DERIVATIONS

Introduction to non-associative algebra

OR

Playing havoc with the product rule?

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OUTLINE OF TODAY'S MEETING

1. SOME SET THEORY
(EQUIVALENCE CLASSES)

2. GROUPS AND THEIR QUOTIENT
GROUPS

3. FIRST COHOMOLOGY GROUP
(Review)

4. SECOND COHOMOLOGY GROUP

Note:
PARTS 1,2,4 WERE NOT DISCUSSED AT
OUR SIXTH MEETING (NOVEMBER 1)

**ONLY PARTS 3 AND 4 WERE
DISCUSSED TODAY**

PART 1 OF TODAY'S TALK

A **partition** of a set X is a disjoint class $\{X_i\}$ of non-empty subsets of X whose union is X

- $\{1, 2, 3, 4, 5\} = \{1, 3, 5\} \cup \{2, 4\}$
- $\{1, 2, 3, 4, 5\} = \{1\} \cup \{2\} \cup \{3, 5\} \cup \{4\}$
- $\mathbf{R} = \mathbf{Q} \cup (\mathbf{R} - \mathbf{Q})$
- $\mathbf{R} = \cdots \cup [-2, -1) \cup [-1, 0) \cup [0, 1) \cup \cdots$

A **binary relation** on the set X is a subset R of $X \times X$. For each ordered pair

$$(x, y) \in X \times X,$$

x is said to be related to y if $(x, y) \in R$.

- $R = \{(x, y) \in \mathbf{R} \times \mathbf{R} : x < y\}$
- $R = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y = \sin x\}$
- For a partition $X = \cup_i X_i$ of a set X, let $R = \{(x, y) \in X \times X : x, y \in X_i \text{ for some } i\}$

An **equivalence relation** on a set X is a relation $R \subset X \times X$ satisfying

reflexive $(x, x) \in R$
symmetric $(x, y) \in R \Rightarrow (y, x) \in R$
transitive $(x, y), (y, z) \in R \Rightarrow (x, z) \in R$

There is a one to one correspondence between equivalence relations on a set X and partitions of that set.

NOTATION

- If R is an equivalence relation we denote $(x, y) \in R$ by $x \sim y$.
- The equivalence class containing x is denoted by $[x]$. Thus

$$[x] = \{y \in X : x \sim y\}.$$

EXAMPLES

- equality: $R = \{(x, x) : x \in X\}$
- equivalence class of fractions
  = rational number:

  $$R = \{(\frac{a}{b}, \frac{c}{d}) : a, b, c, d \in \mathbf{Z}, b \neq 0, d \neq 0, ad = bc\}$$
- equipotent sets: X and Y are equivalent if there exists a function $f : X \to Y$ which is one to one and onto.
- half open interval of length one:
  $R = \{(x, y) \in \mathbf{R} \times \mathbf{R} : x - y$ is an integer$\}$
- integers modulo n:

$R = \{(x, y) \in \mathbf{N} \times \mathbf{N} : x - y$ is divisible by $n\}$

PART 2 OF TODAY'S TALK

A **group** is a set G together with an operation (called *multiplication*) which associates with each ordered pair x, y of elements of G a third element in G (called their *product* and written xy) in such a manner that

- multiplication is *associative*: $(xy)z = x(yz)$
- there exists an element e in G, called the *identity* element with the property that
  $$xe = ex = x \text{ for all } x$$
- to each element x, there corresponds another element in G, called the *inverse* of x and written x^{-1}, with the property that
  $$xx^{-1} = x^{-1}x = e$$

TYPES OF GROUPS

- commutative groups: $xy = yx$
- finite groups $\{g_1, g_2, \cdots, g_n\}$
- infinite groups $\{g_1, g_2, \cdots, g_n, \cdots\}$
- cyclic groups $\{e, a, a^2, a^3, \ldots\}$

EXAMPLES

1. $\mathbf{R}, +, 0, x^{-1} = -x$
2. positive real numbers, $\times, 1, x^{-1} = 1/x$
3. $\mathbf{R}^n$,vector addition,$(0, \cdots, 0)$,
   $(x_1, \cdots, x_n)^{-1} = (-x_1, \cdots, -x_n)$
4. $\mathcal{C}, +, 0, f^{-1} = -f$
5. $\{0, 1, 2, \cdots, m-1\}$, addition modulo m, 0,
   $k^{-1} = m - k$
6. permutations (=one to one onto functions), composition, identity permutation, inverse permutation
7. $M_n(\mathbf{R})$,+,0,$A^{-1} = [-a_{ij}]$
8. non-singular matrices, matrix multiplication, identity matrix, matrix inverse

Which of these are commutative, finite, infinite?

