



Maximal subalgebras of Lie algebras containing Engel subalgebras

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ABSTRACT

Relationships between certain properties of maximal subalgebras of a Lie algebra L and the structure of L itself have been studied by a number of authors. Some maximal subalgebras, however, exert a greater influence on particular results than others. Here we study properties of those maximal subalgebras that contain Engel subalgebras, and of those that also have codimension greater than one in L .

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1. Introduction

Relationships between certain properties of maximal subalgebras of a Lie algebra L and the structure of L itself have been studied by a number of authors. Some maximal subalgebras, however, exert a greater influence on particular results than others. Here, we study properties of those maximal subalgebras that contain Engel subalgebras. This idea is somewhat akin to that of maximal subgroups containing Sylow subgroups as introduced by Bhattacharya and Mukherjee in [8], and further studied in [9,10,15].

For $x \in L$, the Engel subalgebra, $E_L(x)$, is the Fitting null-component relative to $\text{ad } x$. If U is a subalgebra of L , the core of U , U_L , is the largest ideal of L contained in U . We put

$$\mathcal{G} = \{M : M \text{ is a maximal subalgebra of } L \text{ and } E_L(x) \subseteq M \text{ for some } x \in L\},$$

$$G(L) = \bigcap_{M \in \mathcal{G}} M \text{ if } \mathcal{G} \text{ is non-empty; } G(L) = L \text{ otherwise; } \gamma(L) = G(L)_L.$$

In Section 2, we consider the structure of $\gamma(L)$ and its relationship to properties of L itself. It is shown that $\gamma(L)$ is nil on L and that L is nilpotent precisely when $\gamma(L) = L$. When L is solvable, then $\gamma(L) = \tau(L) = T(L)_L$, where $T(L)$ is the intersection of the self-idealising maximal subalgebras of L (see [12]). Necessary and sufficient conditions are found for all maximal subalgebras of L to belong to \mathcal{G} , provided that the underlying field is algebraically closed and of characteristic zero.

In Section 3, necessary and sufficient conditions are found for every maximal subalgebra of L belonging to \mathcal{G} to have codimension one in L . This result generalises [13, Theorem 1]. Now, put

$$\mathcal{H} = \{M : M \text{ is a maximal subalgebra of } L \text{ of codimension } > 1 \text{ in } L\},$$

$$H(L) = \bigcap_{M \in \mathcal{H}} M \text{ if } \mathcal{H} \text{ is non-empty; } H(L) = L \text{ otherwise; } \eta(L) = H(L)_L,$$

$$\mathcal{D} = \{M \in \mathcal{G} : M \in \mathcal{H}\},$$

$$D(L) = \bigcap_{M \in \mathcal{D}} M \text{ if } \mathcal{D} \text{ is non-empty; } D(L) = L \text{ otherwise; } \delta(L) = D(L)_L.$$

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We then investigate some basic properties of $\eta(L)$ and $\delta(L)$, showing, in particular, that if L is solvable then they are supersolvable. Finally, it is shown that all maximal subalgebras M of L with $M \in \mathcal{D}$ are c -ideals of L if and only if L is solvable. This generalises [14, Theorem 3.1].

Throughout, L will denote a finite-dimensional Lie algebra over a field F , which is arbitrary unless restrictions on F are specified. We define the *derived series* for L inductively by $L^{(0)} = L$, $L^{(i+1)} = [L^{(i)}, L^{(i)}]$ for all $i \geq 0$. The symbol \oplus will denote a direct sum of the underlying vector space structure.

2. Maximals containing Engel subalgebras

First, we note that the maximals in \mathcal{G} are precisely those containing Cartan subalgebras, provided that the field has enough elements; also, that $G(L)$ is an ideal when F has characteristic zero.

Lemma 2.1. *Let L be a Lie algebra over a field F with at least $\dim L$ elements. Then $M \in \mathcal{G}$ if and only if M contains a Cartan subalgebra of L .*

Proof. Simply note that the minimal Engel subalgebras are precisely the Cartan subalgebras of L , by [4, Theorem 1]. \square

Lemma 2.2. *Let L be a Lie algebra over a field F of characteristic zero. Then $G(L) = \gamma(L)$.*

Proof. For every automorphism θ of L , and every $x \in L$, $\theta(E_L(x)) = E_L(\theta(x))$, so $G(L)$ is invariant under every automorphism of L . It follows from [11, Corollary 3.2] that it is invariant under all derivations of L , whence the result. \square

Next, we collect together some basic properties of $\gamma(L)$; in particular, parts (iv) and (v) of the first lemma below generalise characterisations of nilpotent Lie algebras due to Barnes (see [2,3]).

