DERIVATIONS

Introduction to non-associative algebra

OR

Playing havoc with the product rule?

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UNIVERSITY STUDIES 4
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PREAMBLE

Much of the algebra taught in the undergraduate curriculum, such as linear algebra (**vector spaces, matrices**), modern algebra (**groups, rings, fields**), number theory (**primes, congruences**) is concerned with systems with one or more associative binary products.

For example, addition and multiplication of matrices is associative:

$$A+(B+C)=(A+B)+C$$

and

$$A(BC)=(AB)C.$$

In the early 20th century, physicists started using the product A.B for matrices, defined by

$$A.B = AB + BA,$$

and called the Jordan product (after the physicist **Pascual Jordan 1902-1980**), to model the observables in quantum mechanics.

Also in the early 20th century both mathematicians and physicists used the product [A,B], defined by

$$[A, B] = AB - BA$$

and called the Lie product (after the mathematician **Sophus Lie 1842-1899**), to study differential equations.

Sophus Lie (1842–1899)

Marius Sophus Lie was a Norwegian mathematician. He largely created the theory of continuous symmetry, and applied it to the study of geometry and differential equations.

Pascual Jordan (1902–1980)

Pascual Jordan was a German theoretical and mathematical physicist who made significant contributions to quantum mechanics and quantum field theory.

Neither one of these products is associative, so they each give rise to what is called a nonassociative algebra, in these cases, called **Jordan algebras** and **Lie algebras** respectively.

Abstract theories of these algebras and other nonassociative algebras were subsequently developed and have many other applications, for example to **cryptography** and **genetics**, to name just two.

Lie algebras are especially important in **particle physics**.

Using only the product rule for differentiation, which every calculus student knows, the seminar will introduce the subject of **nonassociative algebras** as the natural context for derivations.

Before long we shall introduce derivations on other algebraic systems which have a **ternary** rather than a binary product, with special emphasis on Jordan and Lie structures.

Much later on we shall introduce the notion of a **module**, which is usually not taught in an undergraduate curriculum.

This will prepare us for the introduction of the sophisticated subject called **homological algebra**.