We shall consider only commutative groups and we shall denote the multiplication by $+$, the identity by 0, and inverse by -. No confusion should result.

ALERT

Counterintuitively, a very important (commutative) group is a group with one element

Let H be a subgroup of a commutative group G. That is, H is a subset of G and is a group under the same +,0,- as G.

Define an equivalence relations on G as follows: $x \sim y$ if $x - y \in H$.

The set of equivalence classes is a group under the definition of addition given by

$$[x] + [y] = [x + y].$$

This group is denoted by G/H and is called the **quotient group** of G by H.

Special cases:

$H = \{e\}$; $G/H = G$ (isomorphic)

$H = G$; $G/H = \{e\}$ (isomorphic)

EXAMPLES

1. $G = \mathbf{R}, +, 0, x^{-1} = -x$;

   $H = \mathbf{Z}$ or $H = \mathbf{Q}$

2. $\mathbf{R}^n$,vector addition,$(0, \cdots, 0)$,
   $(x_1, \cdots, x_n)^{-1} = (-x_1, \cdots, -x_n)$;

   $H = \mathbf{Z}^n$ or $H = \mathbf{Q}^n$

3. $\mathcal{C}, +, 0, f^{-1} = -f$;

   $H = \mathcal{D}$ or $H =$ polynomials

4. $M_n(\mathbf{R})$,+,0,$A^{-1} = [-a_{ij}]$;

   $H =$symmetric matrices, or
   $H =$anti-symmetric matrices

Part 3 of today's talk

(Review)

COHOMOLOGY OF ASSOCIATIVE ALGEBRAS

(FIRST COHOMOLOGY GROUP)

The basic formula of homological algebra

$$F(x_1, \ldots, x_n, x_{n+1}) =$$
$$x_1 f(x_2, \ldots, x_{n+1})$$
$$-f(x_1 x_2, x_3, \ldots, x_{n+1})$$
$$+f(x_1, x_2 x_3, x_4, \ldots, x_{n+1})$$
$$- \cdots$$
$$\pm f(x_1, x_2, \ldots, x_n x_{n+1})$$
$$\mp f(x_1, \ldots, x_n) x_{n+1}$$

OBSERVATIONS

- n is a positive integer, $n = 1, 2, \cdots$
- f is a function of n variables
- F is a function of $n+1$ variables
- $x_1, x_2, \cdots, x_{n+1}$ belong an algebra A
- $f(y_1, \ldots, y_n)$ and $F(y_1, \cdots, y_{n+1})$ also belong to A

HIERARCHY

- $x_1, x_2, \ldots, x_n$ are points (or vectors)
- f and F are functions—they take points to points
- T, defined by $T(f) = F$ is a transformation—takes functions to functions

- points $x_1, \ldots, x_{n+1}$ and $f(y_1, \ldots, y_n)$ will belong to an algebra A
- functions f will be either constant, linear or multilinear (hence so will F)
- transformation T is linear

SHORT FORM OF THE FORMULA

$$(Tf)(x_1, \ldots, x_n, x_{n+1})$$

$$= x_1 f(x_2, \ldots, x_{n+1})$$

$$+ \sum_{j=1}^{n} (-1)^j f(x_1, \ldots, x_j x_{j+1}, \ldots, x_{n+1})$$

$$+(-1)^{n+1} f(x_1, \ldots, x_n) x_{n+1}$$

FIRST CASES

$\underline{n = 0}$

If f is any constant function from A to A, say, $f(x) = b$ for all x in A, where b is a fixed element of A, we have, consistent with the basic formula,

$$T_0(f)(x_1) = x_1 b - b x_1$$

$\underline{n=1}$

If f is a linear map from A to A, then

$$T_1(f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$$

$\underline{n=2}$

If f is a bilinear map from $A \times A$ to A, then

$$T_2(f)(x_1, x_2, x_3) =$$

$$x_1 f(x_2, x_3) - f(x_1 x_2, x_3)$$

$$+ f(x_1, x_2 x_3) - f(x_1, x_2) x_3$$

Kernel and Image of a linear transformation

- $G : X \to Y$

  Since X and Y are vector spaces, they are in particular, commutative groups.