Lemma 2.3. *Let L be a Lie algebra. Then*

- (i) $\gamma(L)$ is nil on L , and so nilpotent;
- (ii) L is nilpotent if and only if $L = \gamma(L)$;
- (iii) L is nilpotent if and only if $\mathcal{G} = \emptyset$;
- (iv) L is nilpotent if and only if M is an ideal of L for all $M \in \mathcal{G}$; and
- (v) L is nilpotent if and only if L/B is nilpotent for some ideal B of L with $B \subseteq \gamma(L)$.

Proof. (i) Let $x \in \gamma(L)$, and suppose that $E_L(x) \neq L$. Then there is a subalgebra $M \in \mathcal{G}$ such that $E_L(x) \subseteq M$. But $L = E_L(x) \oplus L_1(x)$, where $L_1(x) = \cap_{i=1}^{\infty} L(\text{ad } x)^i \subseteq \gamma(L)$, so $L = M$, a contradiction. The result follows.

(ii) L is nilpotent if and only if $E_L(x) = L$ for all $x \in L$, by Engel's Theorem. But this holds if and only if $\gamma(L) = L$.

(iii) This is clear from (ii).

(iv) If L is nilpotent, then all maximal subalgebras of L are ideals of L (see [2]). Conversely, suppose that all maximal subalgebras of L in \mathcal{G} are ideals of L , and let $M \in \mathcal{G}$. Then $E_L(x) \subseteq M$ for some $x \in L$, so $I_L(M) = M$, by [16, Corollary 4.4.4.4]. It follows that $\mathcal{G} = \emptyset$, and thus that L is nilpotent, by (iii).

(v) Suppose that B is an ideal of L with $B \subseteq \gamma(L)$ such that L/B is nilpotent. Suppose that $\mathcal{G} \neq \emptyset$, and let $M \in \mathcal{G}$. Then there is an $x \in L$ such that $E_L(x) \subseteq M$. It follows from [16, Corollary 4.4.4.4] that $I_L(M) = M$. Now, $B \subseteq \gamma(L) \subseteq M$ and M/B is a maximal subalgebra of L/B , so M/B is an ideal of L/B . It follows that M is an ideal of L , a contradiction. Hence $\mathcal{G} = \emptyset$ and L is nilpotent, by (iii).

The converse is clear. \square

Lemma 2.4. *Let L be a Lie algebra over a field F and let B be an ideal of L . Then*

- (i) $(E_L(x) + B)/B \subseteq E_{L/B}(x + B)$ for all $x \in L$;
- (ii) if $B \subseteq M$ and $M/B \in \mathcal{G}$, then $M \in \mathcal{G}$;
- (iii) $(\gamma(L) + B)/B \subseteq \gamma(L/B)$;
- (iv) if F has at least $\dim L$ elements and $M \in \mathcal{G}$, then $M/B \in \mathcal{G}$ if $B \subseteq M$; and
- (v) if F has at least $\dim L$ elements and $B \subseteq \gamma(L)$, then $\gamma(L)/B = \gamma(L/B)$.

Proof. (i) Let $y \in E_L(x)$. Then $y(\text{ad } x)^n = 0$ for some $n \in \mathbb{N}$, and so $(y+B)(\text{ad } (x+B))^n \subseteq B$. It follows that $y+B \in E_{L/B}(x+B)$.

(ii) This follows from (i).

(iii) This follows from (ii).

(iv) Suppose that $M \in \mathcal{G}$. Then M contains a Cartan subalgebra C of L , by Lemma 2.1. But now $(C+B)/B$ is a Cartan subalgebra of L/B , by [16, Theorem 4.4.5.1], whence $M/B \in \mathcal{G}$.

(v) This follows from (iii) and (iv). \square

The *Frattini subalgebra*, $F(L)$, of L is the intersection of the maximal subalgebras of L , and the *Frattini ideal* is $\phi(L) = F(L)_L$. We say that L is ϕ -free if $\phi(L) = 0$. If U is a subalgebra of L , the *idealiser* of U in L is $I_L(U) = \{x \in L : [x, U] \subseteq U\}$. Next, we see that if L is solvable then the elements of \mathcal{G} are precisely the self-idealising maximal subalgebras of L .

