DERIVATIONS

Introduction to non-associative algebra

OR

Playing havoc with the product rule?

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QUICK REVIEW OF ALGEBRAS
(Meetings 1,2,3)

AXIOMATIC APPROACH

AN ALGEBRA IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH TWO BINARY OPERATIONS, CALLED ADDITION AND MULTIPLICATION

ADDITION IS DENOTED BY

$$a + b$$

AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE

$$a + b = b + a, \quad (a + b) + c = a + (b + c)$$

MULTIPLICATION IS DENOTED BY

$$ab$$

AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION

$$(a+b)c = ac + bc, \quad a(b+c) = ab + ac$$

AN ALGEBRA IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)

(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

Table 2

ALGEBRAS

commutative algebras

$$ab = ba$$

associative algebras

$$a(bc) = (ab)c$$

Lie algebras

$$a^2 = 0$$

$$(ab)c + (bc)a + (ca)b = 0$$

Jordan algebras

$$ab = ba$$

$$a(a^2b) = a^2(ab)$$

DERIVATIONS ON THE SET OF MATRICES

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER

MATRIX ADDITION

$$A + B$$

AND

MATRIX MULTIPLICATION

$$A \times B$$

WHICH IS ASSOCIATIVE BUT NOT COMMUTATIVE.

DEFINITION 2

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \times B) = \delta(A) \times B + A \times \delta(B)$$

.

PROPOSITION 2

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION (WHICH CAN BE NON-ZERO)

THEOREM 2
(1942 Hochschild)

EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

Gerhard Hochschild (1915–2010)

(Photo 1968)
Gerhard Paul Hochschild was an American mathematician who worked on Lie groups, algebraic groups, homological algebra and algebraic number theory.

THE BRACKET PRODUCT ON THE SET OF MATRICES

THE BRACKET PRODUCT ON THE SET $M_n(\mathbf{R})$ OF MATRICES IS DEFINED BY

$$[X, Y] = X \times Y - Y \times X$$

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND BRACKET MULTIPLICATION, WHICH IS NOT ASSOCIATIVE AND NOT COMMUTATIVE.

DEFINITION 3

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO BRACKET MULTIPLICATION

IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$$

.

PROPOSITION 3

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = [A, X] = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION

THEOREM 3

(1942 Hochschild, Zassenhaus)
EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

Hans Zassenhaus (1912–1991)

Hans Julius Zassenhaus was a German mathematician, known for work in many parts of abstract algebra, and as a pioneer of computer algebra.

THE CIRCLE PRODUCT ON THE SET OF MATRICES

THE CIRCLE PRODUCT ON THE SET $M_n(\mathbf{R})$ OF MATRICES IS DEFINED BY

$$X \circ Y = (X \times Y + Y \times X)/2$$

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND CIRCLE MULTIPLICATION, WHICH IS COMMUTATIVE BUT NOT ASSOCIATIVE.

DEFINITION 4

A <u>DERIVATION</u> ON $M_n(\mathbf{R})$ WITH <u>RESPECT TO CIRCLE MULTIPLICATION</u>

IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$

PROPOSITION 4

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION

THEOREM 4

(1972-Sinclair)

EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

REMARK

(1937-Jacobson)

THE ABOVE PROPOSITION AND THEOREM NEED TO BE MODIFIED FOR THE SUBALGEBRA (WITH RESPECT TO CIRCLE MULTIPLICATION) OF SYMMETRIC MATRICES.

Alan M. Sinclair (retired)

Nathan Jacobson (1910–1999)

Nathan Jacobson was an American mathematician who was recognized as one of the leading algebraists of his generation, and he was also famous for writing more than a dozen standard monographs.

Table 1

$M_n(\mathbf{R})$ (ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a, b] = ab - ba$	$a \circ b = ab + ba$
Th. 2	Th.3	Th.4
$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$

END OF REVIEW OF ALGEBRAS

TRIPLE SYSTEMS

IN THIS SEMINAR SO FAR, I WAS MOSTLY INTERESTED IN **NONASSOCIATIVE ALGEBRAS**

WE SHALL NOW STUDY ASSOCIATIVE AND NONASSOCIATIVE **TRIPLE SYSTEMS**

(ASSOCIATIVE AND COMMUTATIVE HAVE TO BE INTERPRETED APPROPRIATELY FOR THE TRIPLE SYSTEMS CONSIDERED WHICH ARE NOT ACTUALLY ALGEBRAS)

DERIVATIONS ON RECTANGULAR MATRICES

MULTIPLICATION DOES NOT MAKE SENSE ON $M_{m,n}(\mathbf{R})$ if $m \neq n$.

NOT TO WORRY!

WE CAN FORM A TRIPLE PRODUCT

$$X \times Y^t \times Z$$

(TRIPLE MATRIX MULTIPLICATION)

COMMUTATIVE AND ASSOCIATIVE DON'T MAKE SENSE HERE. RIGHT?