FIRST MEETING
SEPTEMBER 27, 2012

THE DERIVATIVE

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

DIFFERENTIATION IS A LINEAR PROCESS

$$(f+g)' = f' + g'$$

$$(cf)' = cf'$$

THE SET OF DIFFERENTIABLE FUNCTIONS FORMS AN ALGEBRA $\mathcal{D}$

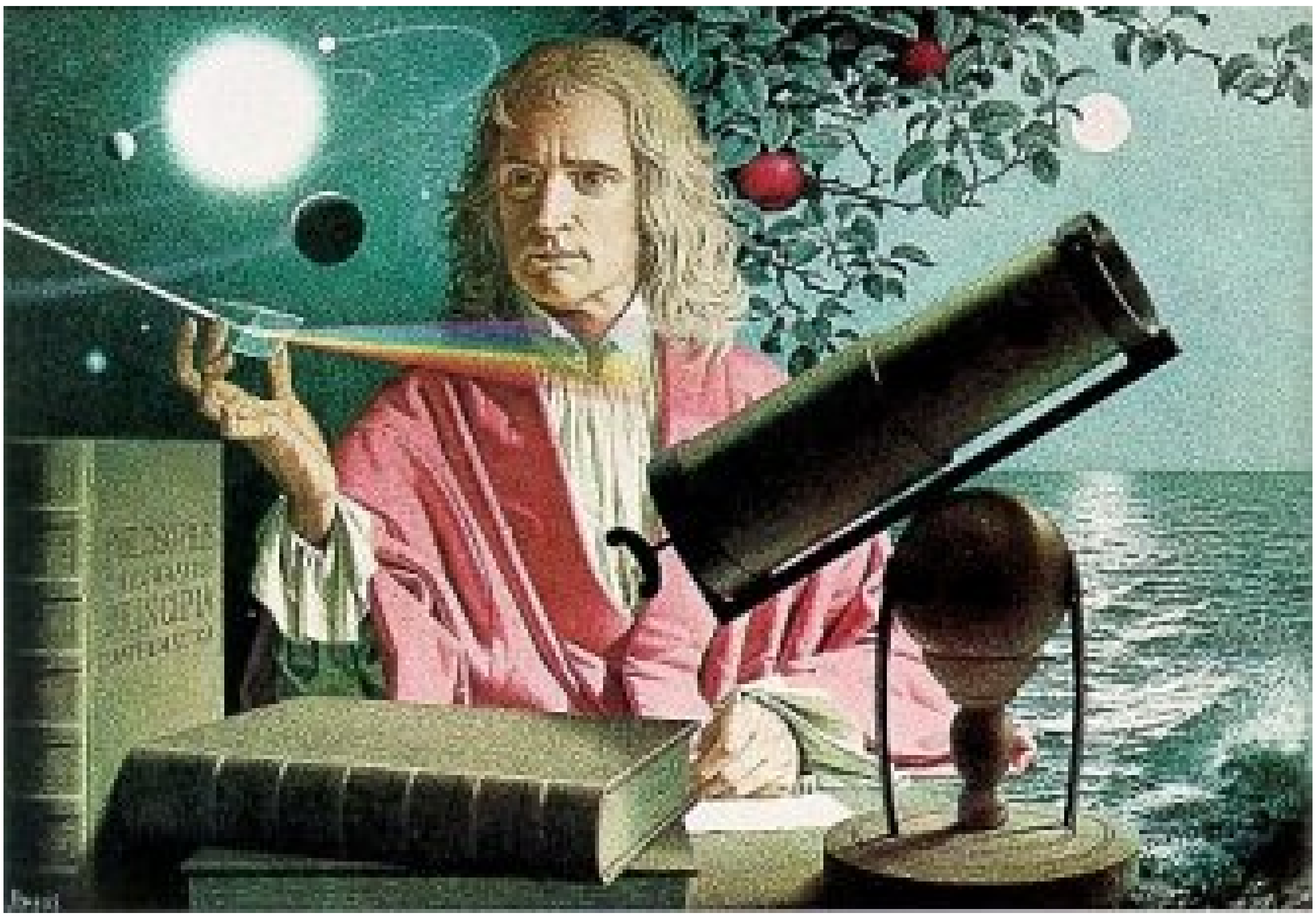
$$(fg)' = fg' + f'g$$

(product rule)

HEROS OF CALCULUS

#1

Sir Isaac Newton (1642-1727)



Isaac Newton was an English physicist, mathematician, astronomer, natural philosopher, alchemist, and theologian, and is considered by many scholars and members of the general public to be one of the most influential people in human history.

LEIBNIZ RULE

$$(fg)' = f'g + fg'$$

(order changed)

****

$$(fgh)' = f'gh + fg'h + fgh'$$

****

$$(f_1 f_2 \cdots f_n)' = f_1' f_2 \cdots f_n + \cdots + f_1 f_2 \cdots f_n'$$

The chain rule,

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

plays no role in this seminar

Neither does the quotient rule

$$(f/g)' = \frac{gf' - fg'}{g^2}$$

#2
Gottfried Wilhelm Leibniz (1646-1716)

Gottfried Wilhelm Leibniz was a German mathematician and philosopher. He developed the infinitesimal calculus independently of Isaac Newton, and Leibniz's mathematical notation has been widely used ever since it was published.

CONTINUITY

$$x_n \to x \Rightarrow f(x_n) \to f(x)$$

THE SET OF CONTINUOUS FUNCTIONS FORMS AN ALGEBRA $\mathcal{C}$

(sums, constant multiples and products of continuous functions are continuous)

$\mathcal{D}$ and $\mathcal{C}$ ARE EXAMPLES OF ALGEBRAS WHICH ARE BOTH **ASSOCIATIVE** AND **COMMUTATIVE**

PROPOSITION 1
EVERY DIFFERENTIABLE FUNCTION IS CONTINUOUS

$\mathcal{D}$ is a subalgebra of $\mathcal{C}$; $\mathcal{D} \subset \mathcal{C}$

DIFFERENTIATION IS A LINEAR PROCESS

LET US DENOTE IT BY D AND WRITE Df for f'

$$D(f+g) = Df + Dg$$

$$D(cf) = cDf$$

$$D(fg) = (Df)g + f(Dg)$$

$$D(f/g) = \frac{g(Df) - f(Dg)}{g^2}$$

IS THE LINEAR PROCESS $D : f \mapsto f'$ CONTINUOUS?

(If $f_n \to f$ in $\mathcal{D}$, does $f_n' \to f'$?)

(ANSWER: NO!)

DEFINITION 1

A DERIVATION ON $\mathcal{C}$ IS A LINEAR PROCESS SATISFYING THE LEIBNIZ RULE:

$$\delta(f+g) = \delta(f) + \delta(g)$$

$$\delta(cf) = c\delta(f)$$

$$\delta(fg) = \delta(f)g + f\delta(g)$$

THEOREM 1

There are no (non-zero) derivations on $\mathcal{C}$.

In other words,
Every derivation of $\mathcal{C}$ is identically zero

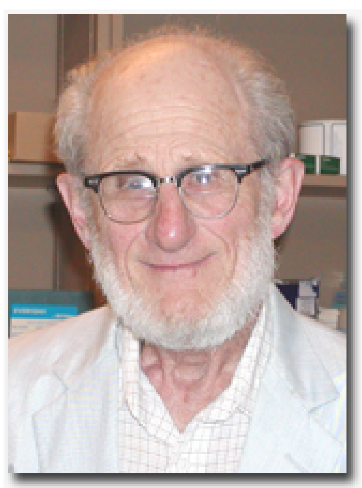
COROLLARY $\mathcal{D} \neq \mathcal{C}$

(NO DUUUH! $f(x) = |x|$)

THEOREM 1A
(1955-Singer and Wermer)
Every continuous derivation on $\mathcal{C}$ is zero.

Theorem 1B
(1960-Sakai)
Every derivation on $\mathcal{C}$ is continuous.

