)}, \cdots \cdots, e_{n}^{(8)}\right) \\
= & \left(e_{1}, e_{2}, \cdots \cdots, e_{n}\right) B B^{(2)} \\
& \cdot\left(e_{1}, e_{2}, \cdots \cdots, e_{n}\right) B B^{(2)} \\
= & \left(e_{1}, e_{2}, \cdots \cdots, e_{n}\right) B\left(B B^{(2)}\right)^{(2)} .
\end{aligned}
$$

Now, we define plenary powers for a matrix as follows:

$$
\begin{aligned}
A^{[1]}= & A \\
A^{[2]}= & A A^{(2)}=A\left(A^{[1]}\right)^{(2)} \\
A^{[3]}= & A\left(A^{[2]}\right)^{(2)}=A\left(A A^{(2)}\right)^{(2)} \\
& \cdots \cdots \cdots \cdots \\
A^{[k+1]}= & A\left(A^{[k]}\right)^{(2)} .
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
& \left(e_{1}^{[m]}, e_{2}^{[m]}, \cdots \cdots, e_{n}^{[m]}\right) \\
= & \left(e_{1}, e_{2}, \cdots \cdots, e_{n}\right) B^{[m]}
\end{aligned}
$$

We note that the matrix representation of the evolution operator $L$ is given by the matrix $B . \rho_{j} L^{k}\left(e_{j}\right) \neq 0$ means the $(j, j)$ entry of $B^{k}$ is not zero. It is not too hard to check that the $(j, j)$ entry of $B^{k}$ is not zero if and only if the $(j, j)$ entry of $B^{[k]}$ is not zero, which means $\rho_{j}\left(e_{j}^{\left(2^{k}\right)}\right) \neq 0$. That also means $e_{j}<e_{j}^{\left(2^{k}\right)}$. This concludes the proof.

From the above proof, we can see that the $k$ th plenary power and the $k$ th action of the evolution operator give us the same information in computing the period of an element. We also obtain the following corollary.
Corollary 8. Generator $e_{j}$ has the period of $d$ if and only if $d$ is the greatest common divisor of the set $\left\{n \mid e_{j}<e_{j}^{[n]}\right\}$, where $e_{j}^{[n]}=e_{j}^{\left(2^{n}\right)}$.
Theorem 7. All generators have the same period in a nonnegative simple evolution algebra.
Proof. Let $e_{i}$ and $e_{j}$ be two generators in a simple evolution algebra $E$. The periods of $e_{i}$ and $e_{j}$ are $d_{i}$ and $d_{j}$ respectively. Since $e_{i}$ must occur in a plenary power of $e_{j}$, say $e_{i}<e_{j}^{[n]}$, and $e_{j}$ must occur in a plenary power of $e_{i}$, say $e_{j}<e_{i}^{[m]}$, from Theorem 3 we have $e_{i}<e_{i}^{[n+m]}$ and $e_{j}<e_{j}^{[n+m]}$. Then $d_{i} \mid n+m$, and $d_{j} \mid n+m$. Since $e_{j}<e_{j}^{\left[d_{j}\right]}$, so $e_{i}<e_{j}^{\left[d_{j}+n\right]}$ and $e_{i}<e_{i}^{\left[d_{j}+n+m\right]}$, then $d_{i} \mid d_{j}+n+m$. Therefore $d_{i} \mid d_{j}$. Similarly, we have $d_{j} \mid d_{i}$. Thus, we get $d_{i}=d_{j}$.

### 3.4.2 Algebraic persistency and algebraic transiency

Let $E$ be an evolution algebra with a generator set $\left\{e_{i} \mid i \in \Lambda\right\}$. We say that generator $e_{j}$ is algebraically persistent if the evolution subalgebra $\left\langle e_{j}\right\rangle$, generated by $e_{j}$, is a simple subalgebra, and $e_{i}$ is algebraically transient if the subalgebra $\left\langle e_{i}\right\rangle$ is not simple. Then, it is obvious that every generator in a simple evolution algebra is algebraically persistent, since each generator generates the same algebra that is simple. We know that if $x$ and $y$ intercommunicate, the evolution subalgebra generated by $x$ is the same as the one generated by $y$. Moreover, we have the following theorem.

Theorem 8. Let $e_{i}$ and $e_{j}$ be generators of an evolution algebra $E$. If $e_{i}$ and $e_{j}$ can intercommunicate and both are algebraically persistent, then they belong to the same simple evolution subalgebra of $E$.

