Math 2250 – Midterm 1: October 2, 2015

This outline is not meant to be an end-all exhaustive study replacement, but rather a structured baseline for your studying. That is, this is a study *companion*. **If something is not on this outline, that does not mean it will not be on the exam!**

The topics covered on the exam are roughly described below. Identify which topics are your own weaknesses. In lab on Thursday, Chee Han and Huachen will review. You should come with questions if you have them. I'll also make an effort to be in my office as much as possible during the week prior to the exam. I'm particularly likely to be in my office if you see nothing on my schedule, found here: `http://www.math.utah.edu/~miles/#contact`.

I strongly recommend looking over the labs and quizzes, as well as making sure you have a good idea of how to solve all of the homework problems listed, even those not collected. Problems from all of these sources are likely to make an appearance in some form on the exam either exactly the same or modified slightly.

The exam will be doable in the 50 minute class period. You will not be allowed (nor will you need): a calculator, notes, or anything really except for a writing utensil. The intent of the exam is to establish your profiency in the ideas discussed in this class, not to test you on algebra or calculus. That said, these basic mechanics are unavoidable, so being comfortable with them is important.

Chapter 1: First-Order Differential Equations

1.1: Differential Equations, Mathematical Models

- fundamentally, what does "rate of change" mean?
- initial value problems = differential equation + initial value
- how to verify something is a solution: plug it in (to the initial value too)
- unknown constant = family of solutions, initial value "chooses" one.

1.2: Integrals as Solutions

- terminology: general vs. particular solution
- if we just have $y' = f(x)$, we can just integrate both sides
- constant acceleration, $dv/dt = a$ is a good example of this
- know how to graphically find velocity, position given graphs of acceleration

1.3: Slope Fields and Solution Curves

- we know what the slope of the solution at a point, so a line is a "local approximation"
- draw a bunch of these lines: we get a slope field
- typical procedure: find $0, \infty$ slope which partitions into regions and then check inside regions
- slope fields can tell us about long term behavior: logistic model
- we have a theorem about existence and uniqueness of solutions, know what it states exactly and what it means. I may ask you to verify that something satisfies the properties, so you must know them!
- Example discussed in class/book: $xy' = 2y$ is great to illustrate subtle issues of existence and uniqueness

1.4: Separable Equations

- if we have $y'(x) = g(x)/f(y)$, we have a separable equation and can integrate naively.
- sometimes we end up with an implicit equation. know what I mean when I request an explicit vs. implicit answer.

- Sometimes our solution is "singular", meaning we don't see it from the integration result, example in class: $y' = 6x(y-1)^{2/3}$.
- natural/exponential growth and decay: $dx/dt = kx$. Know the solution to this as well as how to derive the solution: growth rate proportional to current amount.
- Be able to interpret word problems where you're given weird information about exponential growth/decay (like a half life) and how to get parameters from this.
- Torricelli's Law: you do not need to memorize the derivation, but the result: $A(y)y' = -k\sqrt{y}$. Know what k is in terms of physical objects. Be able to solve this differential equation.
- Newton's Law of Cooling: know the equation $dx/dt = k(A-T)$ and know how to interpret parameters from word problem. Be able to solve this differential equation.

1.5: Linear First-Order Equations

- General technique for solving $y' + P(x)y = Q(x)$ is to find the integrating factor $\mu = \exp\{\int P(x)\,dx\}$ and then we're left with $\frac{1}{dx}\{\mu y\} = Q(x)\mu$ (why?) and we can integrate and maybe solve for y.
- Be able to recognize when to use this technique (or others that are listed above).
- Theorem: solutions obtained with this technique are unique.
- Tank mixture problems: classic example. be able to write down differential equation from a word problem. Solve using integrating factor.
- General idea of tank: rate of change in stuff = rate in - rate out. We're always modeling some solute (salt), not the solvent (mixture). Often rate in = concentration $\times$ flow rate

Chapter 2: Mathematical Models and Numerical Methods

2.1: Population Models

- Our generic population model $dP/Dt = (\beta - \delta)P$, where β, δ measure the number of births/deaths per person, per year. These could be functions of P or t but generically: people die and are born.
- β, δ constant: exponential growth/decay
- $\beta = \beta - \beta_1 P$ leads to logistic equation: $dP/dt = kP(M-P)$. Know this equation well.
- M is the carrying capacity, k is the growth rate. Be able to solve but also study qualitatively.
- As $t \to \infty$, population approaches M, so we call it a "limiting population".
- Can consider a new differential equation $dP/dt = kP(P-M)$, same equilibria but different behavior.
- In new equation: if $P_0 > M$, limiting behavior is $P \to \infty$, "doomsday". In other case, if $P_0 < M$, population dies out, "extinction". Now M is a "threshold" population.

2.2: Equilibria and Stability

- Big idea: sometimes qualitative analysis is more useful or easier than solving the DE
- Autonomous DE: $x'(t) = f(x)$, that is, the rate of change only depends on your current state, not explicitly on time
- critical points of f, denoted c are when $f(c) = 0$, we know then that $x(t) = c$ is an equilibrium (constant) solution to the differential equation
- equilibria can be stable or unstable. informally, this means that that if we're close to a stable equilibria, we'll go near it, and the opposite for unstable.
- semi-stable is also possible, which means it depends on which direction you are close to the equilibria and is stable/unstable from each side.
- phase diagram: tool for studying stability. if arrows point in: stable, out: unstable, both: semi-stable. we just need to check the sign of dx/dt to determine arrow direction
- good example: harvesting equation $dx/dt = kx(M-x) - h$. basically logistic equation with an additional harvesting term. For small h we get a threshold and limiting population.