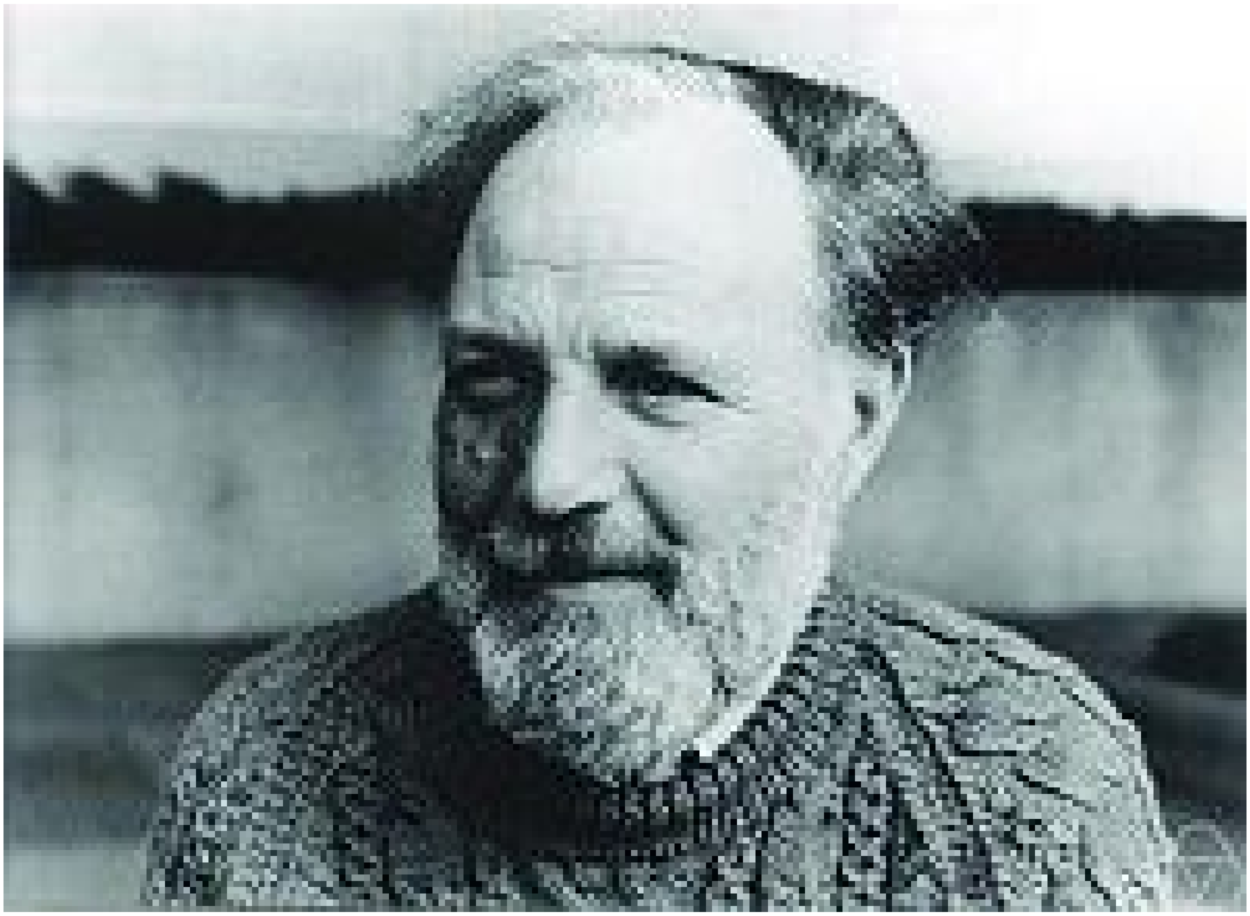
(False for $\mathcal{D}$)



John Wermer (b. 1925)

Soichiro Sakai (b. 1926)

Isadore Singer (b. 1924)



Isadore Manuel Singer is an Institute Professor in the Department of Mathematics at the Massachusetts Institute of Technology. He is noted for his work with Michael Atiyah in 1962, which paved the way for new interactions between pure mathematics and theoretical physics.

DERIVATIONS ON THE SET OF MATRICES

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER

MATRIX ADDITION

$$A + B$$

AND

MATRIX MULTIPLICATION

$$A \times B$$

WHICH IS ASSOCIATIVE BUT NOT COMMUTATIVE.

DEFINITION 2

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS A LINEAR PROCESS δ WHICH SATISFIES THE LEIBNIZ RULE

$$\delta(A \times B) = \delta(A) \times B + A \times \delta(B)$$

.

