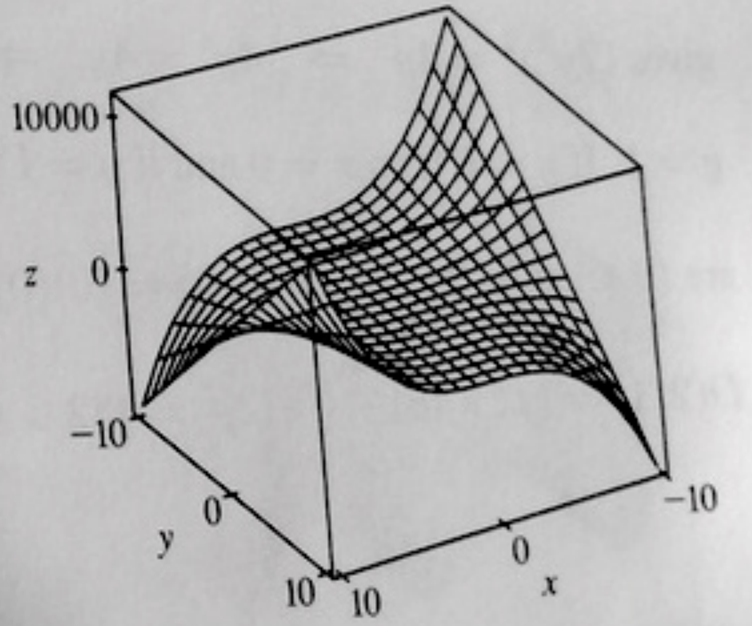
6. $f(x,y) = x^3y + 12x^2 - 8y \quad \Rightarrow \quad f_x = 3x^2y + 24x$,

$f_y = x^3 - 8$, $f_{xx} = 6xy + 24$, $f_{xy} = 3x^2$, $f_{yy} = 0$.

Then $f_y = 0$ implies $x = 2$, and substitution into $f_x = 0$ gives

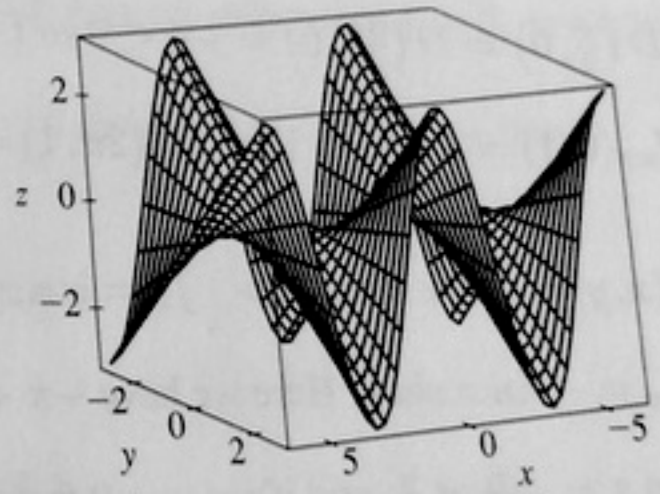
$12y + 48 = 0 \quad \Rightarrow \quad y = -4$. Thus, the only critical point is $(2, -4)$.

$D(2,-4) = (-24)(0) - 12^2 = -144 < 0$, so $(2, -4)$ is a saddle point.

