HW8 Solutions

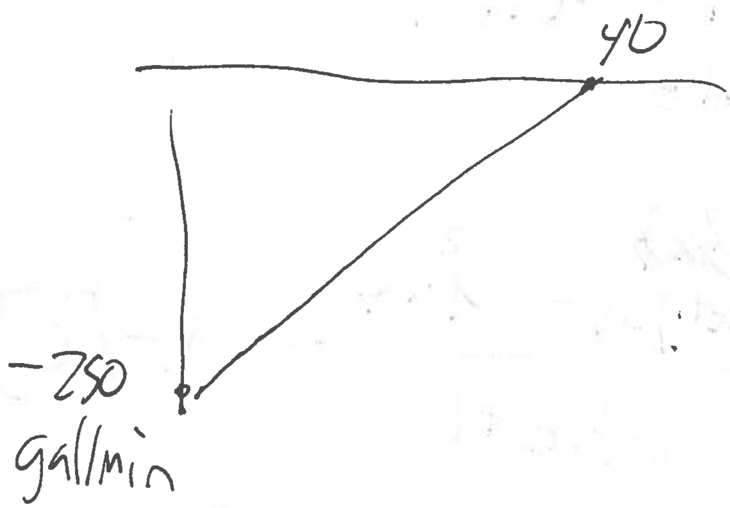
3.8] 18, 22, 34
3.9] 2, 7, 28
4.1] 4, 12, 34

3.8.18] $V = 5000\left(1 - \frac{1}{40}t\right)^2$

$$0 \le t \le 40.$$

We want rate at which tank is draining.

$$\frac{dV}{dt} = 5000 \cdot 2\left(1 - \frac{t}{40}\right) \cdot -\frac{1}{40}$$ by chain rule.



3.8.22] $[C] = a^2kt/(akt+1)$

$$\frac{d[C]}{dt} = \frac{a^2k(akt+1) - (ak)(a^2kt)}{(akt+1)^2}$$

by product rule

$$= \frac{a^2k}{(akt+1)^2}.$$

$$= k\left(\frac{a}{akt+1}\right)^2$$

but note

$$\frac{a}{akt+1} = \frac{(akt+1)a - a^2kt}{akt+1} = a - [C].$$

$$\therefore \frac{d[C]}{dt} = k(a-[C])^2.$$

c.) as $t \to \infty$

$$\lim_{t\to\infty} [C] = \lim_{t\to\infty} \frac{a^2kt}{(akt+1)} \approx \frac{a^2k}{ak} = a.$$

Notize, same power of t so

Conveniently, this makes sense.

$A + B \to C.$

starts with a molecules, how many of C should we end up with? a.

d.) $$\lim_{t\to\infty} \frac{d[C]}{dt} = \lim_{t\to\infty} \frac{a^2k}{(akt+1)^2} \to 0.$$

It reaches chemical equilibrium.

3.8.39] ~~Th~~ I really like this problem. If you're curious how this relates to infectious diseases or chemical reactions, take a mathematical biology class!

~~the book~~ This problem is deeper than it seems. What makes a population stable?

First requirement: $\frac{dP}{dt} = 0$

$$\frac{dP}{dt} = 0 = r_0\left(1 - \frac{P}{P_c}\right)P - \beta P$$

$$= P\left(r_0\left(1 - \frac{P}{P_c}\right) - \beta\right).$$

Obviously $\rho = 0$ is an equilibrium.

$$r_0\left(1-\frac{\rho}{\rho_c}\right)-\beta=0$$

$$\left(1-\frac{\rho}{\rho_c}\right)=\frac{\beta}{r_0}$$

$$\rho = -\frac{\rho_c(\beta - r_0)}{r_0}$$

$= 2000$ with plugged in #s.

Notice these are equilibria because if you are in this state, you can't leave it since $\frac{d\rho}{dt}=0$.

