3. The typical vector in W is of the form $\mathbf{x} = (x_1, 1, x_3)$ with second coordinate 1. But the particular scalar multiple $2\mathbf{x} = (2x_1, 2, 2x_3)$ of such a vector has second coordinate $2 \neq 1$, and thus is not in W. Hence W is not closed under multiplication by scalars, and therefore is not a subspace of $\mathbf{R}^3$. (Since $2\mathbf{x} = \mathbf{x} + \mathbf{x}$, W is not closed under vector addition either.)

4. The typical vector $\mathbf{x} = (x_1, x_2, x_3)$ in W has coordinate sum $x_1 + x_2 + x_3$ equal to 1. But then the particular scalar multiple $2\mathbf{x} = (2x_1, 2x_2, 2x_3)$ of such a vector has coordinate sum

$$2x_1 + 2x_2 + 2x_3 = 2(x_1 + x_2 + x_3) = 2(1) = 2 \neq 1,$$

and thus is not in W. Hence W is not closed under multiplication by scalars, and therefore is not a subspace of $\mathbf{R}^3$. (Since $2\mathbf{x} = \mathbf{x} + \mathbf{x}$, W is not closed under vector addition either.)

5. Suppose $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ are vectors in W, so

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \quad \text{and} \quad y_1 + 2y_2 + 3y_3 + 4y_4 = 0.$$

Then their sum $\mathbf{s} = \mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) = (s_1, s_2, s_3, s_4)$ satisfies the same condition

$$\begin{aligned} s_1 + 2s_2 + 3s_3 + 4s_4 &= (x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) + 4(x_4 + y_4) \\ &= (x_1 + 2x_2 + 3x_3 + 4x_4) + (y_1 + 2y_2 + 3y_3 + 4y_4) = 0 + 0 = 0, \end{aligned}$$

and thus is an element of W. Similarly, the scalar multiple $\mathbf{m} = c\mathbf{x} = (cx_1, cx_2, cx_3, cx_4) = (m_1, m_2, m_3, m_4)$ satisfies the condition

$$m_1 + 2m_2 + 3m_3 + 4m_4 = cx_1 + 2cx_2 + 3cx_3 + 4cx_4 = c(x_1 + 2x_2 + 3x_3 + 4x_4) = 0,$$

and hence is also an element of W. Therefore W is a subspace of $\mathbf{R}^4$.

6. Suppose $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ are vectors in W, so

$$x_1 = 3x_3, \quad x_2 = 4x_4 \quad \text{and} \quad y_1 = 3y_3, \quad y_2 = 4y_4.$$

Then their sum $\mathbf{s} = \mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) = (s_1, s_2, s_3, s_4)$ satisfies the same conditions

$$\begin{aligned} s_1 &= x_1 + y_1 = 3x_3 + 3y_3 = 3(x_3 + y_3) = 3s_3, \\ s_2 &= x_2 + y_2 = 4x_4 + 4y_4 = 4(x_4 + y_4) = 4s_4, \end{aligned}$$

and thus is an element of W. Similarly, the scalar multiple $\mathbf{m} = c\mathbf{x} = (cx_1, cx_2, cx_3, cx_4) = (m_1, m_2, m_3, m_4)$ satisfies the conditions

$$m_1 = cx_1 = c(3x_3) = 3(cx_3) = 3m_3, \quad m_2 = cx_2 = c(4x_4) = 4(cx_4) = 4m_4,$$

and hence is also an element of W. Therefore W is a subspace of $\mathbf{R}^4$.

7. The vectors $\mathbf{x} = (1,1)$ and $\mathbf{y} = (1,-1)$ are in W, but their sum $\mathbf{x}+\mathbf{y} = (2,0)$ is not, because $|2| \neq |0|$. Hence W is not a subspace of $\mathbf{R}^2$.

8. W is simply the zero subspace $\{\mathbf{0}\}$ of $\mathbf{R}^2$.

9. The vector $\mathbf{x} = (1,0)$ is in W, but its scalar multiple $2\mathbf{x} = (2,0)$ is not, because $(2)^2 + (0)^2 = 4 \neq 1$. Hence W is not a subspace of $\mathbf{R}^2$.

10. The vectors $\mathbf{x} = (1,0)$ and $\mathbf{y} = (0,1)$ are in W, but their sum $\mathbf{s} = \mathbf{x}+\mathbf{y} = (1,1)$ is not, because $|1| + |1| = 2 \neq 1$. Hence W is not a subspace of $\mathbf{R}^2$.