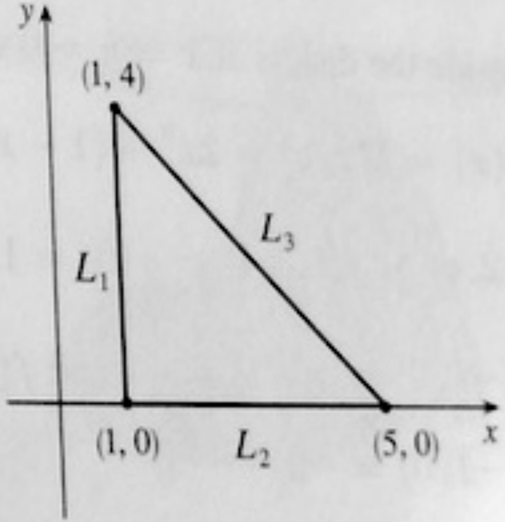


12. $f(x,y) = y\cos x \quad \Rightarrow \quad f_x = -y\sin x$, $f_y = \cos x$, $f_{xx} = -y\cos x$, $f_{xy} = -\sin x$, $f_{yy} = 0$. Then $f_y = 0$ if and only if $x = \frac{\pi}{2} + n\pi$ for n an integer. But $\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$, so $f_x = 0 \quad \Rightarrow \quad y = 0$ and the critical points are $\left(\frac{\pi}{2} + n\pi, 0\right)$, n an integer.

$D\left(\frac{\pi}{2} + n\pi, 0\right) = (0)(0) - (\pm 1)^2 = -1 < 0$, so each critical point is a saddle point.



28. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. $f_x = y - 1$, $f_y = x - 2$, and setting $f_x = f_y = 0$ gives $(2, 1)$ as the only critical point, where $f(2, 1) = 1$. Along L_1: $x = 1$ and $f(1, y) = 2 - y$ for $0 \leq y \leq 4$, a decreasing function in y, so the maximum value is $f(1, 0) = 2$ and the minimum value is $f(1, 4) = -2$. Along L_2: $y = 0$ and $f(x, 0) = 3 - x$ for $1 \leq x \leq 5$, a decreasing function in x, so the maximum value is $f(1, 0) = 2$ and the minimum value is $f(5, 0) = -2$. Along L_3: $y = 5 - x$ and $f(x, 5 - x) = -x^2 + 6x - 7 = -(x - 3)^2 + 2$ for $1 \leq x \leq 5$, which has a maximum at $x = 3$ where $f(3, 2) = 2$ and a minimum at both $x = 1$ and $x = 5$, where $f(1, 4) = f(5, 0) = -2$. Thus the absolute maximum of f on D is $f(1, 0) = f(3, 2) = 2$ and the absolute minimum is $f(1, 4) = f(5, 0) = -2$.



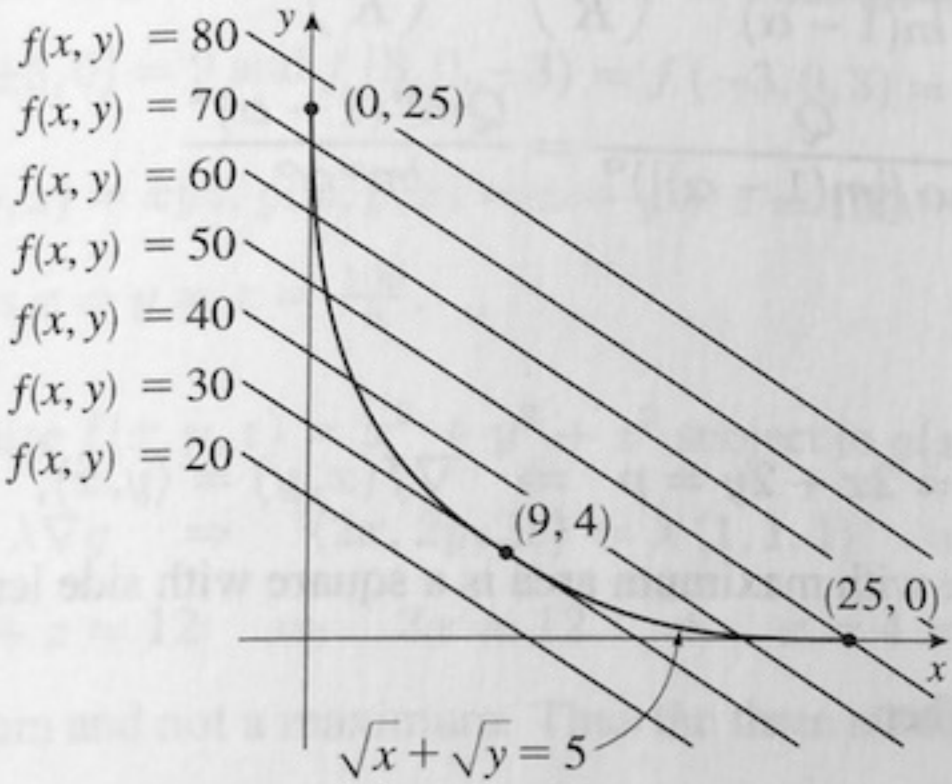
36. Here the distance d from a point on the plane to the point $(1, 2, 3)$ is $d = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$, where $z = 4 - x + y$. We can minimize $d^2 = f(x, y) = (x-1)^2 + (y-2)^2 + (1 - x + y)^2$, so $f_x(x, y) = 2(x-1) + 2(1 - x + y)(-1) = 4x - 2y - 4$ and $f_y(x, y) = 2\,(y - 2) + 2\,(1 - x + y) = 4y - 2x - 2$. Solving $4x - 2y - 4 = 0$ and $4y - 2x - 2 = 0$ simultaneously gives $x = \frac{5}{3}$ and $y = \frac{4}{3}$, so the only critical point is $\left(\frac{5}{3}, \frac{4}{3}\right)$. This point must correspond to the minimum distance, so the point on the plane closest to $(1, 2, 3)$ is $\left(\frac{5}{3}, \frac{4}{3}, \frac{11}{3}\right)$.

4. $f(x,y) = 4x + 6y$, $g(x,y) = x^2 + y^2 = 13 \Rightarrow \nabla f = \langle 4, 6 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2\lambda x = 4$ and $2\lambda y = 6$ imply $x = \frac{2}{\lambda}$ and $y = \frac{3}{\lambda}$. But $13 = x^2 + y^2 = \left(\frac{2}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 \Rightarrow 13 = \frac{13}{\lambda^2} \Rightarrow \lambda = \pm 1$, so f has possible extreme values at the points $(2,3)$, $(-2,-3)$. We compute $f(2,3) = 26$ and $f(-2,-3) = -26$, so the maximum value of f on $x^2 + y^2 = 13$ is $f(2,3) = 26$ and the minimum value is $f(-2,-3) = -26$.

20. (a) $f(x,y)=2x+3y,\ g(x,y)=\sqrt{x}+\sqrt{y}=5 \Rightarrow \nabla f=\langle 2,3\rangle=\lambda\nabla g=\lambda\left\langle \frac{1}{2\sqrt{x}},\frac{1}{2\sqrt{y}}\right\rangle$. Then $2=\frac{\lambda}{2\sqrt{x}}$ and $3=\frac{\lambda}{2\sqrt{y}}$ so $4\sqrt{x}=\lambda=6\sqrt{y} \Rightarrow \sqrt{y}=\frac{2}{3}\sqrt{x}$. With $\sqrt{x}+\sqrt{y}=5$ we have $\sqrt{x}+\frac{2}{3}\sqrt{x}=5 \Rightarrow$ $\sqrt{x}=3 \Rightarrow x=9$. Substituting into $\sqrt{y}=\frac{2}{3}\sqrt{x}$ gives $\sqrt{y}=2$ or $y=4$. Thus the only possible extreme value subject to the constraint is $f(9,4)=30$. (The question remains whether this is indeed the maximum of f.)

(b) $f(25,0)=50$ which is larger than the result of part (a).

(c)



We can see from the level curves of f that the maximum occurs at the left endpoint $(0,25)$ of the constraint curve g. The maximum value is $f(0,25)=75$.

(d) Here ∇g does not exist if $x=0$ or $y=0$, so the method will not locate any associated points. Also, the method of Lagrange multipliers identifies points where the level curves of f share a common tangent line with the constraint curve g. This normally does not occur at an endpoint, although an absolute maximum or minimum may occur there.

(e) Here $f(9,4)$ is the absolute *minimum* of f subject to g.