Midterm Exam II

Math 1310 - Engineering Calculus I
November 14, 2014

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

By signing below, you are acknowledging that you have read and agree to the above paragraph, as well as agree to abide University Honor Code:

Name:____________________________________

Signature:____________________________________

uID:____________________________________

Solutions

Question	Points	Score
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
Total:	105	

Note: there are a total of 105 points available on the exam, **but it will be graded out of 100**.

1. Compute derivatives of the following functions:

5 (a) $y = \dfrac{1}{(2x^4+5x+7)^3}$.

Solution: This is simply an application of the chain rule. Notice we can rewrite our equation as:

$$y = (2x^4+5x+7)^{-3}.$$

From this, we just need to use the formula: $\frac{\mathrm{d}}{\mathrm{d}x}\{[u(x)]^n\} = n[u(x)]^{n-1}u'(x)$, which in this case is:

$$\begin{aligned} y' &= -3(2x^4+5x+7)^{-4}\frac{\mathrm{d}}{\mathrm{d}x}\{2x^4+5x+7\} \\ &= -3(2x^4+5x+7)^{-4}(8x+5) = \frac{-3(8x+5)}{(2x^4+5x+7)^4}. \end{aligned}$$

5 (b) $y = x\ln\left(x^2\right)$.

Solution: The important thing to remember about this problem is the derivative of ln, which is $\frac{\mathrm{d}}{\mathrm{d}x}\ln f(x) = \frac{f'(x)}{f(x)}$. Applying this, we must use the product rule:

$$\begin{aligned} y' &= x\cdot\frac{\mathrm{d}}{\mathrm{d}x}\ln(x^2) + 1\cdot\ln(x^2) \\ &= x\frac{2x}{x^2} + \ln(x^2) \\ &= 2 + \ln(x^2). \end{aligned}$$

5 (c) $y = \cos^{-1}\left(e^{\sqrt{3x}}\right)$.

Solution: We must use the trick here of applying cos to both sides and reduce the problem to implicit differentiation. Notice:

$$y = \cos^{-1}\left(e^{\sqrt{3x}}\right)$$

is the same as:

$$\cos y = e^{\sqrt{3x}},$$

which we can now take the derivative of:

$$-\sin y\cdot y' = e^{\sqrt{3x}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sqrt{3x}\right\}$$

Notice, we have used the fact that $\frac{\mathrm{d}}{\mathrm{d}x}e^{f(x)} = e^{f(x)}f'(x)$. We compute the last derivative using the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sqrt{3x}\right\} = \frac{3}{2}(3x)^{-1/2},$$

which we can now plug in to yield:

$$-\sin y\cdot y' = e^{\sqrt{3x}}\frac{3}{2}(3x)^{-1/2},$$

which implies:

$$y' = -\frac{e^{\sqrt{3x}}\frac{3}{2}(3x)^{-1/2}}{\sin y}.$$

I didn't expect any of you to simplify, but we could notice that $y = \cos^{-1}(e^{3x})$ and therefore (by drawing a triangle) we notice that:

$$\sin y = \sin\cos^{-1} e^{\sqrt{3x}} = \sqrt{1 - e^{2\sqrt{3x}}},$$

which we can now plug in to yield:

$$y' = -\frac{e^{\sqrt{3x}}\frac{3}{2}(3x)^{-1/2}}{\sqrt{1 - e^{2\sqrt{3x}}}}.$$

10 2. (a) Use implicit differentiation to find the slope of the tangent line of the function:

$$e^{xy} = x^2 - y.$$

Solution: Here, we take the derivative of both sides of the equation. Recall again that $\frac{\mathrm{d}}{\mathrm{d}x} e^{f(x)} = e^{f(x)} f'(x)$, yielding:

$$e^{xy}\frac{\mathrm{d}}{\mathrm{d}x}\{xy\} = 2x - y'.$$

To evaluate this final derivative, we need the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{xy\} = 1 \cdot y + xy'.$$

Thus, we have:

$$e^{xy}\left(y + xy'\right) = 2x - y',$$

which suggests that, after solving for y':

$$y' = \frac{2x - e^{xy}y}{e^{xy}x + 1}.$$

5 (b) When does this function have a horizontal tangent line? A vertical tangent line? *Note:* you do not need to provide specific points, just provide a relationship between x and y.

