Math 1320 – Midterm 1: February 27, 2015

This outline is not meant to be an end-all exhaustive study replacement, but rather a structured baseline for your studying. That is, this is a study *companion*.

The topics covered on the exam are roughly described below. Identify which topics are your own weaknesses. In class on Wednesday before the exam, I'll review, so please come with questions. Matthew will also review on the Thursday before the exam. I'll make an effort to be in my office as much as possible during the week prior to the exam. I'm particularly likely to be in my office if you see nothing on my schedule, found here: `http://www.math.utah.edu/~miles/#contact`.

I suggest doing as many of the practice problems as possible, especially the series tests for convergence. I will not supply solutions to the practice problems but will gladly help you through the problem if you stop by my office. You will **not** have the flow chart or any other materials on the exam, so I recommend studying without them. As with the quizzes, there will be easy and difficult questions, but with a deep understanding of the material, all questions will be doable in the time allotted, 50 minutes. **It will be expected that you know the material from last semester.** Do not be surprised if things like integration by parts and partial fractions show up, but I won't make a particular effort to include them. I'll make very clear what my expectations are for each question (such as, how far to simplify).

Chapter 6: Applications of Integration

6.1: Review of Areas

- Area between two curves is: $A = \int f - g$. Understand why.
- Be comfortable integrating dx or dy and understand when each is appropriate. acceleration
- Understand how to find the area enclosed by a parametric curve, $x = f(t), y = g(t)$.

6.2: Volumes, Method of Disks, Washers

- Volume is always roughly area × height, if area changes, this means we need an integral
- To summarize the above, $V = \int A(x)\, dx$, where $A(x)$ is the cross sectional area of a slice at x
- We can rotate a slice (that is perpendicular to axis of rotation) to obtain method of disks.
- Volume of a disk: $\pi r^2 h$. In the classic case, $h = dx$ and $r = f(x)$.
- If there is a hole cut out, new area is $\pi(r_{out}^2 - r_{in}^2)$ and this is called the method of washers.
- We add up the tiny slices by integrating.
- Again, can either integrate dx or dy. Know which is appropriate.
- Know how to rotate around lines other than the axes.
- We can consider other shapes as cross sections and still get volume. See: pyramids.

6.3: Method of Shells

- Rather than rotating a slice that is perpendicular to axis of rotation, we consider a slice parallel to axis of rotation, which forms a shell.
- Volume of a shell = $2\pi r h \Delta r$.
- Often $\Delta r = dx$, but we have to figure out r, h. In the classic case, $h = f(x), r = x$, but this often changes.
- Yet again, we add up the tiny slices by integrating.
- Know how to rotate around lines other than the axes.

6.4: Arc Length

- We can think of the length of a curve as connecting a bunch of line segments that have a particular distance.

- With this idea in mind, we think of the length of a curve as adding a bunch of hypotenuses of triangles with tiny sides dy/dt and dx/dt.
- This results in: $L = \int \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$.
- If we don't have a parametric curve but rather $y = f(x)$ then this simplifies greatly.

6.5: Average Value

- Very similar to the average of a set of points.
- Result is very simple: $f_{ave} = \frac{1}{b-a} \int_a^b f(x)\, dx$.
- Mean value theorem: there is a point c such that: $f(c) = f_{ave}$.

6.6: Applications to Physics and Engineering

- Theme to all of these problems: consider tiny slices. Adding them up is integrating!
- Newton's Second Law: Force = Mass × Acceleration, often written: $F = ma$ or $F = m\frac{d^2x}{dt^2}$.
- Pounds (lb) are a force, where the acceleration is due to gravity. Kilograms are a mass.
- If force and distance are constant, we say Work = Force × Distance. This measures the amount of effort (or energy) we put in to a system.
- Pushing a box from point a to b with non-constant force: $W = \int_a^b f(x)\, dx$.
- Hooke's law: force due to stretching a spring: $F = kx$, where k is the spring constant (or stiffness) and x is the distance the string is stretched from rest. Be able to calculate k given other information.
- Chain problems: we consider a tiny bit of chain of length dx and consider how much force we need to pull it (due to its weight) and how far we pull it
- Bucket problems: we consider pulling the bucket a tiny bit dx and how much force we need to pull it that amount.
- Tank problems: we consider a slice of water or liquid of the tank of height dx and consider the force (due to weight) that we must apply to lift it a particular distance.
- If we have a bunch of points, center of mass is: $\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$.
- $\sum m_i$ = total mass and $\sum m_i x_i$ we can think of as mass × distance. Thus, we can generalize the center of mass as: (sum of mass times distance)/total mass. This is true even when things get more complicated.
- In 2D, we need to measure distance from the x axis, which we'll call $M_x = \sum m_i y_i$ (since it is the y distance) and the distance from the y axis, which we'll call $M_y = \sum m_i x_i$ (since it is the x distance), then $\bar{x} = M_y/m$ and $\bar{y} = M_x/m$.
- Understand how the "mass" part of this formula changes if say, the shape is irregular or the density ρ is non-constant. In these cases, we need an integral which is roughly "adding things up".