Proposition 2.5. *Let L be solvable. Then $M \in \mathcal{G}$ if and only if $I_L(M) = M$.*

Proof. If $M \in \mathcal{G}$, then $E_L(x) \subseteq M$ for some $x \in L$, whence $I_L(M) = M$, by [16, Corollary 4.4.4.4].

Now, let L be a solvable Lie algebra of minimal dimension having a subalgebra M such that $I_L(M) = M$ but $M \notin \mathcal{G}$. Then L is ϕ -free and $M_L = 0$, by Lemma 2.4(ii). Let A be a minimal ideal of L . Then $L = A \oplus M$. Pick a minimal ideal B of M , and let $b \in B$. We have that $L = E_L(b) \oplus L_1(b)$. Moreover, $L_1(b) \subseteq A$, since B is abelian, so $L = E_L(b) + A$. Now, $A + B$ is an ideal of L , and hence so is $[A, B] = [A + B, A + B]$. If $E_L(b) = L$ for all $b \in B$, then $[A, B] = 0$, and B is an ideal of L , contradicting the fact that $M_L = 0$.

It follows that there is a $b \in B$ and a maximal subalgebra K of L such that $E_L(b) \subseteq K$. Now, $L = A \oplus K$ so $E_A(b) = 0$. But it is easy to check that $E_L(b) = E_A(b) \oplus E_M(b) = E_M(b)$, so $M \in \mathcal{G}$. This contradiction establishes the converse. \square

Corollary 2.6. Let L be a solvable Lie algebra. Then L is supersolvable if and only if M has codimension one in L for every $M \in \mathcal{G}$.

Proof. Suppose first that M has codimension one in L for every $M \in \mathcal{G}$. Let K be any maximal subalgebra of L such that $K \notin \mathcal{G}$. Then K is an ideal of L , by Proposition 2.5, and so has codimension one in L . It follows that L is supersolvable, by [3, Theorem 7].

The converse follows immediately from [3, Theorem 7]. \square

Corollary 2.7. Let B be an ideal of L with $B \subseteq \gamma(L)$ such that L/B is supersolvable. Then L is supersolvable.

Following [12], we put

$$\mathcal{T} = \{M : M \text{ is a maximal subalgebra of } L \text{ and } I_L(M) = M\},$$

$$T(L) = \bigcap_{M \in \mathcal{T}} M \text{ if } \mathcal{T} \text{ is non-empty; } T(L) = L \text{ otherwise; } \tau(L) = T(L)_L.$$

Then we have the following result.

Corollary 2.8. If L is solvable, then $\gamma(L) = \tau(L)$.

The hypothesis that L be solvable cannot be removed from Proposition 2.5, as the next result will show. A subalgebra U of a semisimple Lie algebra L is called *regular* if we can choose a basis for U in such a way that every vector of this basis is either a root vector of L corresponding to some Cartan subalgebra C of L , or otherwise belongs to C ; U is an *R-subalgebra* of L if it is contained in a regular subalgebra of L , and is an *S-subalgebra* otherwise (see [6, page 158]).

Proposition 2.9. Let L be a semisimple Lie algebra over an algebraically closed field F of characteristic zero. Then $M \in \mathcal{G}$ if and only if M is a regular maximal subalgebra of L .

Proof. Suppose first that $M \in \mathcal{G}$. Then M contains a Cartan subalgebra of L by Lemma 2.1, and so is clearly regular.

Now, suppose that M is a regular maximal subalgebra of L . Then M is either parabolic or semisimple of maximal rank (see [6]), whence $M \in \mathcal{G}$. \square

In view of the above result, if M is a maximal subalgebra that is an *S-subalgebra* of a semisimple Lie algebra L over an algebraically closed field of characteristic zero, then M is self-idealising but not in \mathcal{G} . This observation yields the following corollary.

Corollary 2.10. Let L be a semisimple Lie algebra over an algebraically closed field F of characteristic zero, and suppose that all maximal subalgebras of L belong to \mathcal{G} . Then $L = \mathfrak{sl}(2, F)$.

Proof. First, suppose that L is simple. If $L \neq \mathfrak{sl}(2, F)$, then L has an *S-subalgebra* isomorphic to $\mathfrak{sl}(2, F)$ (see [6, Section 9, pages 168–175]). This subalgebra must be inside a maximal subalgebra which is itself an *S-subalgebra*, and so is not in \mathcal{G} .