Solution: Notice, we have y' = some fraction. When is a line horizontal? Answer: when $y' = 0$. Thus, we have a horizontal tangent when the numerator is equal to 0, or when:

$$2x - e^{xy}y = 0.$$

Similarly, we have a vertical tangent when $y' = \infty$, so when the denominator is equal to 0, or when:

$$e^{xy}x + 1 = 0.$$

3. Consider the function and corresponding interval:

$$f(x) = \cos x + \sin x, \qquad x \in [0, \pi]$$

8 (a) Compute the critical numbers of $f(x)$ that occur in the designated interval.
Hint: $\tan x = 1$ occurs at $x = \pi/4, 5\pi/4$, etc.

Solution: A critical number is where $f'(x) = 0$ or $f'(x)$ does not exist. In this case, we take the derivative to find:

$$f'(x) = -\sin x + \cos x.$$

If we set this equal to 0, we get:

$$-\sin x + \cos x = 0 \implies \frac{\sin x}{\cos x} = 1 \implies \tan x = 1.$$

Thus, the only critical number that occurs in this region is $\pi/4$.

7 (b) Calculate the global extrema (min/max) of the function $f(x)$ on the interval and state at which x values they occur.
Hint: $\sqrt{2} > 1$.

Solution: Recall that we need only check the value of $f(x)$ at the critical points and our endpoints. Thus:

$$\begin{aligned} f(0) &= \sin 0 + \cos 0 = 1 \\ f(\pi/4) &= \sin \pi/4 + \cos \pi/4 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}. \\ f(\pi) &= \sin \pi + \cos \pi = -1. \end{aligned}$$

Thus, our maximum occurs at $x = \pi/4$ and the minimum occurs at $x = \pi$.

4. Compute the following limits:

7 (a) $\lim_{x\to 0} \frac{e^{x^2} - e^x + \sin x}{5x^2}$.

Solution: If we try plugging in $x = 0$, notice that we have the indeterminant form $\frac{0}{0}$, thus we must apply L'Hôpital's rule:

$$\lim_{x\to 0} \frac{e^{x^2} - e^x + \sin x}{5x^2} = \lim_{x\to 0} \frac{\frac{d}{dx}\{e^{x^2} - e^x + \sin x\}}{\frac{d}{dx}\{5x^2\}} = \lim_{x\to 0} \frac{2xe^{x^2} - e^x + \cos x}{10x}.$$

Notice, when we plug in $x = 0$ we get $\frac{0-1+1}{0} = \frac{0}{0}$, yet another indeterminant form. We must apply L'Hô's again:

$$\lim_{x\to 0} \frac{2xe^{x^2} - e^x + \cos x}{10x} = \lim_{x\to 0} \frac{2x(2x)e^{x^2} + 2e^{x^2} - e^x - \sin x}{10}.$$

Notice now, when we plug in 0, we get:

$$\frac{0 + 2(1) - 1 - 0}{10} = \frac{1}{10},$$

a determinant limit.

8 (b) $\lim_{x\to\infty} x^{\frac{1}{x}}$.

Solution: Notice we have an indeterminant power in this case, as we have: ∞^0. From this, recall our trick that $f(x)^{g(x)} = e^{g(x)\ln f(x)}$, which yields:

$$lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x}.$$

Notice, we can now focus on $\frac{1}{x}\ln x$, as it is indeed an indeterminant product, as it boils down to $0 \times \infty$. We can split this:

$$\lim_{x\to\infty} \frac{1}{x}\ln(x) = \lim_{x\to\infty} \frac{\ln x}{x}.$$

We apply L'Hôital's:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Thus, our original limit becomes $e^0 = 1$.

15 5. After a large number of class action law suits involving exploding cans of canned corn, the aluminum company decides to revise its design to be safer . The company now uses a heavier gauge metal for the top and bottom of the can, costing \$0.004/cm^2 and a cheaper, thinner metal, costing \$0.002/cm^2 for the sides. Find the new dimensions to minimize the cost of the can with a volume of 355 mL. Recall 1 mL = 1 cm^3. **Note: you must include an argument for why this is a minimum for full credit.**

Hint: The volume and surface of a cylinder, respectively, are:

$$V = \pi r^2 h, \qquad SA = 2\pi rh + 2\pi r^2.$$

Note that you'll have to slightly modify one of these to account for the different costs.

