Math 1310 – Midterm 2 Outline

Yet again, this outline is not meant to be an exhaustive list of topics for you to know. It is meant as a baseline study guide to give you a structured start to your studying, not a study replacement.

Look over these list of topics and identify your own weaknesses. In class on Wednesday, we can go over whatever topics you desire. Also, Noah will have a review on lab in Thursday. As always, my office is always open (and encouraged) for questions as well. This exam is not meant to cover topics on the previous exam, but some topics (like the product rule, for instance) are unavoidable and will show up on this exam.

At the end of the outline, there are a number of example questions that are roughly the ballpark of difficulty of problems on the exam. I won't provide answers for these problems. Try them yourself as study practice.

Chapter 3: Computing derivatives

3.3: Derivatives of trig functions

- know the derivatives of $\sin, \cos, \tan$.
- also know derivatives of $\sec, \csc, \cot$.
- knowledge of proofs not necessary.
- table on page 194 in book summarizes all of these

3.4: Chain rule

- Chain rule states: if $F(x) = f(g(x))$, then $F'(x) = f'(g(x))\, g'(x)$
- equivalently: if $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
- in words: derivative of the outside evaluated at the inside, times the derivative of the inside
- know how to take the derivative of trig functions with functions inside, like $\sin(x^2)$
- know the derivative of e^u where u is a function of x, like e^{x^2}
- know the derivative of a^x for constant a
- if we have a a parametric curve $x = f(t), y = f(t)$, know the equation for the tangent slope dy/dx

3.5: Implicit differentiation

- know how to use implicit differentiation to find dy/dx (or y') when we have an implcitly defined function
- example: find y' if $x^3 + y^3 = xy$. Note where y' terms come from.
- keep in mind product rule, other rules like in above example

3.6: Derivatives of inverse trig functions

- know derivatives of inverse trig functions
- alternatively, they are easy to derive so you don't need to memorize
- technique for deriving: we want y' with $\sin^{-1} x = y$, which is the same as $x = \sin y$, now use implicit differentiation. **be able to do this for all trig functions!**
- know how to draw triangles to evaluate things such as $\tan(\sin^{-1} \frac{2}{3})$
- know the limit definition of e and understand how to derive it.

3.7: Derivatives of logs

- similar to previous section, know the derivative of logs by considering $y = \log x$, we want y' so we can consider $e^y = x$ and use implicit differentiation.
- be able to combine this idea with the chain rule, that is compute the derivative of $\ln g(x)$. even more generally, we considered the derivative of $\ln |g(x)|$
- be able to use logs to differentiate gross products/quotients such as $\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$

3.8: Derivatives in science/engineering

- understand that the derivative is fundamentally a rate of change

- this change does NOT need to be in time
- for example, density describes a rate of change in the mass per length
- practice the ability to read word problems and translate into equations

3.9: Linear approximation and differentials

- close to a, the linearization (tangent line) of $f(x)$ very closely describes $f(x)$. Thus, we compute
- in symbols: $f(x) \approx L(x) = f(a) + f'(a)(x - a)$
- differentials are roughly the same idea: we can use the derivative to answer the question: if x changes a little bit ($\mathrm{d}x$), roughly how much does y change ($\mathrm{d}y$)?
- answer to above question: $\mathrm{d}y = f'(x)\,\mathrm{d}x$.
- classic usage of differentials: error in measuring x describes $\mathrm{d}x$, we can figure out maximum possible error in y by $\mathrm{d}y$.

Chapter 4: Applications of derivatives

4.1: Related rates

- be able to interpret classical word problems (ladders sliding, cars moving) and write them as a related rates problem
- classic structure of related rates problem: we're given some equation and a rate of change of one variable, we want the rate of change of another variable
- example, we're given the rate of change of volume of balloon, we want rate of change of radius
- review examples covered in class/lab, particularly ones where there are tricks like similar triangles

4.2: Max/min problems

- critical numbers are defined where $f'(x) = 0$ or $f'(x)$ is undefined
- local min/max occur at critical numbers, but not all critical numbers are min/max
- Extreme Value Theorem: on an interval $[a, b]$, $f(x)$ must attain a min and a max on the interval
- **for a closed interval**, to find extreme max/min we find the critical numbers and check **both** the value of $f(x)$ at the endpoints and the critical numbers to decide min/max

4.3: Derivatives/shape of curves

- Mean Value Theorem: on some interval $[a, b]$, if f is differentiable, then there exists a point c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. understand the picture and convince yourself this theorem is true.
- understand what $f' > 0$, $f' < 0$ and $f' = 0$ tell you about the shape of f
- **First Derivative Test** based on how f' changes sign at a critical number c, it is either a local min, max, or neither.
- concave up: $f'' > 0$ (like a cup), concave down: $f'' < 0$ (like a frown). be able to draw these
- **Second Derivative Test** tells you whether a critical point c is a local min or max depending on whether the function is concave up or down at c. if both $f'(c) = f''(c) = 0$, it is inconclusive.
- be able to sketch functions based on f' and f''.