Proof. Since $e_{i}$ and $e_{j}$ can intercommunicate, $e_{i}$ occurs in $\left\langle e_{j}\right\rangle$ and $e_{j}$ occurs in $\left\langle e_{i}\right\rangle$. Then, there are some powers of $e_{i}$, denoted by $P\left(e_{i}\right)$ and some powers of $e_{j}$, denoted by $Q\left(e_{j}\right)$, such that

$$
\begin{array}{lr}
P\left(e_{i}\right)=a e_{j}+u & a \neq 0, \\
Q\left(e_{j}\right)=b e_{i}+v & b \neq 0 .
\end{array}
$$

Since subalgebras are also ideals in an evolution algebra, we have

$$
\begin{aligned}
& P\left(e_{i}\right) e_{j}=a e_{j}^{2} \in\left\langle e_{i}\right\rangle, \\
& Q\left(e_{j}\right) e_{i}=a e_{i}^{2} \in\left\langle e_{j}\right\rangle .
\end{aligned}
$$

Therefore, $\left\langle e_{i}\right\rangle \cap\left\langle e_{j}\right\rangle \neq\{0\}$. Since $\left\langle e_{i}\right\rangle$ and $\left\langle e_{j}\right\rangle$ are both simple evolution subalgebras, then $\left\langle e_{i}\right\rangle=\left\langle e_{j}\right\rangle$. Thus, $e_{i}$ and $e_{j}$ belong to the same simple evolution subalgebra.

For an evolution algebra, we can give certain conditions to specify whether it is simple or not by the following corollary:

Corollary 9. 1) Let $E$ be a connected evolution algebra, then $E$ has a proper evolution subalgebra if and only if $E$ has an algebraically transient generator.
2) Let $E$ be a connected evolution algebra, then $E$ is a simple evolution algebra if and only if $E$ has no algebraically transient generator.
3) If $E$ has no algebraically transient generator, then $E$ can be written as a direct sum of evolution subalgebras (the number of summands can be one).

Proof. 1) If $E$ has no algebraically transient generator, each generator $e_{i}$ generates a simple evolution subalgebra. These subalgebras are all the same because $E$ is connected. Otherwise, $E$ would be a direct sum of these subalgebras. This means the only nonempty subalgebra of $E$ is itself. On the other hand, if $E$ has an algebraically transient generator $e_{k}$, then the generated evolution
subalgebra $\left\langle e_{k}\right\rangle$ is not simple. This means $\left\langle e_{k}\right\rangle$ has a proper subalgebra, so $E$ has a proper subalgebra.
2) It is obvious from (1).
3) It is also obvious from (1).

Now, the question is, for any evolution algebra, whether there is always an algebraically persistent generator. Generally, this is not true. The following statement tells us that for any finite dimensional evolution algebra, there always is an algebraically persistent generator.

Theorem 9. Any finite dimensional evolution algebra has a simple evolution subalgebra.

Proof. We assume the evolution algebra $E$ is connected, otherwise we just need to consider a component of a direct sum of $E$.

Let $\left\{e_{1}, e_{2}, \cdots, e_{n}\right\}$ be a generator set of $E$. Consider evolution subalgebras generated by each generator

$$
\left\langle e_{1}\right\rangle, \quad\left\langle e_{2}\right\rangle, \cdots \cdots,\left\langle e_{n}\right\rangle
$$

If there is a subalgebra that is simple, it is done. Otherwise, we choose a subalgebra that contains the least number of generators, for example, $\left\langle e_{i}\right\rangle$ and $\left\{e_{i_{1}}, e_{i_{2}}, \cdots, e_{i_{k}}\right\} \subset\left\langle e_{i}\right\rangle$, where $\left\{e_{i_{1}}, e_{i_{2}}, \cdots, e_{i_{k}}\right\}$ is a subset of $\left\{e_{1}, e_{2}\right.$, $\left.\cdots, e_{n}\right\}$. Then, consider

$$
\left\langle e_{i_{1}}\right\rangle, \quad\left\langle e_{i_{2}}\right\rangle, \cdots \cdots,\left\langle e_{i_{k}}\right\rangle
$$

If there is some subalgebra that is simple in this sequence, we are done. Otherwise, we choose a certain $\left\langle e_{i_{j}}\right\rangle$ in the same way as we choose $\left\langle e_{i}\right\rangle$. Since the number of generators is finite, this process will stop. Therefore, we always have a simple evolution subalgebra. Of course, any generator of the simple evolution subalgebra is algebraically persistent.