Now, suppose that L is semisimple, so that $L = S_1 \oplus \cdots \oplus S_n$, where S_i is a simple ideal of L , and let M be a maximal subalgebra of S_i . Then

$$\sum_{j=1, j \neq i}^n S_j \oplus M$$

is a maximal subalgebra of L , and so contains a Cartan subalgebra C of L . But now, if π_i is the projection map of L onto S_i , $\pi_i(C)$ is a Cartan subalgebra of S_i contained in M . It follows that $S_i \cong \mathfrak{sl}(2, F)$ for each $1 \leq i \leq n$.

Finally, let S be $\mathfrak{sl}(2, F)$, let \bar{S} be an isomorphic copy of S , and denote the image of $s \in S$ in \bar{S} by \bar{s} . Put $L = S \oplus \bar{S}$ with $[S, \bar{S}] = 0$. It is easy to check that the diagonal subalgebra $M = \{x \in L : x = s + \bar{s} \text{ for some } s \in S\}$ is maximal in L . However, $s \in E_L(s + \bar{s})$ for every $s \in S$, so $M \notin \mathcal{G}$. \square

The *abelian socle* of L , $\text{Asoc } L$, is the sum of the minimal abelian ideals of L . We can use the above result to classify the Lie algebras all of whose maximal subalgebras belong to \mathcal{G} .

Theorem 2.11. Let L be a Lie algebra over an algebraically closed field F of characteristic zero. Then the following are equivalent:

- (i) all maximal subalgebras of L belong to \mathcal{G} ; and
- (ii) $L/\phi(L) = \text{Asoc } (L/\phi(L)) \oplus S$ where $S \cong \mathfrak{sl}(2, F)$ and each minimal ideal of $L/\phi(L)$ has even dimension.

Proof. Suppose first that (i) holds and that $\phi(L) = 0$. Then we have that $L = \text{Asoc}(L) \oplus (S \oplus B)$, where S is a semisimple subalgebra, B is an abelian subalgebra, and $[S, B] = 0$, by [11, Theorems 7.4 and 7.5]. Let C be a Cartan subalgebra of L , and let $L = C \oplus L_1$ be the Fitting decomposition of L relative to C . Clearly, $L_1 \subseteq L^{(1)} \subseteq \text{Asoc}(L) \oplus S$. If $B \neq 0$, it follows that any maximal subalgebra of L containing $\text{Asoc}(L) \oplus S$ cannot contain a Cartan subalgebra of L . This yields that $B = 0$. That $S \cong \text{sl}(2, F)$ then follows from Lemma 2.4.

Now let A be a minimal ideal of L . Then A is an irreducible $\text{sl}(2, F)$ -module, and so has the structure given in [7, pages 83–86]. Let Fh be a Cartan subalgebra of S , and let D be its centraliser in A . Then $E = Fh + D$ is a Cartan subalgebra of $K = A + S$ (see [5]). If A has odd dimension, Cartan subalgebras of K have dimension greater than 1 (put $i = m/2$ in [7, (36), page 85]). But S is a maximal subalgebra of K and contains no such Cartan subalgebra.

If $\phi(L) \neq 0$, then $L/\phi(L)$ satisfies the hypotheses of (i) by Lemma 2.4, and so (ii) follows.

So now assume that (ii) holds. We can assume that $\phi(L) = 0$, since, if we can prove (ii) for this case, the result will follow from Lemma 2.4. Let $A = \text{Asoc}(L) = A_1 \oplus \cdots \oplus A_n$. It is straightforward to check, using [7, (36), page 85]), that, if Fh is a Cartan subalgebra of S , then $C_A(h) = 0$. It follows from [5] that the Cartan subalgebras of L are one dimensional. Now, the maximal subalgebras of L are of the form $A \oplus K$, where K is a maximal subalgebra of S , or

$$\sum_{j=1, j \neq i}^n A_j \oplus S.$$

Clearly, each of these contains a Cartan subalgebra of L , and (i) follows. \square

3. Maximal subalgebras of codimension one

Our objective here is to generalise Corollary 2.6 and [13, Theorem 1]. First, we recall the definition of the algebras $L_m(\Gamma)$ over a field F of characteristic zero or p , where p is prime, as given by Amayo in [1, page 46]. Let m be a positive integer satisfying

$$\begin{aligned} m &= 1, \\ \text{or if } p \text{ is odd, } m &= p^r - 2 \quad (r \geq 1), \\ \text{or if } p &= 2, \quad m = 2^r - 2 \text{ or } m = 2^r - 3 \quad (r \geq 2). \end{aligned}$$