- Bifurcation theory: study of (equilibria of) differential equations as a parameter changes.
- For example, what are the equilibria of $dx/dt = x(4 - x) - h$ as h changes? In this case: three possible, two, one, or zero equilibria.
- Can also draw a bifurcation diagram: on horizontal axis: parameter h, vertical axis: value of the equilibria. Be able to do this for bifurcations done in class and homework/labs.

2.3: Acceleration-Velocity Models

- Basic idea: Newton's second law mdv/dt =forces. We want to include gravity and drag forces.
- Drag force magnitude is either kv^2 or kv, but direction always has to oppose motion. Keep track of signs!
- Example: if we call our coordinate system $y > 0$ going up and we're and an object is moving up, we have $mdv/dt = -mg - kv$. We can solve this. (how?)
- Key idea: terminal velocity: $\lim_{t \to \infty} v(t)$ or when $v'(t) = 0$, that is, the forces are balanced so velocity stays constant (acceleration is zero).

2.4: Euler's Method

- Sometimes we need to numerically approximate solutions
- Most naive way: Euler's method. Basically the same idea as a slope field. We know the slope at a point, so we approximate the solution by a line.
- Know the Euler's method algorithm! We obtain $y_1, \ldots, y_n$ which are approximations of the true solution $y(x_1), \ldots, y(x_n)$.
- Local error: error introduced at each approximation, cumulative: total error accumulated up to that point of the approximation. Roundoff error: due to limitations of how computers store numbers
- Euler's method doesn't always work: you can think of it not being able to "keep up" sometimes.

2.5: A Closer Look At Euler's Method

- Theorem about the cumulative error: there exists some constant C such that $|error| \leq Ch$.
- In other words: we have a worst case for the cumulative error. We don't know C, but the win here is: we know how the maximum error scales as we change h.
- For example, if we want to reduce error by 1/10, we reduce h by 1/10.
- Improved Euler's method (or Heun's): we take an Euler step normally which predicts the next point but we can then take a correction based on averaging the Euler slope with the predicted next slope.
- Know the improved Euler algorithm!
- Big idea: this algorithm requires twice as many evaluations as $f(x)$. why would we do it? error is of order h^2 in this case. that is, if we reduce h by 1/10, max error reduces by 1/100.

2.6: Runge-Kutta method

- Idea of Runge-Kutta: we can use Simpson's rule to approximate difference in our solution from one point to another
- Once we use Simpson's rule, we use a series of prediction/corrections like in Heun's method.
- Don't memorize this actual algorithm. I'll provide it if necessary.
- Most accurate we discussed: fourth order.

Chapter 3: Linear Systems and Matrices

3.1: Intro to Linear Systems

- we want to study linear algebraic equations like $ax + by + cz = d$. when we have many: this is a system

- We say a linear system is "consistent" if has a solution (or many) and "inconsistent" if it doesn't. Nobody actually says this, I think.
- Think about geometry of 2 equations 2 unknowns: how can 2 lines intersect? Either at a point, parallel, or on top of each other. Therefore, 1, 0, or infinitely many solutions.
- General technique of solving systems: add constant multiples of current row to eliminate current variable from other equations.
- To see structure of infinitely many solutions, we "parameterize".
- Whole process is only elementary operations: multiplying row by a constant, interchanging rows, adding rows.
- Three equations: how can planes intersect? A little bit harder to visualize, but obviously possible: a point, a line.

3.2: Matrices and Gaussian Elimination

- We notice that the coefficients of our linear system are all we care about: why don't we abbreviate? Store numbers in a rectangular array, a matrix!
- $m \times n$ matrix = grid of numbers with m rows, n columns
- matrix with coefficients of left hand side of linear system = coefficient matrix **A**.
- we can add the right hand side b values to get an augmented coefficient matrix $[\mathbf{A}\ \mathbf{b}]$.
- sometimes we write $\mathbf{A} = [a_{ij}]$ which says that A is a matrix whose element in the ith row, jth column is a_{ij}.
- We perform elementary row options, the same as before. Notation: $(-2)R_1 \to R_2$ means take -2 times the first row and add it the second row (and store it there).
- Two matrices are row equivalent if we can get from one to the other via elementary row operations.
- Big idea: if two augmented coefficient matrices are row equivalent, they have the same solutions.
- Echelon matrix: basically every leading term is strictly to the right of the above leading term. Also, all zero rows have to be at the bottom.
- This is the result of our elimination procedure. Variables (columns) without a leading entry are called free variables.
- Once we get into echelon form: we solve via back substitution, that is, plugging in values from bottom to top.
- Procedure of obtaining echelon form: Gaussian elimination. Do not need a leading order 1.

3.3: Reduced Row Echelon Matrices

- We want to go a little further than just Gaussian elimination because we notice that the result of Gaussian elimination is NOT unique.
- Reduced row echelon form: echelon PLUS two new requirements: leading terms are a 1 and they're the only thing in their corresponding column.
- New procedure: Gauss-Jordan elimination to get to reduced row echelon form.
- First step: regular Gaussian elimination, then get leading order 1's and then "clear out" the leading rows.
- The result of Gauss-Jordan elimination is unique!
- From RREF (reduced row echelon form) we can see that there are still only three possibilities: infinite, one, or zero solutions.
- Structure of RREF form gives us this immediately: all diagonal: unique solution. Any free variables: infinitely many. Anything 0 =number, then no solution.
- Homogeneous system: $\mathbf{b} = 0$. Every homogeneous solution has the trivial solution $x_1 = 0, \ldots, x_n = 0$.

- Thus, we know that either every homogeneous solution has infinitely many or one solution, not zero. How do we tell? More variables than equations means infinitely many solutions.
- What if we have equal numbers of equations and variables for a homogeneous system? Depends on if equations are redundant. If we get identity matrix (only diagonal) as our RREF, solution is unique.