PROPOSITION 2

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION (WHICH CAN BE NON-ZERO)

THEOREM 2

(1942 Hochschild)

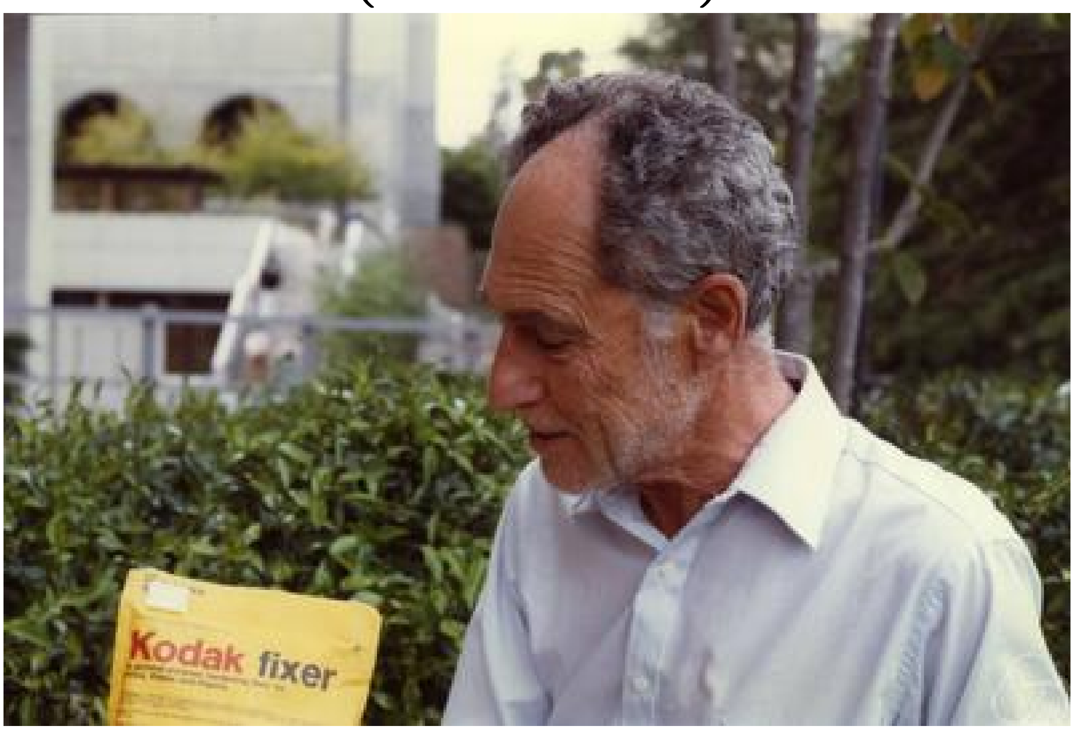
EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

Gerhard Hochschild (1915–2010)

(Photo 1968)

Gerhard Paul Hochschild was an American mathematician who worked on Lie groups, algebraic groups, homological algebra and algebraic number theory.

(Photo 1976)



(Photo 1981)

SECOND MEETING
OCTOBER 4, 2012

THE BRACKET PRODUCT ON THE SET OF MATRICES

THE BRACKET PRODUCT ON THE SET $M_n(\mathbf{R})$ OF MATRICES IS DEFINED BY

$$[X, Y] = X \times Y - Y \times X$$

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND BRACKET MULTIPLICATION, WHICH IS NOT ASSOCIATIVE AND NOT COMMUTATIVE.

DEFINITION 3

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO BRACKET MULTIPLICATION

IS A LINEAR PROCESS δ WHICH SATISFIES THE LEIBNIZ RULE

$$\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$$

.

PROPOSITION 3

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = [A, X] = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION

THEOREM 3

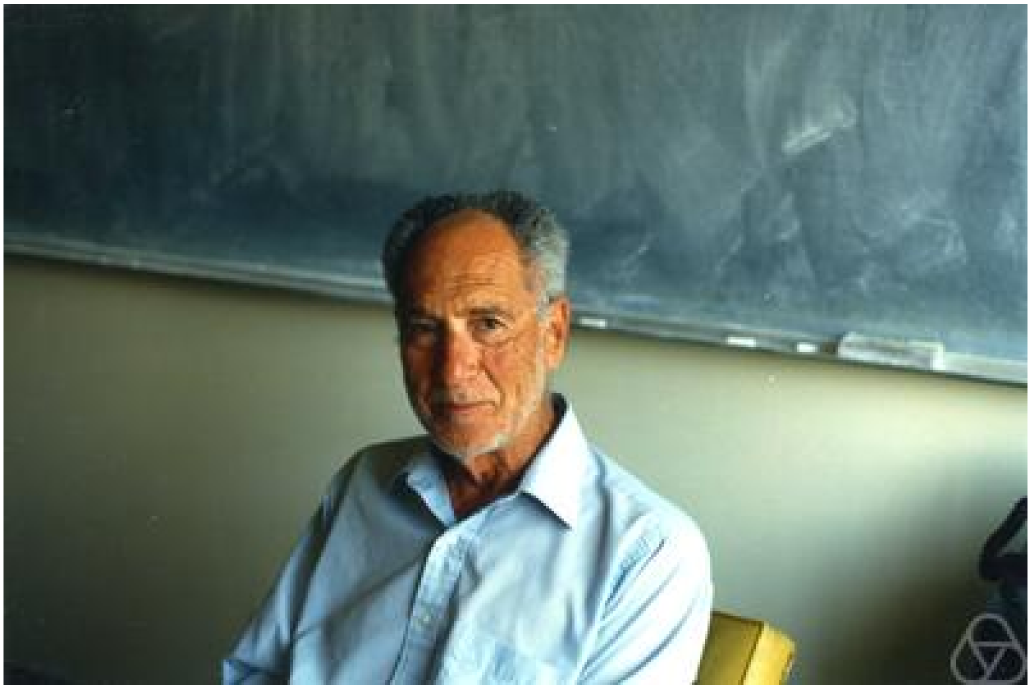
(1942 Hochschild, Zassenhaus)

EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

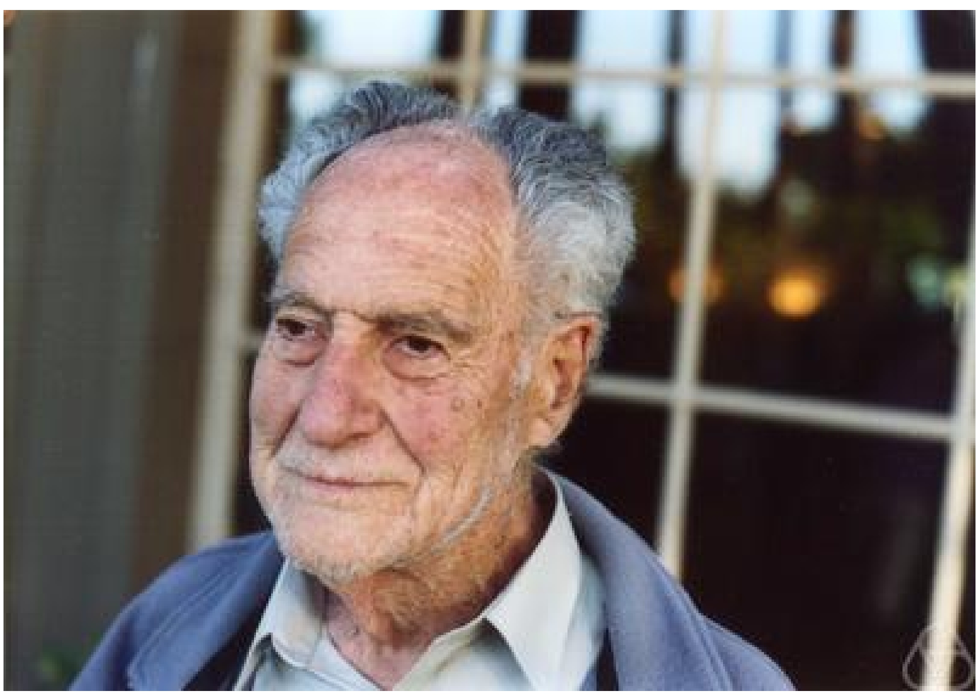
Hans Zassenhaus (1912–1991)

Hans Julius Zassenhaus was a German mathematician, known for work in many parts of abstract algebra, and as a pioneer of computer algebra.

Gerhard Hochschild (1915–2010)



(Photo 1986)



(Photo 2003)

THE CIRCLE PRODUCT ON THE SET OF MATRICES

THE CIRCLE PRODUCT ON THE SET $M_n(\mathbf{R})$ OF MATRICES IS DEFINED BY

$$X \circ Y = (X \times Y + Y \times X)/2$$

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND CIRCLE MULTIPLICATION, WHICH IS COMMUTATIVE BUT NOT ASSOCIATIVE.

DEFINITION 4

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION

IS A LINEAR PROCESS δ WHICH SATISFIES THE LEIBNIZ RULE

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$

PROPOSITION 4

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION

THEOREM 4

(1972-Sinclair)

EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

REMARK

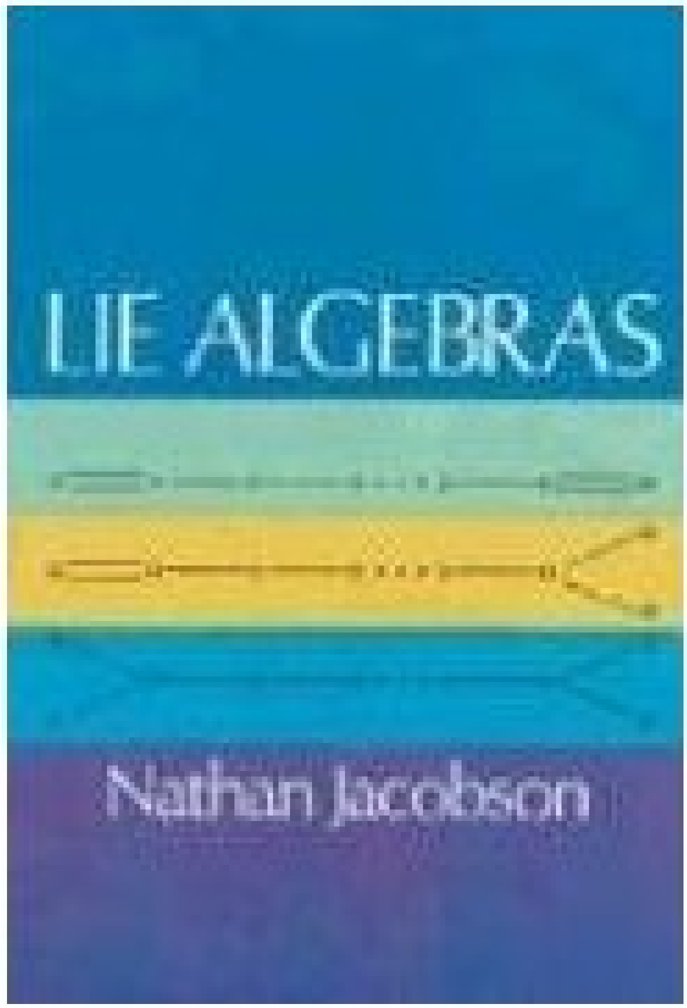
(1937-Jacobson)