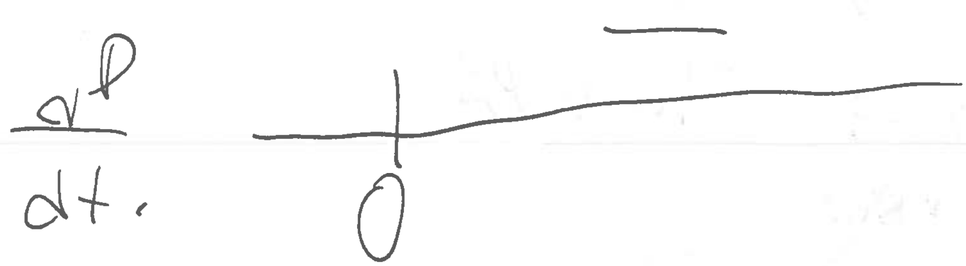
Are they stable? Consider the number line:

$\frac{dP''(t)}{dt}$:

Notice, if the population is say, 1000, $\frac{dP}{dt} > 0$, meaning the population will <u>grow</u> to 2000 and stop. If $P = 3000$, the population will shrink until 2000 and stop.

For this reason, $P = 2000$ is a <u>stable</u> equilibrium and $P = 0$ is an <u>unstable</u> equilibrium.

If we change β to be .05, our number line now becomes



Notice, $\frac{dP}{dt} < 0$ for all P, meaning the fish die out, that is, $P = 0$ is a stable (the only, actually) equilibrium.

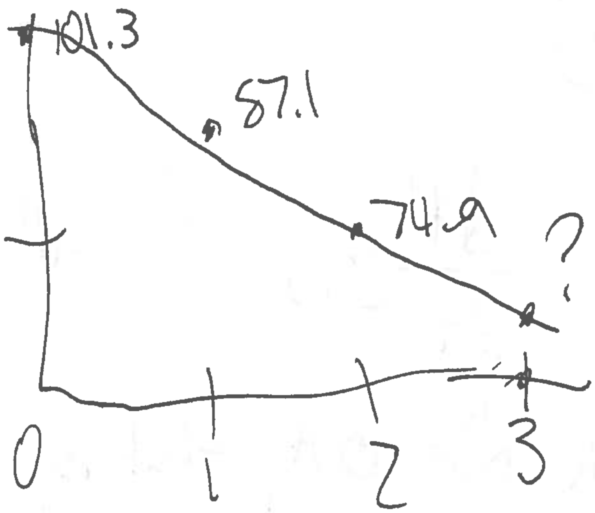
These issues will be looked at much more deeply in your differential equations class. I just wanted to give you a taste.

3.9.2)

$h=0$ $\rightarrow$ 101.3 kPA.

$h=1$ $\rightarrow$ 87.1 kPA

$h=2$ $\rightarrow$ 74.9 kPA



$$f(3) \approx f(2) + f'(2)(3-2)$$

$$f'(2) \approx \frac{74.9-87.1}{2-1} = -12.2$$

$\therefore f(3) \approx 62.7$ kPA,

Just an estimate.

3.9.26] $y=\sqrt{x}$

$dy = f'(x)\cdot dx$

$f'(x) = \frac{1}{2}x^{-1/2}$

$dy = \frac{dx}{2\sqrt{x}}$

$dx = 1$ $\quad dy = \frac{1}{2\sqrt{1}} = \frac{1}{2}$,

$\Delta y = \sqrt{2} - \sqrt{1} = \approx .414$

$\uparrow$

$f(x+\Delta x) - f(x)$

3.9.28] $r = 28cm$, error $= .2cm$.