11. Suppose $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ are vectors in W, so

$$x_1 + x_2 = x_3 + x_4 \quad \text{and} \quad y_1 + y_2 = y_3 + y_4.$$

Then their sum $\mathbf{s} = \mathbf{x}+\mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) = (s_1, s_2, s_3, s_4)$ satisfies the same condition

$$\begin{aligned} s_1 + s_2 &= (x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) \\ &= (x_3 + x_4) + (y_3 + y_4) = (x_3 + y_3) + (x_4 + y_4) = s_3 + s_4 \end{aligned}$$

and thus is an element of W. Similarly, the scalar multiple $\mathbf{m} = c\mathbf{x} = (cx_1, cx_2, cx_3, cx_4) = (m_1, m_2, m_3, m_4)$ satisfies the condition

$$m_1 + m_2 = cx_1 + cx_2 = c(x_1 + x_2) = c(x_3 + x_4) = cx_3 + cx_4 = m_3 + m_4,$$

and hence is also an element of W. Therefore W is a subspace of $\mathbf{R}^4$.

12. The vectors $\mathbf{x} = (1,0,1,0)$ and $\mathbf{y} = (0,2,0,3)$ are in W (because both products are 0 in each case) but their sum $\mathbf{s} = \mathbf{x}+\mathbf{y} = (1,2,1,3)$ is not, because $s_1 s_2 = 2$ but $s_3 s_4 = 3$. Hence W is not a subspace of $\mathbf{R}^4$.

so the linear combination $a\mathbf{u} + b\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is also in W. Hence W is a subspace.

29. If $\mathbf{Ax}_0 = \mathbf{b}$ and $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$, then

$$\mathbf{Ay} = \mathbf{A}(\mathbf{x} - \mathbf{x}_0) = \mathbf{Ax} - \mathbf{Ax}_0 = \mathbf{Ax} - \mathbf{b}.$$

Hence it is clear that $\mathbf{Ay} = \mathbf{0}$ if and only if $\mathbf{Ax} = \mathbf{b}$.

30. Let W denote the intersection of the subspaces U and V. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then these two vectors are both in U and in V. Hence the linear combination $a\mathbf{u} + b\mathbf{v}$ is both in U and in V, and hence is in the intersection W, which therefore is a subspace. If U and V are non-coincident planes through the origin if $\mathbf{R}^3$, then their intersection W is a line through the origin.

31. Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be two vectors in the sum $U + V$. Then $\mathbf{w}_i = \mathbf{u}_i + \mathbf{v}_i$ where $\mathbf{u}_i$ is in U and $\mathbf{v}_i$ is in V $(i = 1, 2)$. Then the linear combination

$$a\mathbf{w}_1 + b\mathbf{w}_2 = a(\mathbf{u}_1 + \mathbf{v}_1) + b(\mathbf{u}_2 + \mathbf{v}_2) = (a\mathbf{u}_1 + \mathrm{b}\mathbf{u}_2) + (a\mathbf{v}_1 + \mathrm{b}\mathbf{v}_2)$$

is the sum of the vectors $a\mathbf{u}_1 + \mathrm{b}\mathbf{u}_2$ in U and $a\mathbf{v}_1 + \mathrm{b}\mathbf{v}_2$ in U, and therefore is an element of $U + V$. Thus $U + V$ is a subspace. If U and V are noncollinear lines through the origin in $\mathbf{R}^3$, then $U + V$ is a plane through the origin.

SECTION 4.3

LINEAR COMBINATIONS AND INDEPENDENCE OF VECTORS

In this section we use two types of computational problems as aids in understanding linear independence and dependence. The first of these problems is that of expressing a vector $\mathbf{w}$ as a linear combination of k given vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ (if possible). The second is that of determining whether k given vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent. For vectors in $\mathbf{R}^n$, each of these problems reduces to solving a linear system of n equations in k unknowns. Thus an abstract question of linear independence or dependence becomes a concrete question of whether or not a given linear system has a nontrivial solution.

1. $\mathbf{v}_2 = \frac{3}{2}\mathbf{v}_1$, so the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent.

2. Evidently the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are not scalar multiples of one another. Hence they are linearly dependent.

3. The three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent, as are any 3 vectors in $\mathbf{R}^2$. The reason is that the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ reduces to a homogeneous linear

9. $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$

$$\mathbf{A} = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 2 & 0 \\ 4 & 5 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 3 equations in 2 unknowns has the unique solution $c_1 = 2, c_2 = -3$, so $\mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2$.

10. $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & 3 \\ 1 & -2 & -1 \\ -2 & 3 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 3 equations in 2 unknowns has the unique solution $c_1 = 7, c_2 = 4$, so $\mathbf{w} = 7\mathbf{v}_1 + 4\mathbf{v}_2$.

