Name:______________________________ Quiz Score:_____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

[4] 1. Find the solution to the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}.$$

Solution: From the previous quiz, we know how to compute the eigenvalues and eigenvectors of a 2×2 matrix. In this particular case, we find we have a complex conjugate pair (of eigenvalues and eigenvectors), which are: $\lambda_{1,2} = -1 \pm 2i$ with corresponding eigenvectors:

$$\mathbf{v}_{1,2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \pm i \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \mathbf{a} \pm i\mathbf{b}$$

Although we have already established the result for how to construct the two linearly independent solutions from this, it's a little easier to derive it than memorize it (in my opinion). If we naïvely construct our solution (using either the $+$ or $-$ conjugate, it doesn't matter but we'll take the $+$):

$$\tilde{\mathbf{x}}(t) = \mathbf{v}_1 e^{\lambda_1 t} = (\mathbf{a} + \mathbf{b}i)e^{(-1+2i)t} = (\mathbf{a} + \mathbf{b}i)e^{-t}(\cos 2t + i \sin 2t) = \mathbf{p}(t) + i\mathbf{q}(t).$$

How do we extract two *real* valued solutions from this single complex valued solution? We simply just split the real and imaginary parts, as they have no influence on each other and are each individually real valued. That is,

$$\mathbf{x}_1(t) = \mathrm{Re}(\tilde{\mathbf{x}}) = \mathbf{p} = \mathbf{a}e^{-t}\cos 2t - \mathbf{b}e^{-t}\sin 2t, \qquad \mathbf{x}_2(t) = \mathrm{Im}(\tilde{\mathbf{x}}) = \mathbf{q} = \mathbf{a}e^{-t}\sin 2t + \mathbf{b}e^{-t}\cos 2t.$$

Thus, our solution is simply a linear combination of these two linearly independent basis functions:

$$\mathbf{x}(t) = c_1 e^{-t}\left[\mathbf{a}\cos 2t - \mathbf{b}\sin 2t\right] + c_2 e^{-t}\left[\mathbf{a}\sin 2t + \mathbf{b}\cos 2t\right].$$

Using the definitions of $\mathbf{a}, \mathbf{b}$, we have

$$\mathbf{x}(t) = c_1 e^{-t}\begin{bmatrix} \sin 2t \\ -\cos 2t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -\cos 2t \\ -\sin 2t \end{bmatrix}.$$

Using the initial condition at $t = 0$, we have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Thus, it's clear $c_1 = 0, c_2 = -1$, so our final solution is

$$\mathbf{x}(t) = e^{-t}\begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}.$$

2. Consider the following non-linear system

$$\dot{x} = x^2 + xy - x, \qquad \dot{y} = yx - y^2 - y.$$

3 (a) Draw the nullclines of the non-linear system and use this information to determine equilibrium solutions.

Solution: For the sake of convenience, call the functions

$$\dot{x} = x^2 + xy - x = f(x, y), \qquad \dot{y} = xy - y^2 - y = g(x, y).$$

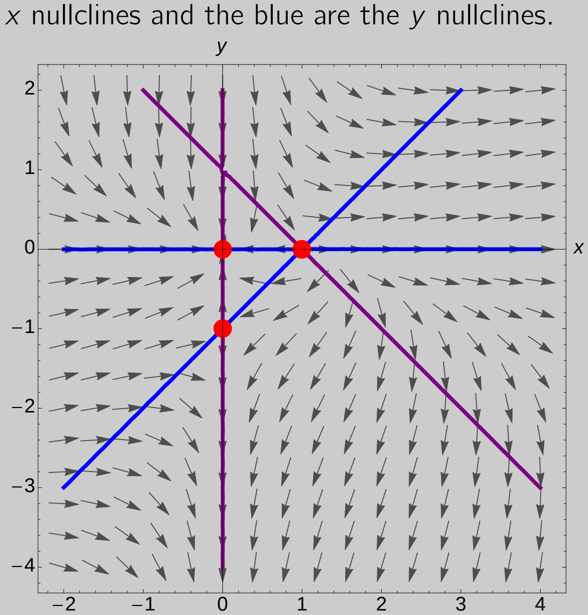
The nullclines of the system are therefore where $f(x, y) = 0$ or where $g(x, y) = 0$. We'll first look at the x nullclines:

$$f(x, y) = 0 = x^2 + xy - x = x(x + y - 1).$$

Thus, this corresponds to two components: either $x = 0$ or $x + y - 1 = 0 \implies y = 1 - x$. We can do the same for the y nullcline:

$$g(x, y) = 0 = xy - y^2 - y = y(x - y - 1),$$

which implies that either $y = 0$ or $y = x - 1$. We can plot these below (along with the slope field, although this would be a bit of a nightmare to draw by hand and is also unnecessary) where the purple lines are the x nullclines and the blue are the y nullclines.



A critical point occurs when all of the derivatives are zero, or, at the intersection of nullclines, which we see here correspond to: $(0, 0), (1, 0)$, and $(0, -1)$. Thus, there are three critical points and therefore three equilibrium solutions.

3 (b) Classify the behavior of the equilibrium solution $(x_\star, y_\star) = (0, 0)$ by linearizing the system and analyzing the Jacobian.

Solution: If we didn't have the slope field above, it would be unclear what type of equilibrium the origin is. Thus, we can use *local* analysis to determine its stability. In other words, consider perturbing slightly away from the origin: what happens? We know that this entirely boils down to the eigenvalues of the Jacobian, which is defined to be:

$$J(x_\star, y_\star) = \begin{bmatrix} \frac{\partial f}{\partial x}(x_\star, y_\star) & \frac{\partial f}{\partial y}(x_\star, y_\star) \\ \frac{\partial g}{\partial x}(x_\star, y_\star) & \frac{\partial g}{\partial y}(x_\star, y_\star) \end{bmatrix}.$$

Computing these partial derivatives:

$$\frac{\partial f}{\partial x} = 2x + y - 1, \qquad \frac{\partial f}{\partial y} = x, \qquad \frac{\partial g}{\partial x} = y, \qquad \frac{\partial g}{\partial y} = x - 2y - 1.$$

Evaluating the Jacobian here:

$$J(0,0) = \begin{bmatrix} \frac{\partial f}{\partial x}(0,0) & \frac{\partial f}{\partial y}(0,0) \\ \frac{\partial g}{\partial x}(0,0) & \frac{\partial g}{\partial y}(0,0) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Here, we don't even need to compute the eigenvalues because we can just read them off $\lambda_1 = -1, \lambda_2 = -1$. Since we have two real eigenvalues with negative parts, the solution to the linear system for the perturbation looks like decaying exponentials, meaning this is a **stable node** or a **sink**, which is also clear from the slope field by noting that trajectories tend toward the origin.