WRONG!!

$$(X \times Y^t \times Z) \times A^t \times B = X \times Y^t \times (Z \times A^t \times B)$$

DEFINITION 5

A <u>DERIVATION</u> ON $M_{m,n}(\mathbf{R})$ WITH <u>RESPECT TO</u>

<u>TRIPLE MATRIX MULTIPLICATION</u>

IS A LINEAR PROCESS δ WHICH SATISFIES THE (TRIPLE) PRODUCT RULE

$$\delta(A \times B^t \times C) =$$
$$\delta(A) \times B^t \times C + A \times \delta(B)^t \times C + A \times B^t \times \delta(C)$$

PROPOSITION 5

FOR TWO MATRICES A, B in $M_{m,n}(\mathbf{R})$,

DEFINE $\delta_{A,B}(X) =$

$$A \times B^t \times X + X \times B^t \times A - B \times A^t \times X - X \times A^t \times B$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION

THEOREM 8

EVERY DERIVATION ON $M_{m,n}(\mathbf{R})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

REMARK

THESE RESULTS HOLD TRUE AND ARE OF INTEREST FOR THE CASE $m = n$.

TRIPLE BRACKET MULTIPLICATION

LET'S GO BACK FOR A MOMENT TO SQUARE MATRICES AND THE BRACKET MULTIPLICATION.

MOTIVATED BY THE LAST REMARK, WE DEFINE THE TRIPLE BRACKET MULTIPLICATION TO BE $[[X,Y],Z]$

DEFINITION 6

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION

IS A LINEAR PROCESS δ WHICH SATISFIES THE TRIPLE PRODUCT RULE

$$\delta([[A,B],C]) =$$
$$[[\delta(A),B],C] + [[A,\delta(B)],C] + [[A,B],\delta(C)]$$

PROPOSITION 6

FIX TWO MATRICES A, B IN $M_n(\mathbf{R})$ AND DEFINE $\delta_{A,B}(X) = [[A, B], X]$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION.

THEOREM 9

EVERY DERIVATION OF $M_n(\mathbf{R})$ WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION IS A SUM OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

TRIPLE CIRCLE MULTIPLICATION

LET'S RETURN TO RECTANGULAR MATRICES AND FORM THE TRIPLE CIRCLE MULTIPLICATION

$$(A \times B^t \times C + C \times B^t \times A)/2$$

For sanity's sake, let us write this as

$$\{A, B, C\} = (A \times B^t \times C + C \times B^t \times A)/2$$

DEFINITION 7

A <u>DERIVATION</u> ON $M_{m,n}(\mathbf{R})$ WITH <u>RESPECT TO</u> <u>TRIPLE CIRCLE MULTIPLICATION</u>

IS A LINEAR PROCESS δ WHICH SATISFIES THE TRIPLE PRODUCT RULE

$$\delta(\{A,B,C\})=$$
$$\{\delta(A), B, C\} + \{A, \delta(B), C\} + \{A, B, \delta(C)\}$$

PROPOSITION 7

FIX TWO MATRICES A, B IN $M_{m,n}(\mathbf{R})$ AND DEFINE

$$\delta_{A,B}(X) = \{A, B, X\} - \{B, A, X\}$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION.

THEOREM 10

EVERY DERIVATION OF $M_{m,n}(\mathbf{R})$ WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

IT IS TIME FOR ANOTHER SUMMARY OF THE PRECEDING

Table 3

$M_{m,n}(\mathbf{R})$ (TRIPLE SYSTEMS)

triple matrix	triple bracket	triple circle
ab^tc	$[[a,b],c]$	$ab^tc + cb^ta$
Th. 8	Th.9	Th.10
$\delta_{a,b}(x)$ $=$ ab^tx $+xb^ta$ $-ba^tx$ $-xa^tb$	$\delta_{a,b}(x)$ $=$ abx $+xba$ $-bax$ $-xab$	$\delta_{a,b}(x)$ $=$ ab^tx $+xb^ta$ $-ba^tx$ $-xa^tb$
(sums)	(sums) $(m=n)$	(sums)

LET'S PUT ALL THIS NONSENSE TOGETHER

Table 1 $M_n(\mathbf{R})$ (ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a, b] = ab - ba$	$a \circ b = ab + ba$
Th. 2	Th.3	Th.4
$\delta_a(x)$ $=$ $ax - xa$	$\delta_a(x)$ $=$ $ax - xa$	$\delta_a(x)$ $=$ $ax - xa$

Table 3 $M_{m,n}(\mathbf{R})$ (TRIPLE SYSTEMS)

triple matrix	triple bracket	triple circle
ab^tc	$[[a, b], c]$	$ab^tc + cb^ta$
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$\delta_{a,b}(x)$ $=$ ab^tx $+xb^ta$ $-ba^tx$ $-xa^tb$	$\delta_{a,b}(x)$ $=$ abx $+xba$ $-bax$ $-xab$	$\delta_{a,b}(x)$ $=$ ab^tx $+xb^ta$ $-ba^tx$ $-xa^tb$
(sums)	(sums) $(m = n)$	(sums)

HEY! IT IS NOT SO NONSENSICAL!

AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

AN TRIPLE SYSTEM IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH ONE BINARY OPERATION, CALLED ADDITION AND ONE TERNARY OPERATION CALLED TRIPLE MULTIPLICATION

ADDITION IS DENOTED BY

$$a + b$$

AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE

$$a + b = b + a, \quad (a + b) + c = a + (b + c)$$

TRIPLE MULTIPLICATION IS DENOTED

$$abc$$

AND IS REQUIRED TO BE LINEAR IN EACH VARIABLE

$$(a + b)cd = acd + bcd$$
$$a(b + c)d = abd + acd$$
$$ab(c + d) = abc + abd$$

SIMPLE BUT IMPORTANT EXAMPLES OF TRIPLE SYSTEMS CAN BE FORMED FROM ANY ALGEBRA

IF ab DENOTES THE ALGEBRA PRODUCT, JUST DEFINE A TRIPLE MULTIPLICATION TO BE $(ab)c$

LET'S SEE HOW THIS WORKS IN THE ALGEBRAS WE INTRODUCED IN PART I

$\mathcal{C}, \mathcal{D}$; $fgh = (fg)h$

$(M_n(\mathbf{R}), \times)$; $abc = a \times b \times c$ or $a \times b^t \times c$

$(M_n(\mathbf{R}), [,])$; $abc = [[a,b],c]$

$(M_n(\mathbf{R}), \circ)$; $abc = (a \circ b) \circ c$ (**NO GO!**)

A TRIPLE SYSTEM IS SAID TO BE <u>ASSOCIATIVE</u> (RESP. <u>COMMUTATIVE</u>) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)

(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

IN THE TRIPLE CONTEXT THIS MEANS THE FOLLOWING

ASSOCIATIVE

$$ab(cde) = (abc)de = a(bcd)e$$

OR $ab(cde) = (abc)de = a(dcb)e$

COMMUTATIVE: $abc = cba$

THE TRIPLE SYSTEMS $\mathcal{C}$, $\mathcal{D}$ AND $(M_n(\mathbf{R}), \times)$ ARE EXAMPLES OF ASSOCIATIVE TRIPLE SYSTEMS.

$\mathcal{C}$ AND $\mathcal{D}$ ARE EXAMPLES OF COMMUTATIVE TRIPLE SYSTEMS.

AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

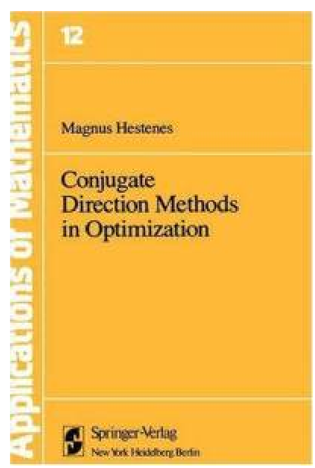
THE AXIOM WHICH CHARACTERIZES TRIPLE MATRIX MULTIPLICATION IS

$$(abc)de = ab(cde) = a(dcb)e$$

THESE ARE CALLED
ASSOCIATIVE TRIPLE SYSTEMS
or
HESTENES ALGEBRAS

Magnus Hestenes (1906–1991)

Magnus Rudolph Hestenes was an American mathematician. Together with Cornelius Lanczos and Eduard Stiefel, he invented the conjugate gradient method.



THE AXIOMS WHICH CHARACTERIZE TRIPLE BRACKET MULTIPLICATION ARE

$$aab = 0$$

$$abc + bca + cab = 0$$

$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

THESE ARE CALLED

LIE TRIPLE SYSTEMS

(NATHAN JACOBSON, MAX KOECHER)

Max Koecher (1924–1990)

Max Koecher was a German mathematician. His main research area was the theory of Jordan algebras, where he introduced the KantorKoecherTits construction.

Nathan Jacobson (1910–1999)

THE AXIOMS WHICH CHARACTERIZE TRIPLE CIRCLE MULTIPLICATION ARE

$$abc = cba$$

$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$

THESE ARE CALLED **JORDAN TRIPLE SYSTEMS**

Kurt Meyberg (living)

Ottmar Loos + Erhard Neher (both living)

YET ANOTHER SUMMARY

Table 4

TRIPLE SYSTEMS

associative triple systems

$$(abc)de = ab(cde) = a(dcb)e$$

Lie triple systems

$$aab = 0$$

$$abc + bca + cab = 0$$

$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

Jordan triple systems

$$abc = cba$$

$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$

FINAL THOUGHT

THE PHYSICAL UNIVERSE SEEMS TO BE ASSOCIATIVE.

HOW THEN, DO YOU EXPLAIN THE FOLLOWING PHENOMENON?

THEOREM 13

(1985 FRIEDMAN-RUSSO)

THE RANGE OF A CONTRACTIVE PROJECTION ON $M_n(\mathbf{R})$ (ASSOCIATIVE) IS A JORDAN TRIPLE SYSTEM (NON-ASSOCIATIVE).

Yaakov Friedman (b. 1948)

Yaakov Friedman is director of research at Jerusalem College of Technology.