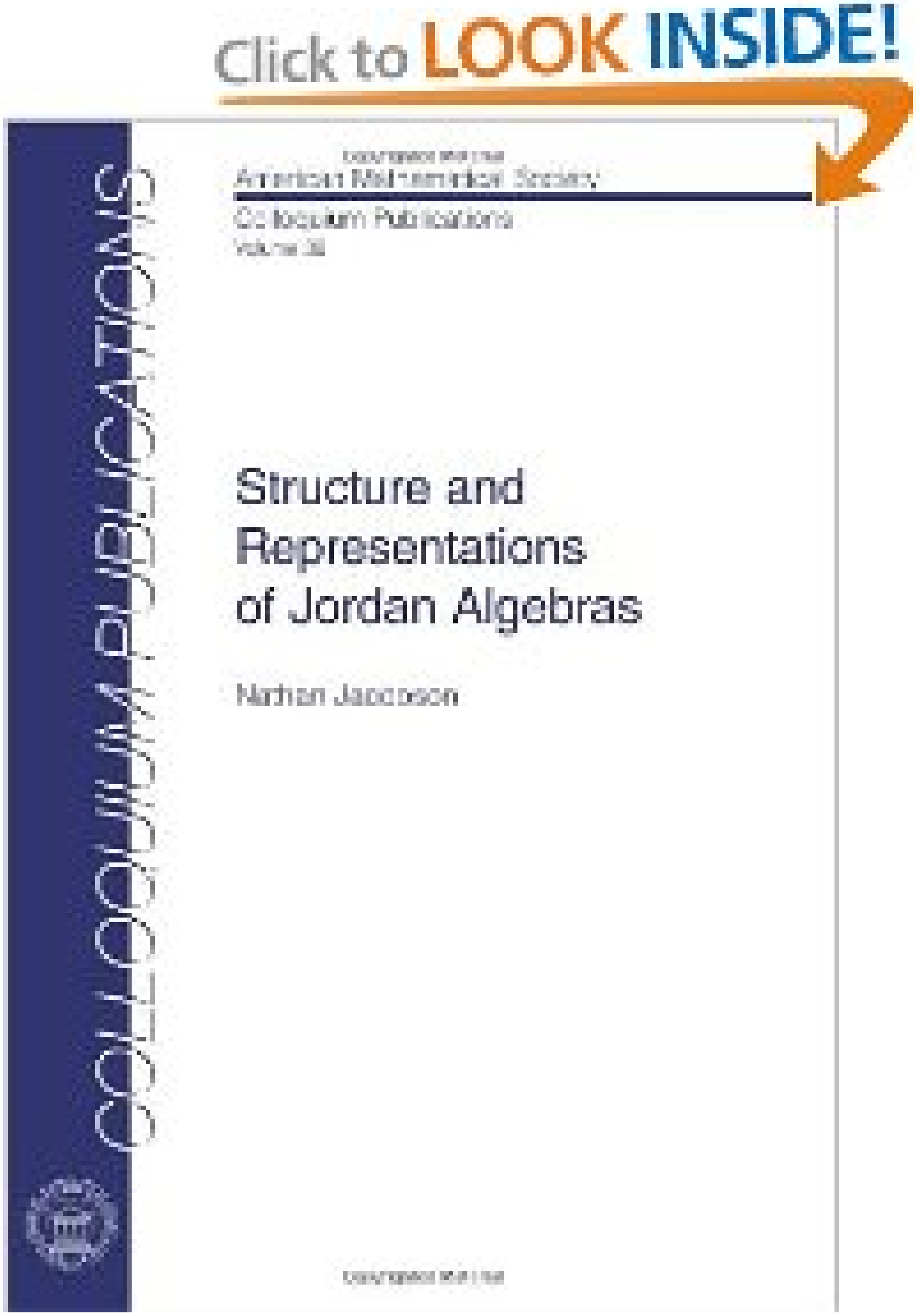
THE ABOVE PROPOSITION AND THEOREM NEED TO BE MODIFIED FOR THE SUBALGEBRA (WITH RESPECT TO CIRCLE MULTIPLICATION) OF SYMMETRIC MATRICES.

Alan M. Sinclair (retired)

Nathan Jacobson (1910–1999)

Nathan Jacobson was an American mathematician who was recognized as one of the leading algebraists of his generation, and he was also famous for writing more than a dozen standard monographs.