- **Kernel** of G (also called **nullspace** of G) is
  $\ker G = \{x \in X : G(x) = 0\}$

  This is a subgroup of X

- **Image** of G is
  $\operatorname{im} G = \{G(x) : x \in X\}$

  This is a subgroup of Y

What is the kernel of D on $\mathcal{D}$?

What is the image of D on $\mathcal{D}$?

(Hint: Second Fundamental theorem of calculus)

We now let $G = T_0, T_1, T_2$

$\underline{G = T_0}$

$X = A$ (the algebra)

$Y = L(A)$ (all linear transformations on A)

$$T_0(f)(x_1) = x_1 b - b x_1$$

$\ker T_0 = \{b \in A : xb - bx = 0$ for all $x \in A\}$ (center of A)

$\operatorname{im} T_0 =$ the set of all linear maps of A of the form $x \mapsto xb - bx$,

in other words, the set of all inner derivations of A

$\ker T_0$ is a subgroup of A

$\operatorname{im} T_0$ is a subgroup of $L(A)$

$\underline{G = T_1}$

$X = L(A)$ (linear transformations on A)

$Y = L^2(A)$ (bilinear transformations on $A \times A$)

$$T_1(f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$$

$\ker T_1 = \{f \in L(A) : T_1 f(x_1, x_2) =$
0 for all $x_1, x_2 \in A\}$ = the set of all
derivations of A

$\operatorname{im} T_1$ = the set of all bilinear maps of $A \times A$
of the form

$$(x_1, x_2) \mapsto x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2,$$

for some linear function $f \in L(A)$.

$\ker T_1$ is a subgroup of $L(A)$

$\operatorname{im} T_1$ is a subgroup of $L^2(A)$

$$L^0(A) \xrightarrow{T_0} L(A) \xrightarrow{T_1} L^2(A) \xrightarrow{T_2} L^3(A) \cdots$$

FACTS:

- $T_1 \circ T_0 = 0$
- $T_2 \circ T_1 = 0$
- $\cdots$
- $T_{n+1} \circ T_n = 0$
- $\cdots$

Therefore

$$\operatorname{im} T_n \subset \ker T_{n+1} \subset L^n(A)$$

and

$\operatorname{im} T_n$ is a subgroup of $\ker T_{n+1}$

- $\operatorname{im} T_0 \subset \ker T_1$

  says

  Every inner derivation is a derivation

- $\operatorname{im} T_1 \subset \ker T_2$

  says

  for every linear map f, the bilinear map F defined by

$$F(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$$

  satisfies the equation

$$x_1 F(x_2, x_3) - F(x_1 x_2, x_3) +$$

$$F(x_1, x_2 x_3) - F(x_1, x_2) x_3 = 0$$

  for every $x_1, x_2, x_3 \in A$.

The cohomology groups of A are defined as the quotient groups

$$H^n(A) = \frac{\ker T_n}{\operatorname{im} T_{n-1}}$$

$$(n = 1, 2, \ldots)$$

Thus

$$H^1(A) = \frac{\ker T_1}{\operatorname{im} T_0} = \frac{\text{derivations}}{\text{inner derivations}}$$

$$H^2(A) = \frac{\ker T_2}{\operatorname{im} T_1} = \frac{?}{?}$$

The theorem that every derivation of $M_n(\mathbf{R})$ is inner (that is, of the form δ_a for some $a \in M_n(\mathbf{R})$) can now be restated as:

"the cohomology group $H^1(M_n(\mathbf{R}))$ is the trivial one element group"

Some facts which may be discussed later on
(M is a module)