Let $\Gamma = \{\gamma_0, \gamma_1, \dots\} \subseteq F$ subject to

$$\begin{aligned} (m+1-i)\gamma_i &= \gamma_{m+i-1} = 0 \quad \text{for all } i \geq 1, \text{ and} \\ \lambda_{i,k+1-i}\gamma_{k+1} &= 0 \quad \text{for all } i, k \text{ with } 1 \leq i \leq k. \end{aligned}$$

Let $L_m(\Gamma)$ be the Lie algebra over F with basis $v_{-1}, v_0, v_1, \dots, v_m$ and products

$$\begin{aligned} [v_{-1}, v_i] &= -[v_i, v_{-1}] = v_{i-1} + \gamma_i v_m, \quad [v_{-1}, v_{-1}] = 0, \\ [v_i, v_j] &= \lambda_{ij} v_{i+j} \quad \text{for all } i, j \text{ with } 0 \leq i, j \leq m, \end{aligned}$$

where $v_{m+1} = \cdots = v_{2m} = 0$.

We also let $H_{m,i}$ be the subspace spanned by v_i, \dots, v_m .

We shall need the following classification of Lie algebras with core-free subalgebras of codimension one as given in [1].

Theorem 3.1. ([1, Theorem 3.1]) Let L have a core-free subalgebra of codimension one. Then either (i) $\dim L \leq 2$, or else (ii) $L \cong L_m(\Gamma)$ for some m and Γ satisfying the above conditions.

We shall also need the following properties of $L_m(\Gamma)$ which are given by Amayo in [1].

Theorem 3.2 ([1, Theorem 3.2]). (i) If $m > 1$ and m is odd, then $L_m(\Gamma)$ is simple, and $H_{m,0}$ is the only subalgebra of codimension one.

(ii) If $m > 1$ and m is even, then $L_m(\Gamma)$ has precisely two subalgebras of codimension one in L_m , namely $L_m^{(1)}$ and $H_{m,0}$.

(iii) $L_1(\Gamma)$ has a basis $\{u_{-1}, u_0, u_1\}$ with multiplication $[u_{-1}, u_0] = u_{-1} + \gamma_0 u_1$ ($\gamma_0 \in F$, $\gamma_0 = 0$ if $\Gamma = \{0\}$), $[u_{-1}, u_1] = u_0$, $[u_0, u_1] = u_1$.

(iv) If F has characteristic different from two, then $L_1(\Gamma) \cong L_1(0) \cong \text{sl}(2, F)$.

(v) If F has characteristic two, then $L_1(\Gamma) \cong L_1(0)$ if and only if γ_0 is a square in F .

Lemma 3.3. Let $x \in L$ be ad-nilpotent. Then $(\text{ad } x)^{\dim L} = 0$.

Proof. It is easy to check that, if $y(\text{ad } x)^n = 0$ but $y(\text{ad } x)^{n-1} \neq 0$, then the elements $y, y(\text{ad } x), \dots, y(\text{ad } x)^{n-1}$ are linearly independent. \square

Lemma 3.4. Let $L \cong L_m(\Gamma)$. Then all maximal subalgebras $M \in \mathcal{G}$ have codimension one in L if and only if $L \cong L_1(0)$.

Proof. Let $L = L_m(\Gamma)$ have all maximal subalgebras $M \in \mathcal{G}$ with codimension one in L , and suppose that $m > 1$. We have that $v_{-1} \notin H_{m,0}$, and it is shown in the proof of [1, Theorem 3.2] that $v_m \notin L_m^{(1)}$, so $v_{-1} + v_m$ does not belong to a subalgebra of codimension one in L . It follows that $E_L(v_{-1} + v_m) = L$ and $v_{-1} + v_m$ is ad-nilpotent. Now,

$$\begin{aligned} [v_i, v_{-1} + v_m] &= -v_{i-1} - \gamma_i v_m \quad (i \geq 1); \\ [v_0, v_{-1} + v_m] &= -v_{-1} - (\gamma_0 + \lambda_{m0})v_m; [v_{-1}, v_{-1} + v_m] = -v_{m-1} - \gamma_m v_m. \end{aligned}$$

This yields that $v_i(\text{ad}(v_{-1} + v_m))^{m+2}$ is $(-1)^{m+2}\gamma_i v_{-1} + f(v_0, v_1, \dots, v_m)$ for $1 \leq i \leq m$. Since $m+2 = \dim L$, it follows from Lemma 3.3 that $\gamma_i = 0$ for $1 \leq i \leq m$. But $v_{-1} + v_0 + v_m$ is also ad-nilpotent, and a similar calculation shows that $v_1(\text{ad}(v_{-1} + v_0 + v_m))^{m+2}$ is $(-1)^{m+2}v_0 + g(v_{-1}, v_1, \dots, v_m)$, which contradicts Lemma 3.3. Hence $m = 1$.