### 3.5 Hierarchy of an Evolution Algebra

The hierarchical structure of an evolution algebra is a remarkable property that gives a picture of the dynamical process when multiplication in the evolution algebra is treated as a discrete time dynamical step. In this section, we study this hierarchy and establish a principal theorem about evolution algebras - the hierarchical structure theorem. Algebraically, this hierarchy is a sequence of semidirect-sum decompositions of a general evolution algebra. It depends upon the "relative" concepts of algebraic persistency and algebraic transiency. By the "relative" concepts here, we mean that we can define higher algebraic persistency and algebraic transiency over the space generated by transient generators in the previous level. The difference between algebraic
persistency and algebraic transiency suggests a sequential semidirect-sum decomposition, or suggests a direction of evolution from the viewpoint of dynamical systems. This hierarchical structure demonstrates that our evolution algebra is a mixed algebraic and dynamical subject. We also establish the structure theorem for simple evolution algebras. A method is given here to reduce a "big" evolution algebra to a "smaller" one, with the hierarchy being the same. This procedure is called reducibility, which gives a rough classification of all evolution algebras - the skeleton-shape classification.

### 3.5.1 Periodicity of a simple evolution algebra

As we know in Section 3.4 Theorem 7, all generators of a nonnegative simple evolution algebra have the same period. It might be well to say that a simple algebra has a period. Thus, simple evolution algebras can be roughly classified as either periodic or aperiodic. The following theorem establishes the structure of a periodic simple evolution algebra.

Theorem 10. Let $E$ be a nonnegative simple evolution algebra with generator set $\left\{e_{i} \mid i \in \Lambda\right\}$, then all generators have the same period, denoted by $d$. There is a partition of generators with $d$ disjointed classes $C_{0}, C_{2}, \cdots, C_{d-1}$, such that $L\left(\Delta_{k}\right) \subseteq \Delta_{k+1}(\bmod d)$, or $\Delta_{k}^{2} \subseteq \Delta_{k+1}(\bmod d), k=1,2, \cdots d-1$, where $\Delta_{k}=\operatorname{Span}\left(C_{k}\right)$ and $L$ is the evolution operator of $E$, mod is taken with respect to the index of the class of generators. There is also a direct sum of linear subspaces

$$
E=\Delta_{0} \oplus \Delta_{1} \oplus \cdots \oplus \Delta_{d-1}
$$

Proof. Since $E$ is simple, if any generator $e_{i}$ has a period of $d$, then every generator has a period of $d$. Set $C_{m}=\left\{e_{j} \mid e_{j}<e_{i}^{[n d+m]}, j \in \Lambda\right\}, 0 \leq m<d$, for any fixed $e_{i}$. Because this evolution algebra is simple, each generator $e_{j}$ will occur in some $C_{m}$. So

$$
\cup_{m=0}^{d-1} C_{m}=\left\{e_{k} \mid k \in \Lambda\right\} .
$$

Claim that these $C_{m}$ are disjoint. We show this as follows: if $e_{j} \in C_{m_{1}} \cap C_{m_{2}}$ for $0 \leq m_{1}, m_{2}<d$, then $e_{j}<e_{i}^{\left[n_{1} d+m_{1}\right]}$, and $e_{j}<e_{i}^{\left[n_{2} d+m_{2}\right]}$ for some integers $n_{1}$ and $n_{2}$. Since $\left\langle e_{i}\right\rangle=\left\langle e_{j}\right\rangle$, so $e_{i}<\left\langle e_{j}\right\rangle$. That is, $e_{i}<e_{j}^{k}$ for some integer $k$. Therefore $e_{i}<e_{i}^{\left[n_{1} d+m_{1}+k\right]}$, and $e_{i}<e_{i}^{\left[n_{2} d+m_{2}+k\right]}$, then we have $d \mid n_{1} d+m_{1}+k$, and $d \mid n_{2} d+m_{2}+k$. Thus $d \mid m_{1}-m_{2}$. But $0 \leq\left|m_{1}-m_{2}\right|<d$, so we have $m_{1}=m_{1}$, then $C_{m_{1}}=C_{m_{2}}$.