11. $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$

$$\mathbf{A} = \begin{bmatrix} 7 & 3 & 1 \\ -6 & -3 & 0 \\ 4 & 2 & 0 \\ 5 & 3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 2 unknowns has the unique solution $c_1 = 1, c_2 = -2$, so $\mathbf{w} = \mathbf{v}_1 - 2\mathbf{v}_2$.

12. $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$

$$\mathbf{A} = \begin{bmatrix} 7 & -2 & 4 \\ 3 & -2 & -4 \\ -1 & 1 & 3 \\ 9 & -3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 2 unknowns has the unique solution $c_1 = 2, c_2 = 5$, so $\mathbf{w} = 2\mathbf{v}_1 + 5\mathbf{v}_2$.

13. $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 5 \\ 5 & -3 & 2 \\ -3 & 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$$

We see that the system of 3 equations in 3 unknowns has the unique solution $c_1 = c_2 = c_3 = 0$, so the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

18. $$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & -5 & 1 \\ -3 & -6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3/5 \\ 0 & 1 & -1/5 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 3 equations in 3 unknowns has a 1-dimensional solution space. If we choose $c_3 = 5$ then $c_1 = 3$ and $c_2 = 1$. Therefore $3\mathbf{v}_1 + \mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$.

19. $$\mathbf{A} = \begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 3 unknowns has the unique solution $c_1 = c_2 = c_3 = 0$, so the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

20. $$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 3 unknowns has the unique solution $c_1 = c_2 = c_3 = 0$, so the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

21. $$\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 3 unknowns has a 1-dimensional solution space. If we choose $c_3 = -1$ then $c_1 = 1$ and $c_2 = -2$. Therefore $\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

22. $$\mathbf{A} = \begin{bmatrix} 3 & 3 & 5 \\ 9 & 0 & 7 \\ 0 & 9 & 5 \\ 5 & -7 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 7/9 \\ 0 & 1 & 5/9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 3 unknowns has a 1-dimensional solution space. If we choose $c_3 = -9$ then $c_1 = 7$ and $c_2 = 5$. Therefore $7\mathbf{v}_1 + 5\mathbf{v}_2 - 9\mathbf{v}_3 = \mathbf{0}$.

5. The three given vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ all lie in the 2-dimensional subspace $x_1 = 0$ of $\mathbf{R}^3$. Therefore they are linearly dependent, and hence do not form a basis for $\mathbf{R}^3$.

6. $\text{Det}([\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]) = -1 \neq 0$, so the three vectors are linearly independent, and hence do form a basis for $\mathbf{R}^3$.

7. $\text{Det}([\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]) = 1 \neq 0$, so the three vectors are linearly independent, and hence do form a basis for $\mathbf{R}^3$.

8. $\text{Det}([\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4]) = 66 \neq 0$, so the four vectors are linearly independent, and hence do form a basis for $\mathbf{R}^4$.

9. The single equation $x - 2y + 5z = 0$ is already a system in reduced echelon form, with free variables y and z. With $y = s$, $z = t$, $x = 2s - 5t$ we get the solution vector

$$(x, y, z) = (2s - 5t, s, t) = s(2,1,0) + t(-5,0,1).$$

Hence the plane $x - 2y + 5z = 0$ is a 2-dimensional subspace of $\mathbf{R}^3$ with basis consisting of the vectors $\mathbf{v}_1 = (2,1,0)$ and $\mathbf{v}_2 = (-5,0,1)$.

10. The single equation $y - z = 0$ is already a system in reduced echelon form, with free variables x and z. With $x = s$, $y = z = t$ we get the solution vector

$$(x, y, z) = (s, t, t) = s(1,0,0) + t(0,1,1).$$

Hence the plane $y - z = 0$ is a 2-dimensional subspace of $\mathbf{R}^3$ with basis consisting of the vectors $\mathbf{v}_1 = (1,0,0)$ and $\mathbf{v}_2 = (0,1,1)$.

11. The line of intersection of the planes in Problems 9 and 11 is the solution space of the system

$$\begin{aligned} x - 2y + 5z &= 0 \\ y - z &= 0. \end{aligned}$$

This system is in echelon form with free variable $z = t$. With $y = t$ and $x = -3t$ we have the solution vector $(-3t, t, t) = t(-3,1,1)$. Thus the line is a 1-dimensional subspace of $\mathbf{R}^3$ with basis consisting of the vector $\mathbf{v} = (-3,1,1)$.