4.5: L'Hôpital's rule

- Know basic indeterminant forms: $0/0, \pm\inf / \pm\inf$
- resolution: if $\lim f(x)/g(x)$ is indeterminant, L'Hôpital's rule says it is equal to $\lim f'(x)/g'(x)$.
- sometimes must be done more than once
- if we have $f(x)g(x)$ as $0 \cdot \infty$, this is indeterminant and can be written as $f/(1/g)$ or $g/(1/f)$ and L'Hôpital's can be applied
- similar idea for $f - g$ where we have $\infty - \infty$. find a common denominator and apply the theorem.
- indeterminant powers: $0^0, \infty^0, 1^\infty$. if $f(x)^{g(x)}$, then rewrite as $e^{g(x)\ln f(x)}$ and apply the product version of L'Hôpital's to exponent

4.6: Optimization

- another word problem section. review previously covered problems to understand general technique for translating word problem into symbols.
- generic formula: we are given some objective (to minimize or maximize) and a constraint.
- usually the objective has 2 variables in it, but we can eliminate one with the constraint
- once down to one variable, turns into simple min/max problem. Find critical numbers and values at the endpoints (if applicable)
- use the first derivative (usually) or second derivative test to argue whether the critical point is a min or max.

4.7: Newton's method

- We want to find roots of some function $f(x)$, that is, we want x values such that $f(x) = 0$.
- To do so, we make an initial guess x_0, we compute the tangent line and find the root of that line and that becomes our next guess x_1. repeat.
- general formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, understand how to derive this from the picture described above
- be able to draw the picture of this process!
- be familiar with cases where Newton's method runs into problems (for instance, when $f' = 0$)

4.8: Antiderivatives

- F is an antiderivative of f if $F' = f$
- If F is a particular antiderivative, so is $F + C$ for a constant C. we call this the general form.
- Easy to derive most antiderivatives. Summary of expected knowledge on page 318. Be able to derive these.
- Application: given acceleration and some starting values, be able to determine position

Practice Problems

Below I have provided a number of practice problems for your benefit. I will attempt to make the exam questions roughly the same difficulty. They are more meant to give you a concrete way to assess your understanding of the material. I will not provide solutions for these problems. Attempt them on your own and I will gladly discuss them with you.

1. Calculate y' if $x^2 \cos y + \sin 2y = xy$.
2. Calculate y' if $y = \ln\left|\frac{x^2-4}{2x+5}\right|$.
3. Calculate y' if $y = \tan^2(\sin\theta)$.
4. Calculate y' if $y = \cos(e^{\sqrt{\tan 3x}})$.
5. Calculate y' if $y = \tan^{-1}(\cos\theta)$
6. Find y'' if $x^6 + y^6 = 1$.
7. The ideal gas law can be summarized by $PV = nRT$, where n is the number of molecules of gas, $R = .0821$ is the universal gas constant, V is the volume, P is the pressure. Suppose that $P = 8$atm and is increasing at a rate .1atm/min. $V = 10$L and is decreasing at a rate .15L/min. Find the rate of change of T at the fixed amount of $n = 10$ molecules.
8. Use linear differentials to approximate $(2.001)^5$.
9. Find the linearization of $f(x) = \cos x$ around $a = \pi/2$.

10. One side of a right triangle is known to be 20 units long and the opposite angle is $30°$ with possible error $\pm 1°$. What is the possible error in computing the hypotenuse? Use differentials.

11. If a snowball melts so that its surface area decreases at a rate of $1 \text{cm}^3/\text{min}$, find the rate at which the diameter decreases when the diameter is 10 cm.

12. Two people start from the same point. One walks east at 3 mi/h and the other walks northeast at 2 mi/h. How fast is the distance between the people changing after 15 minutes?

13. Find the critical numbers of $x^3 + 3x^2 - 24x$.

14. Find the critical numbers of $x^{4/5}(x-4)^2$.

15. Find the critical numbers of $3t - \sin^{-1} t$.

16. Find the absolute min/max of $t\sqrt{4-t^2}$ on the interval $[-1, 2]$.

17. For $f(x) = x\sqrt{x+3}$, find the intervals of increasing, decreasing behavior, concave up and concave down and identify local any local minima or maxima.

18. Do the same for $x^2/(x^2-1)$. Sketch a graph of this function using this information.

19. Compute the following limit: $\lim_{x\to 0^+} (\ln x)/x$.

20. Compute the following limit: $\lim_{x\to 0^+} \cot 2x \sin 6x$.

21. Compute the following limit: $\lim_{x\to 0^+} x^{x^2}$.

22. Compute the following limit: $\lim_{x\to\infty} (xe^{1/x} - x)$.

23. Compute the following limit: $\lim_{x\to 0} (\cot x - 1/x)$.

24. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

25. A box with square base and open top must have volume $32,000$. Find the dimensions of the box that minimize the amount of material used.

26. A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

27. The illumination of an object by a light source is directly proportional (what does this mean?) to the strength of the source, s and inversely proportional to the square of the distance from the source, d.If two light sources, one three times stronger than the other are placed 10 feet apart, where should an object be placed on the line between the sources so as to receive the least illumination?

28. Compute the most general antiderivative of $5x^{1/4} - 7x^{3/4}$.

29. Compute the most general antiderivative of $\cos\theta - 5\sin\theta$.

30. Compute the most general antiderivative of $3e^x + 7\sec^2 x$.