Suppose that $L \not\cong L_1(0)$. Then Theorem 3.2(iii) and (iv) imply that F has characteristic two and L has a basis u_{-1}, u_0, u_1 with multiplication $[u_{-1}, u_0] = u_{-1} + \gamma_0 u_1$, $[u_{-1}, u_1] = u_0$, $[u_0, u_1] = u_1$, where γ_0 is not a square in F . But a simple calculation verifies that Fu_{-1} is a maximal subalgebra of L containing $E_L(u_{-1}) = Fu_{-1}$, contradicting our hypothesis. It follows that $L \cong L_1(0)$.

The converse follows from [13, Theorem 1]. \square

Theorem 3.5. Let L be a Lie algebra over a field with at least $\dim L$ elements. Then the following are equivalent:

- (i) every maximal subalgebra $M \in \mathcal{G}$ has codimension one in L ; and
- (ii) $L/\gamma(L) = S \oplus R$, where $S = S_1 \oplus \dots \oplus S_n$, S_i is a simple ideal of $L/\gamma(L)$ isomorphic to $L_1(0)$ for each $1 \leq i \leq n$, or is $\{0\}$, and R is a supersolvable ideal of $L/\gamma(L)$ (possibly $\{0\}$).

Proof. (i) \Rightarrow (ii): Suppose that (i) holds, and assume first that $\gamma(L) = 0$. If L is solvable it is supersolvable, by Corollary 2.6, so suppose that L is not solvable. Clearly, L has a maximal subalgebra $M_1 \in \mathcal{G}$ (otherwise, L is nilpotent). If all $M \in \mathcal{G}$ have L/M_L solvable, then $L^{(2)} \subseteq \gamma(L) = 0$, and L is solvable, so we can further assume that $L/(M_1)_L \cong L_1(0)$, by Theorem 3.1 and Lemma 3.4. If $(M_1)_L = 0$, we have finished, so suppose that $(M_1)_L \neq 0$.

Let $M_2 \in \mathcal{G}$ be a maximal subalgebra with $(M_1)_L \not\subseteq M_2$. Then $L = (M_1)_L + M_2$. Put $B = (M_1)_L + (M_2)_L$. Since $L/(M_1)_L$ is simple, $B = (M_1)_L$ or $B = L$. The former implies that $(M_1)_L = (M_2)_L = M_2$, a contradiction, so $L = B = (M_1)_L + (M_2)_L$. Now, $L/((M_1)_L \cap (M_2)_L) \cong (L/(M_1)_L) \oplus (L/(M_2)_L)$. Suppose there is such an M_2 with $L/(M_2)_L \cong L_1(0)$. If $(M_1)_L \cap (M_2)_L = 0$, we have finished. If $(M_1)_L \cap (M_2)_L \neq 0$, then choose $M_3 \in \mathcal{G}$ to be a maximal subalgebra with $(M_1)_L \cap (M_2)_L \not\subseteq M_3$. In similar fashion to that above, we find that $L = ((M_1)_L \cap (M_2)_L) + (M_3)_L$. If there is such an M_3 with $L/(M_3)_L \cong L_1(0)$ and $(M_1)_L \cap (M_2)_L \cap (M_3)_L \neq 0$, we continue in the same way.

Eventually, obtain $L = A + (M_n)_L$, where $A = (M_1)_L \cap \dots \cap (M_{n-1})_L$, $M_n \in \mathcal{G}$ is a maximal subalgebra of L with $A \not\subseteq M_n$, $L/(A \cap (M_n)_L) \cong (L/(M_n)_L) \oplus S_1 \oplus \dots \oplus S_{n-1}$, each $S_i \cong L/(M_i)_L \cong L_1(0)$ for $1 \leq i \leq n-1$, and either $L/(M_n)_L \cong L_1(0)$ and $A \cap (M_n)_L = 0$, in which case we have finished, or else there is no $M_n \in \mathcal{G}$ with $L/(M_n)_L \cong L_1(0)$ and $\dim(L/(M_n)_L) \leq 2$. So suppose that the latter holds, in which case $L^{(2)} \subseteq (M_n)_L$. We now have that, for every $M \in \mathcal{G}$, either $A \subseteq M$ or $L^{(2)} \subseteq M$, which yields that $A \cap L^{(2)} \subseteq \gamma(L) = 0$, whence $A^{(2)} = 0$ and A is solvable.