- $H^1(\mathcal{C}) = 0$, $H^2(\mathcal{C}) = 0$
- $H^1(\mathcal{C}, M) = 0$, $H^2(\mathcal{C}, M) = 0$
- $H^n(M_k(\mathbf{R}), M) = 0 \ \forall n \geq 1, k \geq 2$
- $H^n(A) = H^1(A, L(A))$ for $n \geq 2$

Cohomology groups were defined in various contexts as follows

- associative algebras (1945)
- Lie algebras (1952)
- Lie triple systems (1961,2002)
- Jordan algebras (1971)
- associative triple systems (1976)
- Jordan triple systems (1982)

Part 4 of today's meeting

(SECOND COHOMOLOGY GROUP)

$\underline{G = T_2}$

$X = L^2(A)$ (bilinear transformations on $A \times A$)

$Y = L^3(A)$ (trilinear transformations on $A \times A \times A$)

$$T_2(f)(x_1, x_2, x_3) = x_1 f(x_2, x_3) - f(x_1 x_2, x_3) + f(x_1, x_2 x_3) - f(x_1, x_2) x_3$$

$\ker T_2 = \{f \in L(A) : T_2 f(x_1, x_2, x_3) = 0$ for all $x_1, x_2, x_3 \in A\} =?$

$\operatorname{im} T_2 =$ the set of all trilinear maps h of $A \times A \times A$ of the form*

$$h(x_1, x_2, x_3) = x_1 f(x_2, x_3) - f(x_1 x_2, x_3) + f(x_1, x_2 x_3) - f(x_1, x_2) x_3,$$

for some bilinear function $f \in L^2(A)$.

$\ker T_2$ is a subgroup of $L^2(A)$

$\operatorname{im} T_2$ is a subgroup of $L^3(A)$

*we do not use $\operatorname{im} T_2$ in what follows

Homomorphisms of groups

$f : G_1 \to G_2$ is a homomorphism if

$$f(x + y) = f(x) + f(y)$$

- $f(G_1)$ is a subgroup of G_2
- $\ker f$ is a subgroup of G_1
- $G_1 / \ker f$ is isomorphic to $f(G_1)$

(isomorphism =
one to one and onto homomorphism)

Homomorphisms of algebras

$h : A_1 \to A_2$ is a homomorphism if

$$h(x + y) = h(x) + h(y)$$

and

$$h(xy) = h(x)h(y)$$

- $h(A_1)$ is a subalgebra of A_2
- $\ker h$ is a subalgebra of A_1 (actually, an ideal[†] in A_1)
- $A_1/\ker h$ is isomorphic to $h(A_1)$

(isomorphism = one to one and onto homomorphism)

[†] An **ideal** in an algebra A is a subalgebra I with the property that $AI \cup IA \subset I$, that is, $xa, ax \in I$ whenever $x \in I$ and $a \in A$

EXTENSIONS

Let A be an algebra. Let M be another algebra which contains an ideal I and let $g : M \to A$ be a homomorphism.

In symbols,

$$I \overset{\subsetneq}{\to} M \overset{g}{\to} A$$

This is called an **extension of** A **by** I if

- $\ker g = I$
- $\operatorname{im} g = A$

It follows that M/I is isomorphic to A

EXAMPLE 1

Let A be an algebra.

Define an algebra $M = A \oplus A$ to be the set $A \times A$ with addition

$$(a, x) + (b, y) = (a + b, x + y)$$

and product

$$(a, x)(b, y) = (ab, xy)$$

- $\{0\} \times A$ is an ideal in M
- $(\{0\} \times A)^2 \neq 0$
- $g : M \to A$ defined by $g(a, x) = a$ is a homomorphism
- M is an extension of $\{0\} \times A$ by A.

EXAMPLE 2

Let A be an algebra and let $h \in \ker T_2 \subset L^2(A)$.

Recall that this means that for all $x_1, x_2, x_3 \in A$,

$$x_1 f(x_2, x_3) - f(x_1 x_2, x_3)$$

$$+ f(x_1, x_2 x_3) - f(x_1, x_2) x_3 = 0$$

Define an algebra M_h to be the set $A \times A$ with addition

$$(a, x) + (b, y) = (a + b, x + y)$$

and the product

$$(a, x)(b, y) = (ab, ay + xb + h(a, b))$$

Because $h \in \ker T_2$, this algebra is **ASSOCIATIVE!** whenever A is associative.