Therefore, a partition of the set $\left\{e_{k} \mid k \in \Lambda\right\}$ is obtained. We need to prove that if we take $e_{k}$ as a fixed generator that is different from the previous $e_{i}$ for partitioning, we can still get the same partition. Fix $e_{k}$, let $C_{m}^{\prime}=$ $\left\{e_{j} \mid e_{j}<e_{k}^{[n d+m]}, j \in \Lambda\right\}$, where $0 \leq m<d$. Since $E$ is simple, $e_{i}<e_{k}^{[t]}$. If $e_{\alpha}, e_{\beta} \in C_{m}$, then $e_{\alpha}<e_{i}^{\left[n_{1} d+m\right]}$, and $e_{\beta}<e_{i}^{\left[n_{2} d+m\right]}$ for some integers $n_{1}$ and
$n_{2}$. Then $e_{\alpha}<e_{k}^{\left[n_{1} d+m+t\right]}, e_{\beta}<e_{k}^{\left[n_{2} d+m_{2}+k\right]}$. Since $n_{1} d+m+t \equiv n_{2} d+m+t$ (modd), so $e_{\alpha}$ and $e_{\beta}$ are still in the same cell $C_{m}^{\prime}$ of the partition.

Now, if $e_{j} \in C_{k}$, then $e_{i}^{\left(2^{n d+k}\right)}=a e_{j}+v, a \neq 0$. We have $e_{i}^{[k+1]}=a^{2} e_{j}^{2}+$ $v^{2}=a^{2} L\left(e_{j}\right)+v^{2}$, which means that generators occur in $L\left(e_{j}\right) \in C_{k+1}$ or generators occur in $e_{j}^{2} \in C_{k+1}$.

Denote the linear subspace spanned by $C_{k}$ as $\Delta_{k}, k=0,1,2, \cdots d-1$, then we have a direct sum for $E$

$$
E=\Delta_{0} \oplus \Delta_{1} \oplus \cdots \oplus \Delta_{d-1},
$$

and

$$
\begin{aligned}
L: & \Delta_{k} \rightarrow \Delta_{k+1} \quad k=1,2, \cdots d-1 \\
L^{d}: & \Delta_{k} \rightarrow \Delta_{k}, \text { a linear map for each } k
\end{aligned}
$$

Or, we have

$$
\Delta_{k}^{2} \subseteq \Delta_{k+1}, \quad \Delta_{k}^{d} \subseteq \Delta_{k}, \quad k=1,2, \cdots d-1
$$

This concludes the proof.

### 3.5.2 Semidirect-sum decomposition of an evolution algebra

A general evolution algebra has algebraically persistent generators and algebraically transient generators. These two types of generators have distinct "reproductive behavior" - dynamical behavior. Algebraically persistent ones can generate a simple subalgebra. Once an element belongs to the subalgebra, it will never "reproduce" any element that is not in the subalgebra. Or, dynamically, once the dynamical process, represented by the evolution operator $L$, enters a simple evolution subalgebra, it will never escape from it. In contrast, algebraically transient generators behave differently. They generate reducible subalgebras. The following theorem demonstrates how to distinguish these two types of generators algebraically. Actually, it is the starting level of the hierarchy of an evolution algebra, and it can also serve as a sample of structure in each level.

Theorem 11. Let $E$ be a connected evolution algebra. As a vector space, $E$ has a decomposition of direct sum of subspaces:

$$
E=A_{1} \oplus A_{2} \oplus \cdots \oplus A_{n} \dot{+} B
$$

where $A_{i}, i=1,2, \cdots, n$, are all simple evolution subalgebras, $A_{i} \cap A_{j}=\{0\}$ for $i \neq j$, and $B$ is a subspace spanned by algebraically transient generators (which we call a transient space). The summation $A_{1} \oplus A_{2} \oplus \cdots \oplus A_{n}$ is a direct sum of subalgebras. Symbol $\dot{+}$ indicates the summation is not a direct sum of subalgebras, just a direct sum of subspaces. We call this decomposition a semidirect-sum decomposition of an evolution algebra.

Proof. Take a generator set for $E,\left\{e_{i} \mid i \in \Lambda\right\}$, where $\Lambda$ is a finite index set, then we will have two categories of generators: algebraically transient generators and algebraically persistent generators. Let

$$
B=\operatorname{Span}\left(e_{k} \mid e_{k} \text { is algebraically transient }\right)
$$

Take any algebraically persistent element $e_{i_{1}}$, let $A_{1}=\left\langle e_{i_{1}}\right\rangle$. Again take any algebraically persistent element $e_{i_{2}}$ that does not occur in $A_{1}$, let $A_{2}=\left\langle e_{i_{2}}\right\rangle$. Keep doing in this way. Since $\Lambda$ is finite, we will end up with some $A_{n}=\left\langle e_{i_{n}}\right\rangle$.