Chapter 7: Differential Equations

7.1: Modeling with Differential Equations

- Motivation: sometimes it's easy to observe how things change in time (like a population of bunnies)
- Classic example: $dP/dt = kP$. Here, $P(t)$ is the population at time t. We only know from this formulation the rate of change. This is an example of a differential equation.
- Differential equations have solutions. The one to the above is $P(t) = Ce^{kt}$.
- Understand how to verify a solution to a differential equation: plug it in.
- We have an unknown constant: to solve, we need another condition, typically an initial value like $P(0) = p_0$.
- Differential equation + initial value = initial value problem.

7.2: Direction Fields, Euler's Method

- Often we can't solve DEs directly so we analyze behavior geometrically
- Differential equation tells us a "slope" so we can draw these slopes
- The direction or slope field we get shows us the behavior of the family of solutions
- How do we find a single solution? Start at a point and follow the arrows.
- Know how to construct a slope field and solution. Understand how this represents solution behavior.
- Euler's method is a way of computing a trajectory along a slope field by taking tiny steps.
- $y_{n+1} = y_n + hF(x_n, y_n)$. Understand the geometry and derivation of this statement.

7.3: Separable Equations

- We know how to solve a whole family of differential equations of the form $dy/dx = g(x)/h(y)$, the solution is of the form: $\int h(y)\,dy = \int g(x)\,dx$.
- We got the above by separating the x, y to be isolated and integrating.
- Be able to separate even if it isn't obviously in this form.
- Classic problem: mixing problem in a tank. Be familiar with the setup and how to solve.
- Mixing problem: dx/dt = rate in - rate out. Rate is often concentration × flow rate.

7.4: Exponential Growth, Decay

- We know the solution to the IVP: $dy/dt = ky, y(0) = y_0$, it is: $y(t) = y_0 e^{kt}$. Know how to get this.
- We call this either exponential growth $k > 0$ or exponential decay $k < 0$.
- Growth is a good model for population growth. Know how to obtain k given information. v
- Decay is a good model for radioactive decay, related to half life.
- Newton's Law of Cooling states that the temperature of an object T satisfies: $dT/dt = k(T - T_s)$, where T_s is the surrounding temperature.
- In this case, we assume that T_s is smaller and therefore T is going down (cooling).
- We could also consider T_s being larger, in which case T would go up.
- In both cases, same differential equation with $k < 0$. Why is this? Consider whether the derivative is positive or negative.

Chapter 8: Sequences and Series

8.1: Sequences

- A sequence is just a list of numbers in a particular order, typically denoted $\{a_n\}$.
- Given the terms of the sequence, be able to come up with the general formula. This could also be recursive like Fibonacci.
- Understand what it means to be the limit of a sequence or what the terms go to as $n \to \infty$. Understand why this is different than our limit as $x \to \infty$ but why we can compute it the same way.
- Be able to use the squeeze theorem that states $|a_n| \to 0 \implies a_n \to 0$. Why is this true?
- Understand what it means to for a sequence to be bounded above or below, that is $a_n \leq M$ or $M \leq a_n$.
- A sequence can be monotonically (or exclusively) increasing or decreasing. Big theorem: monotonic + bounded = convergent. Why? Be able to argue this geometrically.
- A geometric sequence r^n converges when $|r| < 1$. Why?

8.2: Series

- An (infinite) series is when we add terms of an infinite sequence.

- We often consider the partial sums $s_n = \int_{i=1}^{n} a_i = a_1 + \cdots + a_n$. Note that the partial sums form a sequence.
- If the series converges, it converges to the sum of the series.
- Geometric series: $\sum_{n=1}^{\infty} ar^{n-1} = a/(1-r)$ if $|r| < 1$ and diverges otherwise. Be able to manipulate to this form.
- The harmonic series $\sum 1/n$ diverges.
- Theorem: if $\sum a_n$ converges, $a_n \to 0$. The converse is not always true: see the harmonic series.
- Test for divergence (contrapositive of above statement): if $a_n \not\to 0$, then $\sum a_n$ does not converge.