12. The typical vector in $\mathbf{R}^4$ of the form (a, b, c, d) with $a = b + c + d$ can be written as

$$\mathbf{v} = (b + c + d, b, c, d) = b(1,1,0,0) + c(1,0,1,0) + d(1,0,0,1).$$

5. The three given vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ all lie in the 2-dimensional subspace $x_1 = 0$ of $\mathbf{R}^3$. Therefore they are linearly dependent, and hence do not form a basis for $\mathbf{R}^3$.

6. $\operatorname{Det}([\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]) = -1 \neq 0$, so the three vectors are linearly independent, and hence do form a basis for $\mathbf{R}^3$.

7. $\operatorname{Det}([\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]) = 1 \neq 0$, so the three vectors are linearly independent, and hence do form a basis for $\mathbf{R}^3$.

8. $\operatorname{Det}([\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4]) = 66 \neq 0$, so the four vectors are linearly independent, and hence do form a basis for $\mathbf{R}^4$.

9. The single equation $x - 2y + 5z = 0$ is already a system in reduced echelon form, with free variables y and z. With $y = s,\ z = t,\ x = 2s - 5t$ we get the solution vector

$$(x, y, z) = (2s - 5t, s, t) = s(2,1,0) + t(-5,0,1).$$

Hence the plane $x - 2y + 5z = 0$ is a 2-dimensional subspace of $\mathbf{R}^3$ with basis consisting of the vectors $\mathbf{v}_1 = (2,1,0)$ and $\mathbf{v}_2 = (-5,0,1)$.

10. The single equation $y - z = 0$ is already a system in reduced echelon form, with free variables x and z. With $x = s,\ y = z = t$ we get the solution vector

$$(x, y, z) = (s, t, t) = s(1,0,0) + t(0,1,1).$$

Hence the plane $y - z = 0$ is a 2-dimensional subspace of $\mathbf{R}^3$ with basis consisting of the vectors $\mathbf{v}_1 = (1,0,0)$ and $\mathbf{v}_2 = (0,1,1)$.

11. The line of intersection of the planes in Problems 9 and 11 is the solution space of the system

$$\begin{aligned} x - 2y + 5z &= 0 \\ y - z &= 0. \end{aligned}$$

This system is in echelon form with free variable $z = t$. With $y = t$ and $x = -3t$ we have the solution vector $(-3t, t, t) = t(-3,1,1)$. Thus the line is a 1-dimensional subspace of $\mathbf{R}^3$ with basis consisting of the vector $\mathbf{v} = (-3,1,1)$.

12. The typical vector in $\mathbf{R}^4$ of the form (a, b, c, d) with $a = b + c + d$ can be written as

$$\mathbf{v} = (b + c + d, b, c, d) = b(1,1,0,0) + c(1,0,1,0) + d(1,0,0,1).$$

Thus the solution space of the given system is 2-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (-11, -3, 1, 0)$ and $\mathbf{v}_2 = (-11, -5, 0, 1)$.

18. $\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 6 & 9 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 25 \\ 0 & 0 & 1 & -5 \end{bmatrix} = \mathbf{E}$

With free variables $x_2 = s,\ x_4 = t$ and with $x_1 = -3s - 25t,\ x_3 = 5t$ we get the solution vector

$$\mathbf{x} = (-3s - 25t, s, 5t, t) = s(-3, 1, 0, 0) + t(-25, 0, 5, 1).$$

Thus the solution space of the given system is 2-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (-3, 1, 0, 0)$ and $\mathbf{v}_2 = (-25, 0, 5, 1)$.

19. $\mathbf{A} = \begin{bmatrix} 1 & -3 & -8 & -5 \\ 2 & 1 & -4 & 11 \\ 1 & 3 & 3 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$

With free variables $x_3 = s,\ x_4 = t$ and with $x_1 = 3s - 4t,\ x_2 = -2s - 3t$ we get the solution vector

$$\mathbf{x} = (3s - 4t, -2s - 3t, s, t) = s(3, -2, 1, 0) + t(-4, -3, 0, 1).$$

Thus the solution space of the given system is 2-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (3, -2, 1, 0)$ and $\mathbf{v}_2 = (-4, -3, 0, 1)$.

20. $\mathbf{A} = \begin{bmatrix} 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \\ 1 & 3 & 8 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$

With free variables $x_3 = s,\ x_4 = t$ and with $x_1 = s - 2t,\ x_2 = -3s + t$ we get the solution vector

$$\mathbf{x} = (s - 2t, -3s + t, s, t) = s(1, -3, 1, 0) + t(-2, 1, 0, 1).$$

Thus the solution space of the given system is 2-dimensional with basis consisting of the vectors $\mathbf{v}_1 = (1, -3, 1, 0)$ and $\mathbf{v}_2 = (-2, 1, 0, 1)$.

21. $\mathbf{A} = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$