LIE ALGEBRAS
Nathan Jacobson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American Mathematical Society
Colloquium Publications
COLLOQUIUM PUBLICATIONS
Structure and
Representations
of Jordan Algebras
Nathan Jacobson

IT IS TIME FOR A SUMMARY OF THE PRECEDING

Table 1

$M_n(\mathbf{R})$ (ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a, b] = ab - ba$	$a \circ b = ab + ba$
Th. 2	Th.3	Th.4
$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$

THIRD MEETING
OCTOBER 11, 2012

AXIOMATIC APPROACH

AN ALGEBRA IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH TWO BINARY OPERATIONS, CALLED ADDITION AND MULTIPLICATION

ADDITION IS DENOTED BY

$$a + b$$

AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE

$$a + b = b + a, \quad (a + b) + c = a + (b + c)$$

MULTIPLICATION IS DENOTED BY

$$ab$$

AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION

$$(a + b)c = ac + bc, \quad a(b + c) = ab + ac$$

AN ALGEBRA IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)

(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

THE ALGEBRAS $\mathcal{C}$, $\mathcal{D}$ AND $M_n(\mathbf{R})$ ARE EXAMPLES OF ASSOCIATIVE ALGEBRAS.

$\mathcal{C}$ AND $\mathcal{D}$ ARE COMMUTATIVE, AND $M_n(\mathbf{R})$ IS NOT COMMUTATIVE.

IN THIS SEMINAR, I AM MOSTLY INTERESED IN ALGEBRAS (PART I) AND TRIPLE SYSTEMS (PART II) WHICH ARE NOT ASSOCIATIVE, ALTHOUGH THEY MAY OR MAY NOT BE COMMUTATIVE.

(ASSOCIATIVE AND COMMUTATIVE HAVE TO BE INTERPRETED APPROPRIATELY FOR THE TRIPLE SYSTEMS CONSIDERED WHICH ARE NOT ACTUALLY ALGEBRAS)

LET'S START AT THE BEGINNING

THE AXIOM WHICH CHARACTERIZES ASSOCIATIVE ALGEBRAS IS

$$a(bc) = (ab)c$$

THESE ARE CALLED
ASSOCIATIVE ALGEBRAS

THE AXIOM WHICH CHARACTERIZES COMMUTATIVE ALGEBRAS IS

$$ab = ba$$

THESE ARE CALLED (you guessed it)
COMMUTATIVE ALGEBRAS

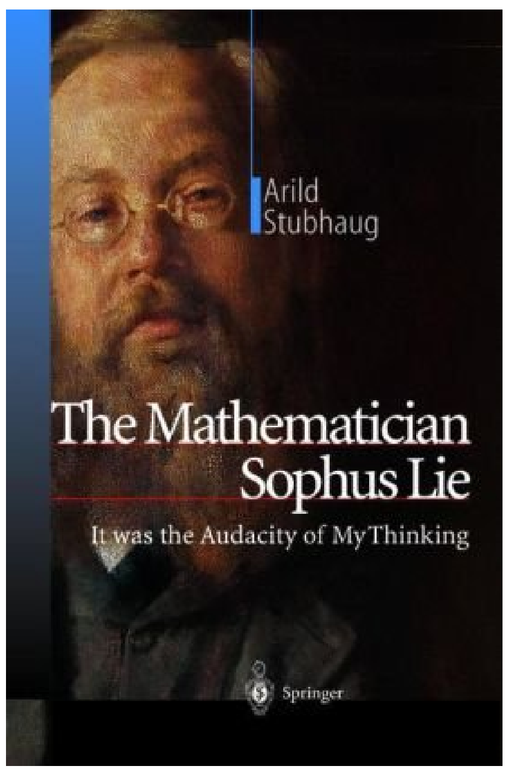
HOWEVER, THESE TWO CONCEPTS ARE TOO GENERAL TO BE OF ANY USE BY THEMSELVES

THE AXIOMS WHICH CHARACTERIZE BRACKET MULTIPLICATION ARE

$$a^2 = 0$$

$$(ab)c + (bc)a + (ca)b = 0$$

THESE ARE CALLED
LIE ALGEBRAS



Sophus Lie (1842–1899)

Marius Sophus Lie was a Norwegian mathematician. He largely created the theory of continuous symmetry, and applied it to the study of geometry and differential equations.

THE AXIOMS WHICH CHARACTERIZE CIRCLE MULTIPLICATION ARE

$$ab = ba$$

$$a(a^2b) = a^2(ab)$$

THESE ARE CALLED
JORDAN ALGEBRAS

Pascual Jordan (1902–1980)

Pascual Jordan was a German theoretical and mathematical physicist who made significant contributions to quantum mechanics and quantum field theory.