Now

$$L/A \cong \frac{L/(A \cap (M_n)_L)}{A/(A \cap (M_n)_L)} \cong S_1 \oplus \dots \oplus S_{n-1}$$

(since $A/(A \cap (M_n)_L) \cong L/(M_n)_L$), so $L = L^{(2)} \oplus A$. Moreover, A is supersolvable, by Corollary 2.6, and $L^{(2)} \cong L/A$, which completes the proof for the case $\gamma(L) = 0$.

Now, suppose that $\gamma(L) \neq 0$. Then, if $M/\gamma(L) \in \mathcal{G}$, we have $M \in \mathcal{G}$, by Lemma 2.4(ii), and so $M/\gamma(L)$ has codimension one in $L/\gamma(L)$. Thus (ii) follows from the argument above.

(ii) \Rightarrow (i): So now suppose that (ii) holds, and let $M \in \mathcal{G}$. Then $L/\gamma(L)$ is ϕ -free, so $\bar{L} = L/\gamma(L) = S \oplus R = S \oplus (A \oplus B)$, where $A = A_1 \oplus \dots \oplus A_k = \text{Asoc } L$ and B is abelian, by [11, Theorems 7.3 and 7.4]. Since $R = A \oplus B$ is supersolvable, $\dim A_i = 1$ for each $1 \leq i \leq k$. Clearly, $\bar{M} = M/\gamma(L)$ is a maximal subalgebra of \bar{L} . If $A \not\subseteq \bar{M}$, then there is an $A_i \not\subseteq \bar{M}$ for some $1 \leq i \leq k$. But then $\bar{L} = A_i + \bar{M}$, and \bar{M} has codimension one in \bar{L} , whence M has codimension one in L . So assume that $A \subseteq \bar{M}$.

Suppose that $B \not\subseteq \bar{M}$. Then there is an element $b \in B$ such that $b \notin \bar{M}$. But $[B, \bar{L}] \subseteq A \subseteq \bar{M}$, so $\bar{L} = \bar{M} + Fb$, and again M has codimension one in L .

So suppose that $R \subseteq \bar{M}$. Suppose further that there exist i, j with $1 \leq i, j \leq n$ and $S_i \not\subseteq \bar{M}$, $S_j \not\subseteq \bar{M}$. Then $\bar{L} = \bar{M} + S_i = \bar{M} + S_j$. Moreover, $[\bar{L}, \bar{M} \cap S_i] = [\bar{M} + S_j, \bar{M} \cap S_i] = [\bar{M}, \bar{M} \cap S_i] \subseteq \bar{M} \cap S_i$, so $\bar{M} \cap S_i$ is an ideal of \bar{L} . It follows that $\bar{M} \cap S_i = 0$. Since $\bar{M} \in \mathcal{G}$, by Lemma 2.4(iv), $E_L(r + \sum_{t=1}^n x_t) \subseteq \bar{M}$ for some $r \in R$, $x_t \in S_t$. But now $x_i \in E_L(r + \sum_{t=1}^n x_t) \cap S_i = 0$ for each $x_i \in S_i$, which yields that $S_i \subseteq E_L(r + \sum_{t=1}^n x_t) \cap S_i = 0$, a contradiction. We therefore have that there is just one S_i with $S_i \not\subseteq \bar{M}$, in which case

$$\bar{M} = R \oplus \sum_{j=1, j \neq i}^n S_j \oplus K,$$

where K is a maximal subalgebra of S_i . It follows from Lemma 3.4 that M has codimension one in L . \square

Then we have the following corollary to Theorem 3.5.

Corollary 3.6. *Let L be a Lie algebra over a field with at least $\dim L$ elements. Then L is supersolvable if and only if $\delta(L) = L$ and $L/\gamma(L)$ has no ideals isomorphic to $L_1(0)$.*

Next, we have some basic properties of $\eta(L)$ and $\delta(L)$.

Lemma 3.7. *Let L be a Lie algebra over a field F , and let B be an ideal of L . Then*

- (i) $(\eta(L) + B)/B \subseteq \eta(L/B)$ and $(\delta(L) + B)/B \subseteq \delta(L/B)$;
- (ii) if $B \subseteq \eta(L)$, then $\eta(L)/B = \eta(L/B)$; and
- (iii) if F has at least $\dim L$ elements and $B \subseteq \delta(L)$, then $\delta(L)/B = \delta(L/B)$.