By our construction, each $A_{k}$ is simple, since $e_{i_{k}}$ is algebraically persistent. And $A_{i} \cap A_{j}=\{0\}$ for $i \neq j$, since they are simple. Finally, as a vector space $E$, $A_{1} \oplus A_{2} \oplus \cdots \oplus A_{n} \dot{+} B$ is a direct sum decomposition, since $A_{i} \cap B=\{0\}, i=$ $1,2, \cdots, n$. But $B$ is not a subalgebra; it is just a linear subspace. Therefore, as an algebra $E$, we just say that it is a semidirect-sum decomposition.

Note, if $E$ is simple, $n$ is 1 and $B=\phi$. Otherwise, $B$ is not zero.

### 3.5.3 Hierarchy of an evolution algebra

1). The 0 th structure of an evolution algebra $E$ : the 0 th decomposition of $E$ is given by Theorem 11 as

$$
E=A_{1} \oplus A_{2} \oplus \cdots \oplus A_{n_{0}} \stackrel{\bullet}{+} B_{0}
$$

where $B_{0}$ is the subspace spanned by algebraically transient generators of $E$, we call it the $0 t h$ transient space.
2). The 1 st structure of $E$, which is the decomposition of the 0 th transient space $B_{0}$.

Although the 0th transient space $B_{0}$ is not an evolution subalgebra, it inherits evolution algebraic structure from $E$ if the algebraic multiplication is confined within $B_{0}$. We shall make this point clear.

- The induced multiplication: we write generators for $B_{0}$ as $e_{0, k}$ and $k \in \Lambda_{0}$, where $\Lambda_{0} \subset \Lambda$ is a subset of the index set. Actually, they are algebraic transient generators. Then, we have the induced multiplication on $B_{0}$, denoted by ${ }^{1}$, as follows

$$
\begin{aligned}
& e_{0, i} \stackrel{1}{\cdot} e_{0, j}=0 \quad \text { if } \quad i \neq j, \\
& e_{0, i} \cdot e_{0, i}=\rho_{B_{0}}\left(e_{0, i} \cdot e_{0, i}\right)
\end{aligned}
$$

and linearly extend onto $B_{0} \times B_{0}$, where $\rho_{B_{0}}$ is the projection from $E$ to $B_{0}$. It is not hard to check that $B_{0}$ is an evolution algebra, which we call the first induced evolution algebra.

- The first induced evolution operator in $B_{0}$ is given by

$$
L_{B_{0}}=\rho_{B_{0}} L
$$

Then, we have

$$
L_{B_{0}}^{2}=\left(\rho_{B_{0}} L\right)\left(\rho_{B_{0}} L\right)=\rho_{B_{0}} L^{2}
$$

and for any positive integer $n$, we have

$$
L_{B_{0}}^{n}=\rho_{B_{0}} L^{n}
$$

- First induced evolution subalgebras generated by some generators of $B_{0}$ : Denote the evolution subalgebra generated by $e_{0, i}$ in $B_{0}$ by $\left\langle e_{0, i} \mid B_{0}\right\rangle$ (using multiplication ${ }^{1}$ in $B_{0}$ ). Sometimes we just use $\left\langle e_{0, i}\right\rangle_{1}$ for this subalgebra. (It may be a nilpotent subalgebra).
- First algebraically persistent generators in $B_{0}$ :

We say $e_{0, i}$ is a first algebraically persistent if $\left\langle e_{0, i}\right\rangle_{1}$ is a simple subalgebra. Otherwise, we say $e_{0, i}$ is a first algebraically transient.
$B_{0}$ is called irreducible (simple) if it has no proper first induced evolution subalgebra. Similarly, we have a first reducible evolution subalgebra.
$B_{0}$ is connected if $B_{0}$ can not be decomposed as a direct sum of two first induced evolution subalgebras.

- The 1 st decomposition of $E$, the decomposition of $B_{0}$ :

We state the decomposition theorem for the $0 t h$ transition space $B_{0}$ here. The proof is essentially a repeat of that of the 0 th decomposition theorem. We therefore skip the proof.