THE PLOT THICKENS

- $\{0\} \times A$ is an ideal in M_h
- $(\{0\} \times A)^2 = 0$
- $g : M_h \to A$ defined by $g(a, x) = a$ is a homomorphism
- M_h is an extension of $\{0\} \times A$ by A.

EQUIVALENCE OF EXTENSIONS

Extensions

$$I \stackrel{\subsetneq}{\to} M \stackrel{g}{\to} A$$

and

$$I \stackrel{\subsetneq}{\to} M' \stackrel{g'}{\to} A$$

are said to be equivalent if

there is an isomorphism $\psi : M \to M'$

such that

- $\psi(x) = x$ for all $x \in I$
- $g = g' \circ \psi$

(Is this an equivalence relation?)

EXAMPLE 2—continued

Let $h_1, h_2 \in \ker T_2$.

We then have two extensions of A by $\{0\} \times A$, namely

$$\{0\} \times A \stackrel{\subsetneq}{\to} M_{h_1} \stackrel{g_1}{\to} A$$

and

$$\{0\} \times A \stackrel{\subsetneq}{\to} M_{h_2} \stackrel{g_2}{\to} A$$

Now suppose that h_1 is equivalent‡ to h_2, $h_1 - h_2 = T_1 f$ for some $f \in L(A)$

- The above two extensions are equivalent.
- We thus have a mapping from $H^2(A, A)$ into the set of equivalence classes of extensions of A by the ideal $\{0\} \times A$

‡This is the same as saying that $[h_1] = [h_2]$ as elements of $H^2(A, A) = \ker T_2 / \operatorname{im} T_1$

GRADUS AD PARNASSUM (COHOMOLOGY)

1. Verify that there is a one to one correspondence between partitions of a set X and equivalence relations on that set.
   Precisely, show that
   - If $X = \cup X_i$ is a partition of X, then $R := \{(x, y) \times X : x, y \in X_i \text{ for some } i\}$ is an equivalence relation whose equivalence classes are the subsets X_i.
   - If R is an equivalence relation on X with equivalence classes X_i, then $X = \cup X_i$ is a partition of X.

2. Verify that $T_{n+1} \circ T_n = 0$ for $n = 0, 1, 2$. Then prove it for all $n \geq 3$.

3. Show that if $f : G_1 \to G_2$ is a homomorphism of groups, then $G_1/\ker f$ is isomorphic to $f(G_1)$
   Hint: Show that the map $[x] \mapsto f(x)$ is an isomorphism of $G_1/\ker f$ onto $f(G_1)$

4. Show that if $h : A_1 \to A_2$ is a homomorphism of algebras, then $A_1/\ker h$ is isomorphic to $h(A_1)$
   Hint: Show that the map $[x] \mapsto h(x)$ is an isomorphism of $A_1/\ker h$ onto $h(A_1)$

5. Show that the algebra M_h in Example 2 is associative.
   Hint: You use the fact that A is associative AND the fact that, since $h \in \ker T_2$, $h(a,b)c + h(ab.c) = ah(b,c) + h(a,bc)$

6. Show that equivalence of extensions is actually an equivalence relation.
   Hint:
   - reflexive: $\psi : M \to M$ is the identity map
   - symmetric: replace $\psi : M \to M'$ by its inverse $\psi^{-1} : M' \to M$
   - transitive: given $\psi : M \to M'$ and $\psi' : M' \to M''$ let $\psi'' = \psi' \circ \psi : M \to M''$

7. Show that in example 2, if h_1 and h_2 are equivalent bilinear maps, that is, $h_1 - h_2 = T_1 f$ for some linear map f, then M_{h_1} and M_{h_2} are equivalent extensions of $\{0\} \times A$ by A. **Hint:** $\psi : M_{h_1} \to M_{h_2}$ is defined by
   $$\psi(a,x) = (a, x + f(a))$$