Proof. This is straightforward. \square

Define the series $\{Z_i : i \geq 0\}$ inductively by $Z_0 = \{0\}$, $Z_i/Z_{i-1} = Z(L/Z_{i-1})$ for all $i \geq 1$, where $Z(L)$ is the centre of L . Then the hypercentre of L is $Z_\infty = \bigcup_{i=0}^\infty Z_i$.

Proposition 3.8. *For any Lie algebra L , $Z_\infty \subseteq \eta(L) \subseteq \delta(L)$.*

Proof. Suppose that $Z_\infty \not\subseteq \eta(L)$. Then there is a maximal subalgebra $M \in \mathcal{H}$ and $k \geq 1$ such that $Z_k \not\subseteq M$, whereas $Z_{k-1} \subseteq M$. But now $L = M + Z_k$, from which it follows that M is an ideal of L , and so has codimension one in L , a contradiction. The other inclusion is clear. \square

Proposition 3.9. *For any solvable Lie algebra L , $\delta(L)$ (and hence also $\eta(L)$) is supersolvable.*

Proof. Let L be a minimal counter-example. Suppose first that $\gamma(L) \neq 0$. Then $\delta(L)/\gamma(L) \subseteq \delta(L/\gamma(L))$ is supersolvable. But

$$\frac{\delta(L)}{\gamma(L)} \cong \frac{(\delta(L)/\phi(L))}{(\gamma(L)/\phi(L))} \cong \frac{(\delta(L)/\phi(L))}{(\tau(L)/\phi(L))} = \frac{(\delta(L)/\phi(L))}{Z(L/\phi(L))},$$

by Corollary 2.8 and [12, Theorem 2.8]. It follows that $\delta(L)/\phi(L)$ is supersolvable, and hence that $\delta(L)$ is supersolvable, by [3, Theorem 6].

So suppose now that $\gamma(L) = 0$. Let A be a minimal ideal of L with $A \subseteq \delta(L)$. Then there is a maximal subalgebra $M \in \mathcal{G}$ of L with $L = A \oplus M$. If $\dim A > 1$, then $A \subseteq \delta(L) \subseteq M$, a contradiction. Hence $\dim A = 1$. But $\delta(L)/A \subseteq \delta(L/A)$ is supersolvable, whence $\delta(L)$ is supersolvable. This contradiction completes the proof. \square

A subalgebra U of L is called a *c-ideal* of L if there is an ideal C of L such that $L = U + C$ and $U \cap C \leq U_L$. Finally, we have the following generalisation of [14, Theorem 3.1].

Theorem 3.10. *Let L be a Lie algebra over any field F . Then all maximal subalgebras M of L with $M \in \mathcal{D}$ are c-ideals of L if and only if L is solvable.*

Proof. This follows closely that of [14, Theorem 3.1], but we include the details for the convenience of the reader. Let L be a non-solvable Lie algebra of smallest dimension in which all maximal subalgebras M of L with $M \in \mathcal{D}$ are c-ideals of L . Clearly, all maximal subalgebras M of L with $M \in \mathcal{G}$ are c-ideals of L and $\mathcal{G} \neq \emptyset$. Then all proper factor algebras of L are solvable, by [14, Lemma 2.1(ii)] and Lemma 2.4. Suppose first that L is simple. Let M be a maximal subalgebra of L with $M \in \mathcal{G}$. Then M is a c-ideal, so there is an ideal C of L such that $L = M + C$ and $M \cap C \leq M_L = 0$, as L is simple. This yields that C is a non-trivial proper ideal of L , a contradiction. If L has two minimal ideals B_1 and B_2 , then L/B_1 and L/B_2 are solvable and $B_1 \cap B_2 = 0$, so L is solvable. Hence L has a unique minimal ideal B and L/B is solvable.

Suppose that there is an element $b \in B$ such that $\text{ad}_L b$ is not nilpotent. Let $L = E_L(b) \oplus L_1$ be the Fitting decomposition of L relative to $\text{ad}_L b$. Then $L \neq E_L(b)$, so let M be a maximal subalgebra of L containing $E_L(b)$. As $M \in \mathcal{G}$, it is a c-ideal, and so there is an ideal C of L such that $L = M + C$ and $M \cap C \leq M_L$. Now $L_1 \leq B$, so $B \not\leq M_L$. It follows that $M_L = 0$, whence $M = E_L(b)$ and $B = C = L_1$. But $b \in M \cap B = 0$. Hence every element of B is ad-nilpotent, yielding that B is nilpotent, and so L is solvable, a contradiction.

The converse follows from [14, Theorem 3.1]. \square

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