$A = \pi r^2$

$dA = \pi \cdot 2r \cdot dr$.

at $r = 24$

$$A = \pi(24)^2$$

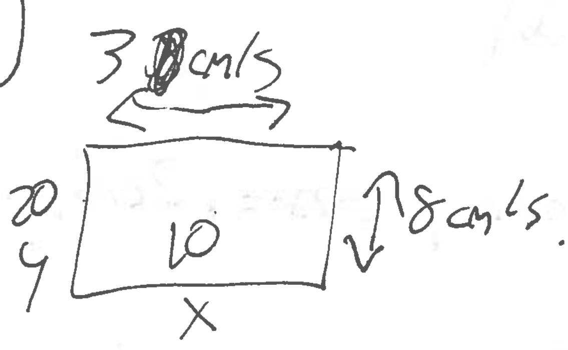
$$dA = 2\pi(24)\cdot(.02)$$

$$\approx .96\pi \approx 3.01 \text{ cm}^2 \text{ possible error.}$$

Relative: $\dfrac{dA}{A} = \dfrac{3.01}{\pi(24)^2} \approx .0016$

$\approx .16\%$ error.

4.1.4

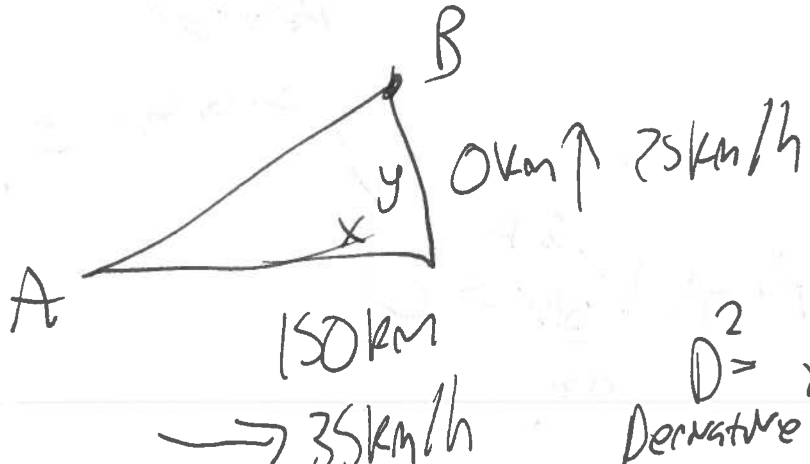


$$A = x\cdot y$$

$x = 10$, $y = 20$, $\dfrac{dy}{dt} = 8$, $\dfrac{dx}{dt} = 3$

$$\frac{dA}{dt} = \frac{dx}{dt}\cdot y + \frac{dy}{dt}\cdot x = 140 \text{ cm}^2/\text{s}$$

4.1.12)



$\frac{dx}{dt} = -35$

$\frac{dy}{dt} = 25.$

At 4 hours:

$x = 150$
$(-35)\cdot 4$
$= 10$ km

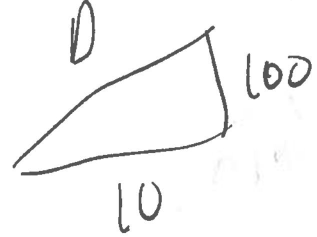
$y = 0 + (25 \cdot 4)$
$= 100$ km

$$D^2 = x^2 + y^2$$

Derivative:

$$2D \cdot \frac{dD}{dt} = 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}.$$

at $t = 4$ hours:



$D = \sqrt{100^2 + 10^2}$

$= 10\sqrt{101}$

$$\frac{dD}{dt} = \frac{2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}}{2D}.$$

$$= \frac{215}{\sqrt{10\cancel{0}1}} \approx 21.\cancel{9}39 \text{ km/h}.$$

4.1.34) $PV^{1.4} = C.$

$V = 400, \quad P = 80, \quad \frac{dP}{dt} = -10.$

what is $\frac{dV}{dt}$?

Take derivative,

<u>Product rule!</u>

Deriv of constant = 0

$$\frac{dP}{dt}\cdot V^{1.4} + P\cdot\frac{1}{4}V^{0.4}\cdot\frac{dV}{dt} = 0$$

$$\frac{dV}{dt} = \frac{-\frac{dP}{dt}\cdot V^{1.4}}{P\cdot\frac{1}{4}\cdot V^{0.4}}$$

$$= 200 \text{ cm}^3/\text{min}$$