Solution: We first modify the surface area formula to account for the cost

$$SA = (.002)2\pi rh + (.004)2\pi r^2,$$

where we make this adjustment to account for the outside of the cylinder costing \$0.002/cm^2 and the top and bottom costing \$0.004/cm^2. Thus, this is our objective equation as we want to minimize it. We must consider our constraint equation, which becomes:

$$V = 355 = \pi r^2 h,$$

which is our constraint because the volume remains fixed. Notice, as we typically do, we can eliminate a h from both equations by using our constraint, which suggests:

$$h = \frac{355}{\pi r^2}.$$

We now plug this into the surface area equation to yield:

$$SA = (.002)2\pi r\frac{355}{\pi r^2} + (.004)2\pi r^2 = \frac{.004(355)}{r} + .008\pi r^2.$$

We now take a derivative to find critical points:

$$SA' = -\frac{.004(355)}{r^2} + .016\pi r.$$

We want to set this equal to 0 to find the critical value. Notice, we can multiply both sides by r^2 to avoid dividing by r:

$$-\frac{.004(355)}{r^2} + .016\pi r = 0 = .016\pi r^3 - .004(355),$$

which suggests that:

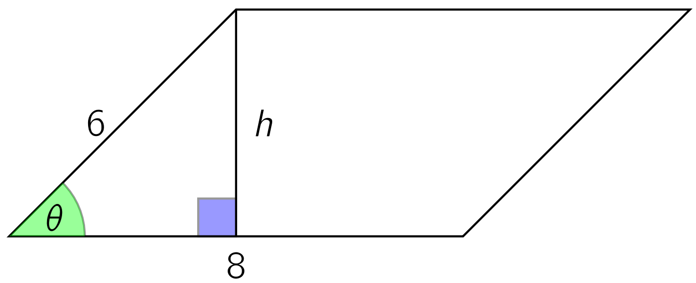
$$r = \sqrt[3]{\frac{.004(355)}{.016}} \quad \text{cm}.$$

How do we know this is a minimum? Notice we can compute the second derivative:

$$SA'' = 2\frac{.004(355)}{r^3},$$

which for any $r > 0$ (which only makes physical sense), $SA'' > 0$, meaning it is concave up, like a cup, which suggests we have a minimum. An argument about the 1st derivative was also acceptable.

15 6. Suppose that a parallelogram has sides of length 8m and 6m and that the angle θ of the bottom left corner is decreasing at a rate of 1/10 radians/second. How fast is the area of the parallelogram changing when the angle is currently at $\theta = \pi/3$?



Hint: The area of this parallelogram is $A = 8h$, but how does θ relate to h?

Solution: By the properties of right triangles, we observe that:

$$\sin\theta = \frac{h}{6} \implies h = 6\sin\theta.$$

That is, opposite to θ is h and the hypotenuse of the triangle is always fixed to be 6. Using this expression for the area, we now have:

$$A = 8h = 48\sin\theta.$$

We now want to know how this changes in time, so we take a derivative of both sides, using the chain rule:

$$\frac{dA}{dt} = 48\cos\theta\frac{d\theta}{dt}.$$

We know the rate at which the angle is changing, and we observe that $\cos(\pi/3) = 1/2$, to yield:

$$\frac{dA}{dt} = 48\cos\theta\frac{d\theta}{dt} = 48\frac{1}{2}\frac{1}{10} = 2.4 \quad \text{m}^2.$$

5 7. (a) Write down the general formula for computing the $n+1$th step of Newton's method (x_{n+1}) assuming you know the nth (x_n).

Solution: We know from class that Newton's method stems from computing the root of the tangent line, which yields:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

10 (b) For the function and initial guess, draw a rough sketch on the graph describing the steps of performing Newton's method to estimate the root:

$$f(x) = x^3 - 8, \qquad x_0 = 5.$$

Solution: As mentioned above, the geometrical technique to plotting Newton's method is to draw the tangent line, see where it intersects the x axis, and that is your new guess. The result looks something roughly like: