3. Characteristic polynomial: $p(\lambda) = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5)$

Eigenvalues: $\lambda_1 = 2, \quad \lambda_2 = 5$

With $\lambda_1 = 2$: $\left.\begin{aligned} 6a - 6b &= 0 \\ 3a - 3b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 5$: $\left.\begin{aligned} 3a - 6b &= 0 \\ 3a - 6b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

4. Characteristic polynomial: $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

Eigenvalues: $\lambda_1 = 2, \quad \lambda_2 = 5$

With $\lambda_1 = 1$: $\left.\begin{aligned} 3a - 3b &= 0 \\ 2a - 2b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 2a - 3b &= 0 \\ 2a - 3b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

5. Characteristic polynomial: $p(\lambda) = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 4$

With $\lambda_1 = 1$: $\left.\begin{aligned} 9a - 9b &= 0 \\ 6a - 6b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 4$: $\left.\begin{aligned} 6a - 9b &= 0 \\ 6a - 9b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

6. Characteristic polynomial: $p(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 4$

With $\lambda_1 = 2$: $\left.\begin{aligned} 4a - 4b &= 0 \\ 3a - 3b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 3$: $\left.\begin{aligned} 3a - 4b &= 0 \\ 3a - 4b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

7. Characteristic polynomial: $p(\lambda) = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$

Eigenvalues: $\lambda_1 = 2, \quad \lambda_2 = 4$

12. Characteristic polynomial: $p(\lambda) = \lambda^2 - 7\lambda + 212 = (\lambda-3)(\lambda-4)$

Eigenvalues: $\lambda_1 = 3, \quad \lambda_2 = 4$

With $\lambda_1 = 3$: $\left.\begin{aligned} 10a - 15b &= 0 \\ 6a - 9b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

With $\lambda_2 = 4$: $\left.\begin{aligned} 9a - 15b &= 0 \\ 6a - 10b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

13. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 3\lambda^2 - 2\lambda = -\lambda(\lambda-1)(\lambda-2)$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 2$

With $\lambda_1 = 0$: $\left.\begin{aligned} 2a &= 0 \\ 2a - 2b - c &= 0 \\ -2a + 6b + 3c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

With $\lambda_2 = 1$: $\left.\begin{aligned} a &= 0 \\ 2a - 3b - c &= 0 \\ -2a + 6b + 2c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

With $\lambda_3 = 2$: $\left.\begin{aligned} 0 &= 0 \\ 2a - 4b - c &= 0 \\ -2a + 6b + c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

14. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 7\lambda^2 - 10\lambda = -\lambda(\lambda-2)(\lambda-5)$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 2, \quad \lambda_3 = 5$

With $\lambda_1 = 0$: $\left.\begin{aligned} 5a &= 0 \\ 4a - 4b - 2c &= 0 \\ -2a + 12b + 6c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 3a &= 0 \\ 4a - 6b - 2c &= 0 \\ -2a + 12b + 4c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

With $\lambda_3 = 5$: $\left.\begin{aligned} 0 &= 0 \\ 4a - 9b - 2c &= 0 \\ -2a + 12b + c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

15. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 3\lambda^2 - 2\lambda = -\lambda(\lambda-1)(\lambda-2)$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 2$

With $\lambda_1 = 0$:
$$\left.\begin{aligned} 2a-2b &= 0 \\ 2a-2b-c &= 0 \\ -2a+2b+3c &= 0 \end{aligned}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

With $\lambda_2 = 1$:
$$\left.\begin{aligned} a-2b &= 0 \\ 2a-3b-c &= 0 \\ -2a+2b+2c &= 0 \end{aligned}\right\} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

With $\lambda_3 = 2$:
$$\left.\begin{aligned} -2b &= 0 \\ 2a-4b-c &= 0 \\ -2a+2b+c &= 0 \end{aligned}\right\} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

16. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 4\lambda^2 - 3\lambda = -\lambda(\lambda-1)(\lambda-3)$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 3$

With $\lambda_1 = 0$:
$$\left.\begin{aligned} a-c &= 0 \\ -2a+3b-c &= 0 \\ -6a+6b &= 0 \end{aligned}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

With $\lambda_2 = 1$:
$$\left.\begin{aligned} -c &= 0 \\ -2a+2b-c &= 0 \\ -6a+6b-c &= 0 \end{aligned}\right\} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

With $\lambda_3 = 3$:
$$\left.\begin{aligned} -2a-c &= 0 \\ -2a-c &= 0 \\ -6a+6b-3c &= 0 \end{aligned}\right\} \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

17. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = -(\lambda-1)(\lambda-2)(\lambda-3)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$

With $\lambda_1 = 1$:
$$\left.\begin{aligned} 2a+5b-2c &= 0 \\ b &= 0 \\ 2b &= 0 \end{aligned}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

With $\lambda_4 = 2$:
$$\left.\begin{aligned} 2a-3d &= 0 \\ 0 &= 0 \\ -3c &= 0 \\ 6a-7d &= 0 \end{aligned}\right\} \qquad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

27. Characteristic polynomial: $p(\lambda) = \lambda^2 + 1$

Eigenvalues: $\lambda_1 = -i, \quad \lambda_2 = +i$

With $\lambda_1 = -i$:
$$\left.\begin{aligned} ia+b &= 0 \\ -a+ib &= 0 \end{aligned}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

With $\lambda_2 = +i$:
$$\left.\begin{aligned} -ia+b &= 0 \\ -a-ib &= 0 \end{aligned}\right\} \qquad \mathbf{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

28. Characteristic polynomial: $p(\lambda) = \lambda^2 + 36$

Eigenvalues: $\lambda_1 = -6i, \quad \lambda_2 = +6i$

With $\lambda_1 = -6i$:
$$\left.\begin{aligned} 6ia-6b &= 0 \\ 6a+6ib &= 0 \end{aligned}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

With $\lambda_2 = +6i$:
$$\left.\begin{aligned} -6ia-6b &= 0 \\ 6a-6ib &= 0 \end{aligned}\right\} \qquad \mathbf{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

29. Characteristic polynomial: $p(\lambda) = \lambda^2 + 36$

Eigenvalues: $\lambda_1 = -6i, \quad \lambda_2 = +6i$

With $\lambda_1 = -6i$:
$$\left.\begin{aligned} 6ia-3b &= 0 \\ 12a+6ib &= 0 \end{aligned}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} -i \\ 2 \end{bmatrix}$$

With $\lambda_2 = +6i$:
$$\left.\begin{aligned} -6ia-3b &= 0 \\ 12a-6ib &= 0 \end{aligned}\right\} \qquad \mathbf{v}_2 = \begin{bmatrix} i \\ 2 \end{bmatrix}$$

30. Characteristic polynomial: $p(\lambda) = \lambda^2 + 144$

Eigenvalues: $\lambda_1 = -12i, \quad \lambda_2 = +12i$

With $\lambda_1 = -12i$:
$$\left.\begin{aligned} 12ia-12b &= 0 \\ 12a+12ib &= 0 \end{aligned}\right\} \qquad \mathbf{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

With $\lambda_2 = +12i$:
$$\left.\begin{aligned} -12ia-12b &= 0 \\ 12a-12ib &= 0 \end{aligned}\right\} \qquad \mathbf{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

SECTION 6.2

DIAGONALIZATION OF MATRICES

In Problems 1–28 we first find the eigenvalues and associated eigenvectors of the given $n \times n$ matrix $\mathbf{A}$. If $\mathbf{A}$ has n linearly independent eigenvectors, then we can proceed to set up the desired diagonalizing matrix $\mathbf{P} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$ and diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$. If you write the eigenvalues in a different order on the diagonal of $\mathbf{D}$, then naturally the eigenvector columns of $\mathbf{P}$ must be rearranged in the same order.

1. Characteristic polynomial: $p(\lambda) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 3$

With $\lambda_1 = 1$: $\left.\begin{aligned} 4a - 4b &= 0 \\ 2a - 2b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 3$: $\left.\begin{aligned} 2a - 4b &= 0 \\ 2a - 4b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

2. Characteristic polynomial: $p(\lambda) = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 2$

With $\lambda_1 = 0$: $\left.\begin{aligned} 6a - 6b &= 0 \\ 4a - 4b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 4a - 6b &= 0 \\ 4a - 6b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

3. Characteristic polynomial: $p(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$

Eigenvalues: $\lambda_1 = 2, \quad \lambda_2 = 3$

With $\lambda_1 = 2$: $\left.\begin{aligned} 3a - 3b &= 0 \\ 2a - 2b &= 0 \end{aligned}\right\}$ $\qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 3$: $\left.\begin{aligned} 2a - 3b &= 0 \\ 2a - 3b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

4. Characteristic polynomial: $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 4a - 4b &= 0 \\ 3a - 3b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 3a - 4b &= 0 \\ 3a - 4b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

5. Characteristic polynomial: $p(\lambda) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 3$

With $\lambda_1 = 1$: $\left.\begin{aligned} 8a - 8b &= 0 \\ 6a - 6b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

With $\lambda_2 = 3$: $\left.\begin{aligned} 6a - 8b &= 0 \\ 6a - 8b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

6. Characteristic polynomial: $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 9a - 6b &= 0 \\ 12a - 8b &= 0 \end{aligned}\right\}$ $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 8a - 6b &= 0 \\ 12a - 9b &= 0 \end{aligned}\right\}$ $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

11. Characteristic polynomial: $p(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$

Eigenvalues: $\lambda_1 = 2, \quad \lambda_2 = 2$

With $\lambda_1 = 2$: $\left.\begin{aligned} 3a + b &= 0 \\ -9a - 3b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Because the given matrix $\mathbf{A}$ has only the single eigenvector $\mathbf{v}_1$, it is not diagonalizable.

12. Characteristic polynomial: $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$

Eigenvalues: $\lambda_1 = -1, \quad \lambda_2 = -1$

With $\lambda_1 = -1$: $\left.\begin{aligned} 12a + 9b &= 0 \\ -16a - 12b &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

Because the given matrix $\mathbf{A}$ has only the single eigenvector $\mathbf{v}_1$, it is not diagonalizable.

13. Characteristic polynomial: $p(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda - 1)(\lambda - 2)^2$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 2$

With $\lambda_1 = 1$: $\left.\begin{aligned} 3b &= 0 \\ b &= 0 \\ c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

With $\lambda_2 = 2$: $\left.\begin{aligned} 3b - a &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

The eigenspace of $\lambda_2 = 2$ is 2-dimensional. We get the eigenvector $\mathbf{v}_2$ with $b = 0,\ c = 1$, and the eigenvector $\mathbf{v}_3$ with $b = 1,\ c = 0$.

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

14. Characteristic polynomial: $p(\lambda) = -\lambda^3 + \lambda^2 = -\lambda^2(\lambda - 1)$

Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 1$

With $\lambda_1 = 0$: $\left.\begin{aligned} 2a - 2b + c &= 0 \\ 2a - 2b + c &= 0 \\ 2a - 2b + c &= 0 \end{aligned}\right\}$ $\quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

31. If $\mathbf{A}$ is similar to $\mathbf{B}$ so $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$ then $\mathbf{A}^{-1} = (\mathbf{P}^{-1}\mathbf{BP})^{-1} = \mathbf{P}^{-1}\mathbf{B}^{-1}\mathbf{P}$, so $\mathbf{A}^{-1}$ is similar to $\mathbf{B}^{-1}$.

32. If $\mathbf{A}$ is similar to $\mathbf{B}$ so $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$, then $|\mathbf{P}^{-1}||\mathbf{P}| = |\mathbf{P}^{-1}\mathbf{P}| = |\mathbf{I}| = 1$ so

$$\begin{aligned}|\mathbf{A} - \lambda\mathbf{I}| &= |\mathbf{P}^{-1}||\mathbf{A} - \lambda\mathbf{I}||\mathbf{P}| = |\mathbf{P}^{-1}(\mathbf{A} - \lambda\mathbf{I})\mathbf{P}| \\ &= |\mathbf{P}^{-1}\mathbf{AP} - \lambda\mathbf{P}^{-1}\mathbf{IP}| = |\mathbf{B} - \lambda\mathbf{I}|.\end{aligned}$$

Thus $\mathbf{A}$ and $\mathbf{B}$ have the same characteristic polynomial.

33. If $\mathbf{A}$ and $\mathbf{B}$ are similar with $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$, then

$$|\mathbf{A}| = |\mathbf{P}^{-1}\mathbf{BP}| = |\mathbf{P}^{-1}||\mathbf{B}||\mathbf{P}| = |\mathbf{P}|^{-1}|\mathbf{B}||\mathbf{P}| = |\mathbf{B}|.$$

Moreover, by Problem 32 the two matrices have the same eigenvalues, and by Problem 39 in Section 6.1, the trace of a square matrix with real eigenvalues is equal to the sum of those eigenvalues. Therefore trace $\mathbf{A}$ = (eigenvalue sum) = trace $\mathbf{B}$.

34. The characteristic equation of the 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$, and the discriminant of this quadratic equation is

$$\Delta = (a+d)^2 - 4(ad - bc) = (a-d)^2 + 4bc.$$

(a) If $\Delta > 0$, then $\mathbf{A}$ has two distinct eigenvalues and hence has two linearly independent eigenvectors, and is therefore diagonalizable.

(b) If $\Delta < 0$, then $\mathbf{A}$ has no (real) eigenvalues and hence no real eigenvectors, and therefore is not diagonalizable.

(c) Finally, note that $\Delta = 0$ for both

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

but $\mathbf{A}$ has only the single eigenvalue $\lambda = 1$ and the single eigenvector $\mathbf{v} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, and is therefore not diagonalizable.

35. Three eigenvectors associated with three distinct eigenvalues can be arranged in six different orders as the column vectors of the diagonalizing matrix $\mathbf{P} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}^T$.

CHAPTER 7

LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

SECTION 7.1

FIRST-ORDER SYSTEMS AND APPLICATIONS

1. Let $x_1 = x$ and $x_2 = x_1' = x'$, so $x_2' = x'' = -7x - 3x' + t^2$.

Equivalent system:

$x_1' = x_2$, $\quad x_2' = -7x_1 - 3x_2 + t^2$

2. Let $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2' = x''$, $x_4 = x_3' = x'''$, so $x_4' = x^{(4)} = x + 3x' - 6x'' + \cos 3t$.

Equivalent system:

$x_1' = x_2$, $\quad x_2' = x_3$, $\quad x_3' = x_4$, $\quad x_4' = -x_1 + 3x_2 - 6x_3 + \cos 3t$

3. Let $x_1 = x$ and $x_2 = x_1' = x'$, so $x_2' = x'' = \left[\left(1-t^2\right)x - tx'\right]/t^2$.

Equivalent system:

$x_1' = x_2$, $\quad t^2 x_2' = (1 - t^2)x_1 - tx_2$

4. Let $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2' = x''$, so $x_3' = x''' = \left(-5x - 3tx' - 2t^2x'' + \ln t\right)/t^3$.

Equivalent system:

$x_1' = x_2$, $\quad x_2' = x_3$, $\quad t^3 x_3' = -5x_1 - 3tx_2 + 2t^2x_3 + \ln t$

5. Let $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2' = x''$, so $x_3' = x''' = \left(x'\right)^2 + \cos x$.

Equivalent system:

$x_1' = x_2$, $\quad x_2' = x_3$, $\quad x_3' = x_2^2 + \cos x_1$

6. Let $x_1 = x$, $x_2 = x_1' = x'$, $y_1 = y$, $y_2 = y_1' = y'$ so $x_2' = x'' = 5x - 4y$, $y_2' = y'' = -4x + 5y$.

Equivalent system:

$$x_1' = x_2, \qquad x_2' = 5x_1 - 4y_1$$
$$y_1' = y_2, \qquad y_2' = -4x_1 + 5y_1$$

7. Let $x_1 = x,\ x_2 = x_1' = x',\ y_1 = y,\ y_2 = y_1' = y'$ so $x_2' = x'' = -kx/(x^2+y^2)^{3/2}$, $y_2' = y'' = -ky/(x^2+y^2)^{3/2}$.