LET'S SUMMARIZE AGAIN

Table 2

ALGEBRAS

commutative algebras

$$ab = ba$$

associative algebras

$$a(bc) = (ab)c$$

Lie algebras

$$a^2 = 0$$

$$(ab)c + (bc)a + (ca)b = 0$$

Jordan algebras

$$ab = ba$$

$$a(a^2 b) = a^2(ab)$$

DERIVATIONS ON C^*-ALGEBRAS

THE ALGEBRA $M_n(\mathbf{R})$, WITH MATRIX MULTIPLICATION, AS WELL AS THE ALGEBRA $\mathcal{C}$, WITH ORDINARY MULTIPLICATION, ARE EXAMPLES OF C^*-ALGEBRAS.

THE FOLLOWING THEOREM THUS EXPLAINS THEOREM 1.

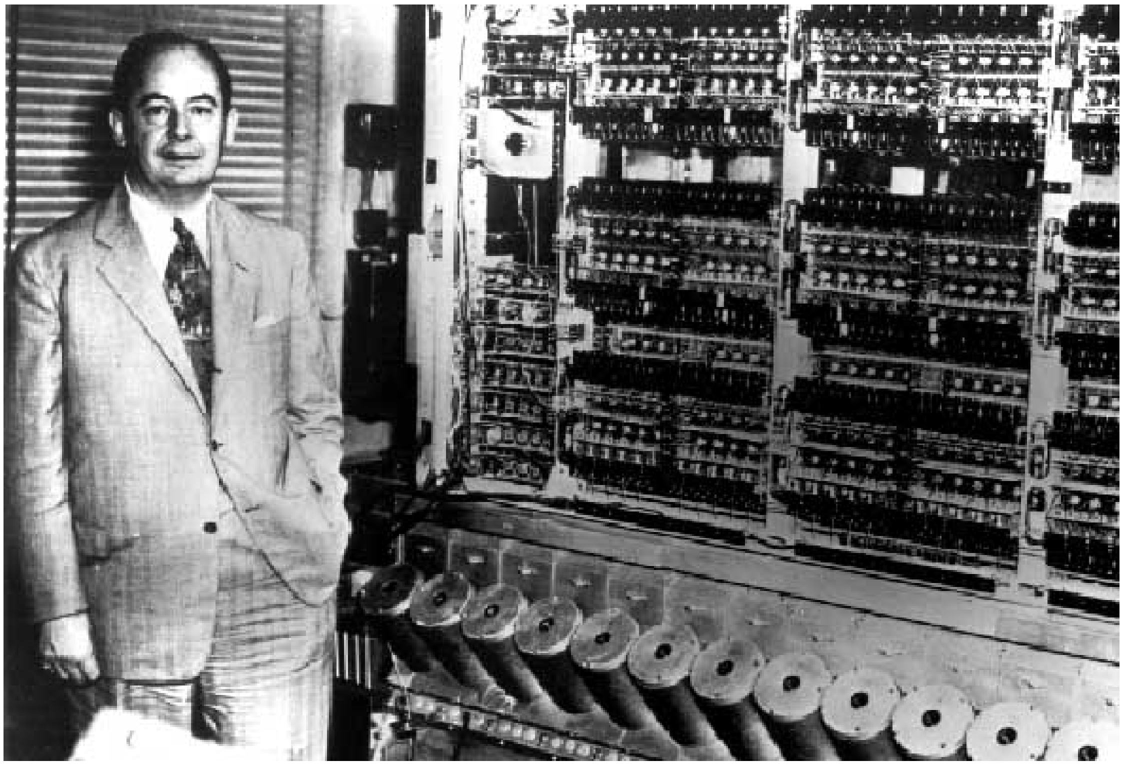
THEOREM 5 (1966-Sakai, Kadison) EVERY DERIVATION OF A C^*-ALGEBRA IS OF THE FORM $x \mapsto ax - xa$ FOR SOME a IN THE C^*-ALGEBRA

KEY POINT: C*-ALGEBRAS CAN BE INFINITE DIMENSIONAL!

Richard Kadison (b. 1925)

Richard V. Kadison is an American mathematician known for his contributions to the study of operator algebras.

John von Neumann (1903–1957)



John von Neumann was a Hungarian American mathematician who made major contributions to a vast range of fields, including set theory, functional analysis, quantum mechanics, ergodic theory, continuous geometry, economics and game theory, computer science, numerical analysis, hydrodynamics, and statistics, as well as many other mathematical fields. He is generally regarded as one of the greatest mathematicians in modern history

AUTOMATIC CONTINUITY REVISITED

THEOREM 6
(SAKAI 1960)

EVERY DERIVATION OF A C*-ALGEBRA IS CONTINUOUS

DERIVATIONS INTO A MODULE: COHOMOLOGY THEORY

THEOREM 7
(RINGROSE 1972)

EVERY DERIVATION OF A C*-ALGEBRA INTO A MODULE IS CONTINUOUS

WHAT IS A MODULE ANYWAY?
WHAT IS COHOMOLOGY?

John Ringrose (b. 1932)

John Ringrose is a leading world expert on non-self-adjoint operators and operator algebras. He has written a number of influential texts including Compact non-self-adjoint operators (1971) and, with R V Kadison, Fundamentals of the theory of operator algebras in four volumes published in 1983, 1986, 1991 and 1992.