

Name: _____

Quiz Score: _____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

- 7 1. Compute the eigenvalues and eigenvectors of the following matrix

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

Solution: We are looking for all eigenvalues λ and eigenvectors \mathbf{v} that satisfy the relationship

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Rearranging this, we have

$$B\mathbf{v} = (A - \lambda I)\mathbf{v} = \mathbf{0}.$$

Since we want non-trivial ($\mathbf{v} \neq \mathbf{0}$) solutions to this equation, we require $\det B = \det(A - \lambda I) = 0$. This determinant is a polynomial called the **characteristic polynomial**. The roots of the characteristic polynomial are the eigenvalues we seek!

Actually computing this:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -2 \\ 1 & 4 - \lambda \end{vmatrix} = (1 - \lambda)(4 - \lambda) - (-2) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

From this, we can conclude our two eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. We can now compute the eigenvectors corresponding to these eigenvalues.

- (1)** $\lambda_1 = 2$. We consider the relationship we started with, now with this particular value of λ :

$$(A - \lambda_1 I)\mathbf{v}_1 = \mathbf{0}.$$

Using the value of $\lambda_1 = 2$ and calling $\mathbf{v} = [a \ b]^T$, we find

$$\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We note that the resulting two equations are redundant, but this is expected since the determinant condition we imposed forces the matrix $A - \lambda I$ to be singular (and therefore produce an infinite number of solutions to the homogeneous system). Writing one of these equations (the second, out of convenience):

$$a + 2b = 0.$$

Since we have an infinite family of a, b that satisfy this, we have a ton of freedom, but note an easy choice is $b = 1$, which yields $a = -2$, thus our eigenvector corresponding to $\lambda_1 = 2$ is

$$\mathbf{v}_1 = [-2 \ 1]^T.$$

(2) $\lambda_2 = 3$. We repeat the same process but for the second eigenvalue, thus

$$(A - \lambda_2 I)\mathbf{v}_2 = \mathbf{0}$$

turns into

$$\begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The relationship here is then

$$a + b = 0 \quad \implies \quad \mathbf{v}_2 = [1 \quad -1]^T.$$

3 2. Using your answer to question 1, write out the **diagonalization** of A.

That is, write the matrix A as a product of matrices and clearly specify what these matrices are.

Solution: The diagonalization of a matrix A is of the following form

$$A = PDP^{-1},$$

where P is a matrix comprised of the eigenvectors and D is a diagonal matrix of the eigenvalues:

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad P = \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix}.$$

Using the particular values found in problem 1 (and the useful hint below to compute the inverse), we see

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, \quad P^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}.$$

If we multiplied out these three matrices, we would recover A back.

Useful reminder

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$