Theorem 12. The 1 st structure of an evolution algebra $E$ : the 1 st decomposition of $E$ is given by

$$
B_{0}=A_{1,1} \oplus A_{1,2} \oplus A_{1,3} \oplus \cdots \oplus A_{1, n_{1}} \dot{+} B_{1}
$$

where $A_{1, i}, i=1,2, \cdots, n_{1}$, are all first simple evolution subalgebras of $B_{0}, A_{1, i} \cap A_{1, j}=\{0\}$, if $i \neq j$, and $B_{1}$ is the first transient space spanned by the first algebraically transient generators.

- The first induced periodicity and intercommunication:

The following is the definition of the first induced period

$$
\text { The period of } e_{0, i}=\operatorname{gcd}\left\{n \mid e_{0, i}<e_{0, i}^{[n]_{0}}\right\}
$$

where $e_{0, i}^{[n]}$ means that the plenary powers are taken within space $B_{0}$. We have a theorem about the intercommunications within the space $B_{0}$. The proof is the same as that at the 0th level. We will not give it here.

Theorem 13. If $e_{0, i}$ and $e_{0, j}$ intercommunicate, then they have the same first induced periods.

- The decomposition of a first simple periodical evolution subalgebra:

Theorem 14. If $A_{1, k}$ is a first nonnegative simple periodic reduced evolution subalgebra and some $e_{0, i}$ of its generator has a period of $d$, then it can be written as a direct sum

$$
A_{1, k}=\Delta_{1,0} \oplus \Delta_{1,1} \oplus \cdots \oplus \Delta_{1, d-1} .
$$

(The proof is the same as that in the 0th level.)
3). We can construct the $2 n d$ induced evolution algebra over the first transient space $B_{1}$, if $B_{1}$ is connected and not simple. If the $k t h$ transient space $B_{k}$ is disconnected and each component is simple, we will stop with a direct sum of $(k+1)$ th simple evolution subalgebras. Otherwise, we can continue to construct evolution subalgebras until we reach a level where each evolution subalgebra is simple. Now, we have the hierarchy as follows

$$
\begin{aligned}
E= & A_{0,1} \oplus A_{0,2} \oplus \cdots \oplus A_{0, n_{0}} \dot{+} B_{0} \\
B_{0}= & A_{1,1} \oplus A_{1,2} \oplus \cdots \oplus A_{1, n_{1}} \dot{+} B_{1} \\
B_{1}= & A_{2,1} \oplus A_{2,2} \oplus \cdots \oplus A_{2, n_{2}} \dot{+} B_{2} \\
& \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdot \\
B_{m-1}= & A_{m, 1} \oplus A_{m, 2} \oplus \cdots \oplus A_{m, n_{m}} \dot{+} B_{m} \\
B_{m}= & B_{m, 1} \oplus B_{m, 2} \oplus \cdots \oplus B_{m, h},
\end{aligned}
$$

where $A_{k, l}$ is a $k t h$ simple evolution subalgebra, $A_{k, l} \cap A_{k, l \prime}=\{0\}$ if $l \neq l \prime, B_{k}$ is the $k t h$ transient space. $B_{m}$ can be decomposed as a direct sum of $(m+1) t h$ simple evolution subalgebras. We may call these $(m+1)$ th simple evolution subalgebras the heads of the hierarchy, and $h$ is the number of heads.

Example 2. Let's look at an evolution algebra $E$ with dimension 25 . The generator set is $e_{1}, e_{2}, \cdots, e_{25}$. The defining relation are given: $e_{i} e_{j}=0$ if $i \neq j$; when $i=j$, they are

$$
\begin{aligned}
e_{1}^{2} & =e_{2}+2 e_{3}+e_{4}+3 e_{5}, e_{2}^{2}=2 e_{3}+7 e_{6}+e_{9}, \\
e_{3}^{2} & =e_{2}+5 e_{7}+e_{8}+9 e_{9}, e_{4}^{2}=7 e_{5}+e_{9}+e_{10}+10 e_{11}, \\
e_{5}^{2} & =e_{4}+7 e_{9}+5 e_{12}, e_{6}^{2}=e_{7}+e_{8}+7 e_{13}, \\
e_{7}^{2} & =6 e_{6}+e_{8}+2 e_{13}, e_{8}^{2}=e_{6}+3 e_{7}+e_{13}+2 e_{14}, \\
e_{9}^{2} & =3 e_{15}+2 e_{14}, e_{10}^{2}=4 e_{11}+e_{12}+2 e_{16}, \\
e_{11}^{2} & =6 e_{10}+e_{12}+5 e_{15}, e_{12}^{2}=e_{10}+4 e_{11}+2 e_{15}+e_{16}, \\
e_{13}^{2} & =e_{14}+5 e_{17}+3 e_{18}+e_{21},
\end{aligned}
$$