Equivalent system:

$$x_1' = x_2, \qquad x_2' = -kx_1/\left(x_1^2 + y_1^2\right)^{3/2}$$
$$y_1' = y_2, \qquad y_2' = -ky_1/\left(x_1^2 + y_1^2\right)^{3/2}$$

8. Let $x_1 = x,\ x_2 = x_1' = x',\ y_1 = y,\ y_2 = y_1' = y'$ so $x_2' = x'' = -4x + 2y - 3x'$, $y_2' = y'' = 3x - y - 2y' + \cos t$.

Equivalent system:

$$x_1' = x_2, \qquad x_2' = -4x_1 + 2y_1 - 3x_2$$
$$y_1' = y_2, \qquad y_2' = 3x_1 - y_1 - 2y_2 + \cos t$$

9. Let $x_1 = x,\ x_2 = x_1' = x',\ y_1 = y,\ y_2 = y_1' = y',\ z_1 = z,\ z_2 = z_1' = z'$, so $x_2' = x'' = 3x - y + 2z,\ y_2' = y'' = x + y - 4z,\ z_2' = z'' = 5x - y - z.$

Equivalent system:

$$x_1' = x_2, \qquad x_2' = 3x_1 - y_1 + 2z_1$$
$$y_1' = y_2, \qquad y_2' = x_1 + y_1 - 4z_1$$
$$z_1' = z_2, \qquad z_2' = 5x_1 - y_1 - z_1$$

10. Let $x_1 = x,\ x_2 = x_1' = x',\ y_1 = y,\ y_2 = y_1' = y'$ so $x_2' = x'' = x(1-y),\ y_2' = y'' = y(1-x)$.

Equivalent system:

$$x_1' = x_2, \qquad x_2' = x_1(1 - y_1)$$
$$y_1' = y_2 \qquad y_2' = y_1(1 - x_1)$$

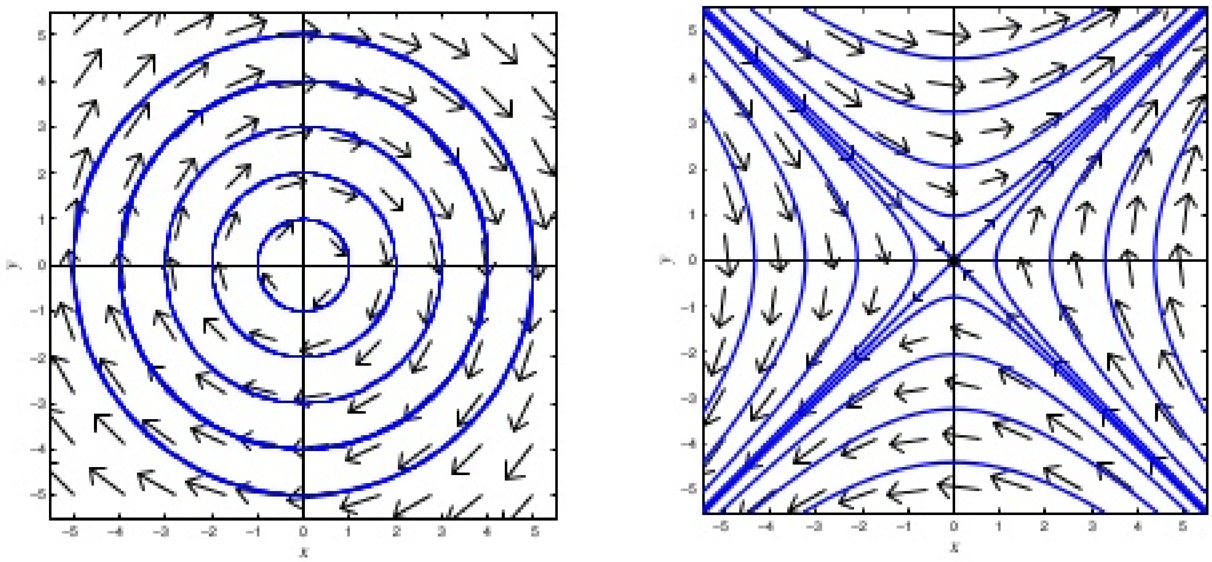
11. The computation $x'' = y' = -x$ yields the single linear second-order equation $x'' + x = 0$ with characteristic equation $r^2 + 1 = 0$ and general solution

