

Name: \_\_\_\_\_

Quiz Score: \_\_\_\_/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

1. Consider the differential equation with a parameter (constant)  $k$ :

$$\frac{dy}{dx} = (2 - y)y - k.$$

- 3 (a) **First, assume that  $k < 1$ .** What are the equilibria of the differential equation?

**Solution:** To find the equilibria of the system, we find the critical points of the differential equation. That is, if we have an *autonomous* (know what this means), first order differential equation:

$$\frac{dy}{dx} = f(y)$$

then if  $f(c) = 0$ , then the solution  $y(x) \equiv c$  is an equilibrium. In this particular case, we find

$$0 = (2 - y)y - k = -y^2 + 2y - k.$$

Using the quadratic equation, denoting our two roots  $y_+$  and  $y_-$  we find

$$y_{\pm} = 1 \pm \sqrt{1 - k}.$$

Note, from this form it should be clear that something exciting happens when  $k = 1$ , but here we are assuming that  $k > 0$ , meaning that  $1 - k > 0$  and therefore we have two real roots, and therefore two equilibria at these values.

- 3 (b) Draw a phase diagram and label the stability of each of the equilibria.

**Solution:**

Here, we have two equilibria,  $y_+ = 1 + \sqrt{1 - k}$  and  $y_- = 1 - \sqrt{1 - k}$ .

We can check what  $dy/dx$  is in each of the regions inbetween these equilibria. For instance, if  $y > y_+$ , then we can think of  $y$  being very very large and we have

$$\frac{dy}{dx} = \underbrace{(2 - y)}_{-} \underbrace{y}_{+} - k,$$

which is negative, so  $dy/dx < 0$ .

Similarly, if  $y < y_-$ , we can think of  $y$  as very very negative and we get

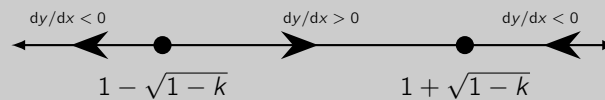
$$\frac{dy}{dx} = \underbrace{(2 - y)}_{+} \underbrace{y}_{-} - k,$$

which again produces  $dy/dx < 0$ .

The middle case is perhaps a little trickier to see, but notice that we have, for  $y_- < y < y_+$ :

$$\frac{dy}{dx} = - \underbrace{(y - y_+)}_{-} \underbrace{(y - y_-)}_{+},$$

so this says that  $dy/dx > 0$ . From this, we can produce the phase diagram which you can see below.



2 (c) **Now, consider  $k > 1$ .** What are the equilibria in this case?

**Solution:** For this part, we see that the equilibria still occur at

$$y_{\pm} = 1 \pm \sqrt{1 - k},$$

but if  $k > 0$ , then  $1 - k < 0$  and therefore  $\sqrt{1 - k}$  is imaginary. Thus, there are no real roots and the system has no equilibria.

2 (d) For this equation,  $k = 1$  is a *bifurcation point*. Draw the corresponding bifurcation diagram and interpret it.

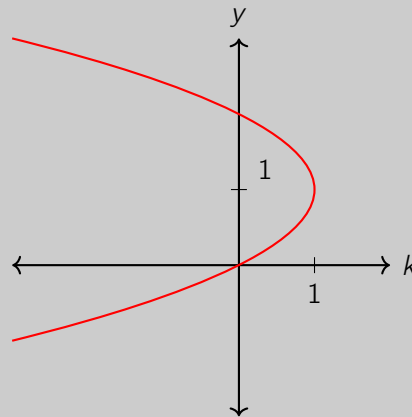
**Solution:** The whole idea of a bifurcation diagram is to plot the equilibria as a function of the parameter  $k$ . Thus, the only time we have equilibria are when  $k \geq 1$ , and in this case,

$$y = 1 \pm \sqrt{1 - k} \implies y - 1 = \pm \sqrt{1 - k},$$

and squaring both sides yields

$$(y - 1)^2 = 1 - k,$$

which is a parabola that opens left and has a vertex at  $(k, y) = (1, 1)$ , the plot of which looks like:



Note that the interpretation of the graph is exactly the behavior of our equilibria as  $k$  changes. We see that when  $k > 1$ , there are no equilibria, and then when  $k \leq 1$ , there are clearly two and the actual value of the equilibria can be read off from the graph.