$$
\begin{aligned}
& e_{14}^{2}=e_{13}+4 e_{17}+e_{18}+5 e_{19}+e_{20}, \\
& e_{15}^{2}=8 e_{16}+e_{20}+e_{21}+7 e_{22}, \\
& e_{16}^{2}=9 e_{15}+e_{23}+10 e_{24}+e_{25}, \\
& e_{17}^{2}=3 e_{17}+2 e_{18}, e_{18}^{2}=4 e_{17}+2 e_{18}, e_{19}^{2}=3 e_{19}+e_{20}, \\
& e_{20}^{2}=e_{19}, e_{21}^{2}=3 e_{22}+e_{21}, e_{22}^{2}=2 e_{22}+5 e_{21}, \\
& e_{23}^{2}=e_{25}+4 e_{24}, e_{24}^{2}=2 e_{25}, \quad e_{25}^{2}=e_{23}+8 e_{24} .
\end{aligned}
$$

The 0th evolution subalgebras are $A_{0,1}=\left\langle e_{17}, e_{18}\right\rangle, A_{0,2}=\left\langle e_{19}, e_{20}\right\rangle$, $A_{0,3}=\left\langle e_{21}, e_{22}\right\rangle$, and $A_{0,4}=\left\langle e_{23}, e_{24}, e_{25}\right\rangle$. The 0th transient space is spane $_{1}, e_{2}, \cdots, e_{16}$. The 1st evolution subalgebras are $A_{1,1}=\left\langle e_{13}, e_{14}\right\rangle$ and $A_{1,2}=\left\langle e_{15}, e_{16}\right\rangle$. The 2nd evolution subalgebras are $A_{2,1}=\left\langle e_{6}, e_{7}, e_{8}\right\rangle$, $A_{2,2}=\left\langle e_{9}\right\rangle$, and $A_{2,3}=\left\langle e_{10}, e_{11}, e_{12}\right\rangle$. The 3rd evolution subalgebras are $A_{3,1}=\left\langle e_{2}, e_{3}\right\rangle$ and $A_{3,2}=\left\langle e_{4}, e_{5}\right\rangle$. The 3rd transient space, the head of the hierarchy given by the algebra $B_{3}$, is $\operatorname{span}\left\{e_{1}\right\}$. Figure 3.1 shows the hierarchical structure.

### 3.5.4 Reducibility of an evolution algebra

From the hierarchy of an evolution algebra, we get an impression about the dynamical flow of an algebra. That is, if we start at a high level, a big index level, the dynamical flow will automatically go down to a low level, it may also sojourn in a simple evolution subalgebra at each level. It is reasonable to view each simple evolution subalgebra at each level as one point or one-dimensional subalgebra. The big evolution picture still remains. If we call this remained hierarchy the skeleton of the original evolution algebra, all evolution algebras that possess the same skeleton will have a similar dynamical behavior. We call this procedure the reducibility of an evolution algebra and write it as a statement.

Theorem 15. Every evolution algebra $E$ can be reduced to a unique evolution algebra $E_{r}$ such that its evolution subalgebras in its hierarchy are all one-dimensional subalgebras. We call such a unique evolution algebra $E_{r}$ the skeleton of $E$.

Example 3. The skeleton $E_{r}$ of the algebra $E$ in Example 2 is the evolution algebra generated by $\eta_{1}, \eta_{2}, \cdots, \eta_{12}$ that are subject to the following defining relations:

$$
\begin{aligned}
& \eta_{1}^{2}=\eta_{2}+\eta_{3}, \eta_{2}^{2}=\eta_{4}+\eta_{5}, \eta_{3}^{2}=\eta_{5}+\eta_{6} \\
& \eta_{4}^{2}=\eta_{7}, \eta_{5}^{2}=\eta_{7}+\eta_{8}, \eta_{6}^{2}=\eta_{8} \\
& \eta_{7}^{2}=\eta_{9}+\eta_{10}+\eta_{11}, \eta_{8}^{2}=\eta_{12}+\eta_{10}+\eta_{11} \\
& \eta_{9}^{2}=\eta_{9}, \eta_{10}^{2}=\eta_{10}, \eta_{11}^{2}=\eta_{11}, \eta_{12}^{2}=\eta_{12}
\end{aligned}
$$



Fig. 3.1. The hierarchy of the Example 2

The Fig. 3.2 shows the hierarchical structure of $E_{r}$. Comparing with Fig. 3.1, these two have the same dynamical shape.