$$x(t) = A\cos t + B\sin t.$$

Then the original first equation $y = x'$ gives

$$y(t) = B\cos t - A\sin t.$$

The figure on the left below shows a direction field and typical solution curves (obviously circles?) for the given system.



12. The computation $x'' = y' = x$ yields the single linear second-order equation $x'' - x = 0$ with characteristic equation $r^2 - 1 = 0$ and general solution

$$x(t) = Ae^t + Be^{-t}.$$

Then the original first equation $y = x'$ gives

$$y(t) = Ae^t - Be^{-t}.$$

The figure on the right above shows a direction field and some typical solution curves of this system. It appears that the typical solution curve is a branch of a hyperbola.

13. The computation $x'' = -2y' = -4x$ yields the single linear second-order equation $x'' + 4x = 0$ with characteristic equation $r^2 + 4 = 0$ and general solution

$$x(t) = A\cos 2t + B\sin 2t.$$

Then the original first equation $y = -x'/2$ gives

$$y(t) = -B\cos 2t + A\sin 2t.$$

Finally, the condition $x(0) = 1$ implies that $A = 1$, and then the condition $y(0) = 0$ gives $B = 0$. Hence the desired particular solution is given by

$$16x^2 + y^2 = C^2,$$

the equation of an ellipse with semi-axes 1 and 4.

23. When we solve Equations (20) and (21) in the text for e^{-t} and e^{2t} we get

$$2x - y = 3Ae^{-t} \text{ and } \qquad x + y = 3Be^{2t}.$$

Hence

$$(2x - y)^2(x + y) = (3Ae^{-t})^2(3Be^{2t}) = 27A^2B = C.$$

Clearly $y = 2x$ or $y = -x$ if $C = 0$, and expansion gives the equation $4x^3 - 3xy^2 + y^3 = C$.

24. Looking at Fig. 7.1.9 in the text, we see that the first spring is stretched by x_1, the second spring is stretched by $x_2 - x_1$, and the third spring is compressed by x_2. Hence Newton's second law gives $m_1 x_1'' = -k_1(x_1) + k_2(x_2 - x_1)$ and $m_2 x_2'' = -k_2(x_2 - x_1) - k_3(x_2)$.

25. Looking at Fig. 7.1.10 in the text, we see that

$$my_1'' = -T\sin\theta_1 + T\sin\theta_2 \approx -T\tan\theta_1 + T\tan\theta_2 = -Ty_1/L + T(y_2 - y_1)/L,$$

$$my_2'' = -T\sin\theta_2 - T\sin\theta_3 \approx -T\tan\theta_2 - T\tan\theta_3 = -T(y_2 - y_1)/L - Ty_2/L.$$

We get the desired equations when we multiply each of these equations by L/T and set $k = mL/T$.

26. The concentration of salt in tank i is $c_i = x_i/100$ for $i = 1, 2, 3$ and each inflow-outflow rate is $r = 10$. Hence

$$x_1' = -rc_1 + rc_3 = \tfrac{1}{10}(-x_1 + x_3),$$

$$x_2' = +rc_1 - rc_2 = \tfrac{1}{10}(x_1 - x_2),$$

$$x_3' = +rc_2 - rc_3 = \tfrac{1}{10}(x_2 - x_3).$$

27. If θ is the polar angular coordinate of the point (x, y) and we write $F = k/(x^2 + y^2) = k/r^2$, then Newton's second law gives

$$mx'' = -F\cos\theta = -(k/r^2)(x/r) = -kx/r^3,$$

$$my'' = -F\sin\theta = -(k/r^2)(y/r) = -ky/r^3,$$