8.3: Integral Test, Comparison Tests

- The integral of a function describes an area. We can think of the series as an area.
- Integral test: if f is continuous, positive, decreasing function, and $a_n = f(n)$ then $\sum a_n$ either converges or diverges if $\int f(x)\,dx$ does.
- Make sure you can argue and verify that the conditions of the test are true.
- If the sum doesn't start at 1 then just start the integral later. $\sum_{n=5}^{\infty} \implies \int_{5}^{\infty}$.
- p-series: $\sum 1/n^p$ converges when $p > 1$ and diverges otherwise. Very important.
- Comparison test: if $\sum b_n$ converges and $a_n \leq b_n$ then $\sum a_n$ converges. Also if $\sum b_n$ diverges and $a_n \geq b_n$ then $\sum a_n$ diverges. Why?
- Limit comparison test: if $\lim_{n\to\infty} a_n/b_n = c$ then the series share the same behavior. This is often useful if the other comparison test fails.
- We can estimate the remainder of a series, defined to be $R_n = s - s_n$ by the integrals: $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$. Understand the geometry of this.

8.4: Other Convergence Tests

- Alternating series test: an alternating series of the form $\sum(-1)^{n-1}b_n$ converges if $b_{n+1} \leq b_n$ and $b_n \to 0$. Be able to verify this.
- Often it's helpful to think of b_n as $f(x)$ and prove $f'(x) < 0$.
- A series is absolutely convergent if $\sum |a_n|$ converges.
- Theorem: If $\sum a_n$ is absolutely convergent, then it is convergent.
- Ratio test: $|a_{n+1}/a_n| = L$. If $L > 1$: diverges, $L < 1$: absolutely convergent (and therefore convergent), $L = 1$: inconclusive.
- Ratio test is typically very helpful if series has a factorial or powers of n.

8.5: Power Series

- We can think of series having a variable in it, we call a power series centered at $x = a$: $\sum c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$.
- Almost always use the ratio test to determine convergence. A series converges for either all, one, or some x values.
- If it converges for some, we consider the interval of convergence and the radius of convergence to be the distance from a that it converges.
- We often need to consider endpoints separately since ratio test does not handle well.

8.6: Representing Functions as Power Series

- We can use geometric series to understand when we can write a function as a series.
- For instance, $1/(1-x) = 1 + x + x^2 + \cdots$ when $|x| < 1$.
- We can manipulate functions that don't obviously look of this form, like $2x^2/(2+x^2)$. Know how to do this.

- Theorem: if we can represent a function as a power series, we can compute the derivative and integral by taking the term-wise derivative and integral. Very useful for constructing new series like $\ln(1+x)$.

8.7: Taylor Series, Maclaurin Series

- We make the observation that if $f(x)$ = power series, then we know the coefficients $c_n = f^{(n)}(a)/n!$.
- This actually gives us a recipe for constructing series: we call this the Taylor series. $f(x) = \sum f^{(n)}(a)(x-a)^n/n! = f(a) + f'(a)(x-a)/1! + f''(a)(x-a)^2/2! + \cdots$.
- When $a = 0$, we have a special cased called a Maclaurin series.
- Note the technicality: if the Taylor series exists, it is of this form, but we don't know that every function has a Taylor series.
- $e^x = 1 + x/1! + x^2/2! + x^3/3! + \cdots$ for all x.
- A function is equal to its Taylor series if the remainder $R_n(x) = f(x) - T_n(x) \to 0$.
- Taylor's Inequality: if $|f^{(n+1)}(x)| < M$ then $|R_n| < \frac{M}{(n+1)!}|x-a|^{n+1}$. That is, if the derivatives of a function are bounded, the remainder goes to 0 as we hope.
- Be able to compute a Taylor series and argue (via Taylor's theorem) that the series is equal to the function on some interval.
- Know the Taylor series for $\sin x, \cos x, e^x$. Often helpful starting points.
- Multiplying and dividing series is somewhat tricky but useful. For instance $e^x \sin x$.

8.8: Applications of Taylor Series

- Given some error tolerance, compute how many terms are needed to approximate a function.
- Be familiar with applications of Taylor series in computing limits, integrals, sums, and applications to physics.