The concept of the skeleton can be utilized to give a rough classification of all evolution algebras. From Examples 2 and 3, we can see that two types of numbers, the number of levels $m$ and the numbers $n_{k}$ of simple evolution subalgebras at each level $k$, can roughly determine the shape of the hierarchy of an evolution algebra, ignoring the flow relations between two different levels. Note that at level $(m+1)$, the number $n_{m+1}$ is $h$, the number of heads, in our notation. We give the criterions for classification of evolution algebras. That is, if two evolution algebras have the same number $m$ of levels and the numbers $n_{k}$ of simple evolution subalgebras at each level $k$, we say these two evolution algebras belong to the same class of skeleton-shape. Furthermore, we say two evolution algebras belong to the same class of skeleton if they belong to the same class of skeleton-shape and the flow relations between any two different levels are the same correspondingly.


Fig. 3.2. The hierarchy of the Example 3

Now, there are two basic questions related to our classifications that should be answered.

The first one stated as follows: given the level number $m$ and the total number $n$ of simple evolution subalgebras (including heads) wherever they are, how many classes of skeleton-shapes of evolution algebras can we have? The answer is a famous number in number theory, $p_{m+1}(n)$, the number of partitions of $n$ into $m+1$ cells. For $n<m+1, p_{m+1}(n)=0$ and $p_{m+1}(m+1)=1$. Generally, we have the recursion

$$
p_{m+1}(n)=p_{m+1}(n-m-1)+p_{m}(n-m-1)+\cdots+p_{1}(n-m-1) .
$$

We list here the answers for the question when the hierarchy has small levels as follows:

$$
\begin{gathered}
p_{1}(n)=1, m=0 ; \\
p_{2}(n)=\left\{\begin{array}{l}
\frac{n}{2}, n \equiv 0(2), \\
\frac{n-1}{2}, n \equiv 1(2),
\end{array}\right. \\
p_{3}(n)=\left\{\begin{array}{l}
\frac{n^{2}}{12}, n \equiv 0(6), \\
\frac{n^{2}}{12}-\frac{1}{12}, n \equiv 1(6), \\
\frac{n^{2}}{12}-\frac{1}{3}, n \equiv 2(6), \\
\frac{n^{2}}{12}+\frac{1}{4}, n \equiv 3(6), \\
\frac{n^{2}}{12}-\frac{1}{3}, n \equiv 4(6), \\
\frac{n^{2}}{12}-\frac{1}{12}, n \equiv(6),
\end{array}\right.
\end{gathered}
$$

Generally, we have

$$
p_{m+1}(n)=\frac{n^{m}}{m!(m-1)!}+R_{m-1}(n), n \equiv n^{\prime}((m+1)!)
$$

where $R_{m-1}(n)$ is a polynomial in $n$ of degree at most $m-1$. Therefore, by the number of levels and the numbers of simple evolution subalgebras, we can determine any evolution algebra up to its skeleton-shape. Thus, we obtain a skeleton-shape classification of all evolution algebras.

The second problem is that, given the level number $m$ and the numbers $n_{k}$ of simple evolution subalgebras at each level, how many classes of skeletons of evolution algebras can we have? We will use a formula that gives the number $b p(n, m)$ of bipartite graphs with two given disjoint vertex sets, $V_{1}$ and $V_{2}$, and $\left|V_{1}\right|=n\left|V_{2}\right|=m$. This formula is given by Winfried Just:

$$
b p(n, m)=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\left(\sum_{l=0}^{m}(-1)^{l}\binom{m}{l} 2^{(n-k)(m-l)}\right) .
$$

Then, the number of classes of skeletons of evolution algebras with $m$ levels and $n_{k}$ subalgebras at each level is

$$
b p\left(n_{0}, n_{1}\right) b p\left(n_{1}, n_{2}\right) \cdots b p\left(n_{m-1}, n_{m}\right)=\prod_{i=1}^{m-1} b p\left(n_{i}, n_{i+1}\right)
$$

Therefore, by the number of levels and subalgebras at each level, we can determine any evolution algebra up to its skeleton.

