

Name: \_\_\_\_\_

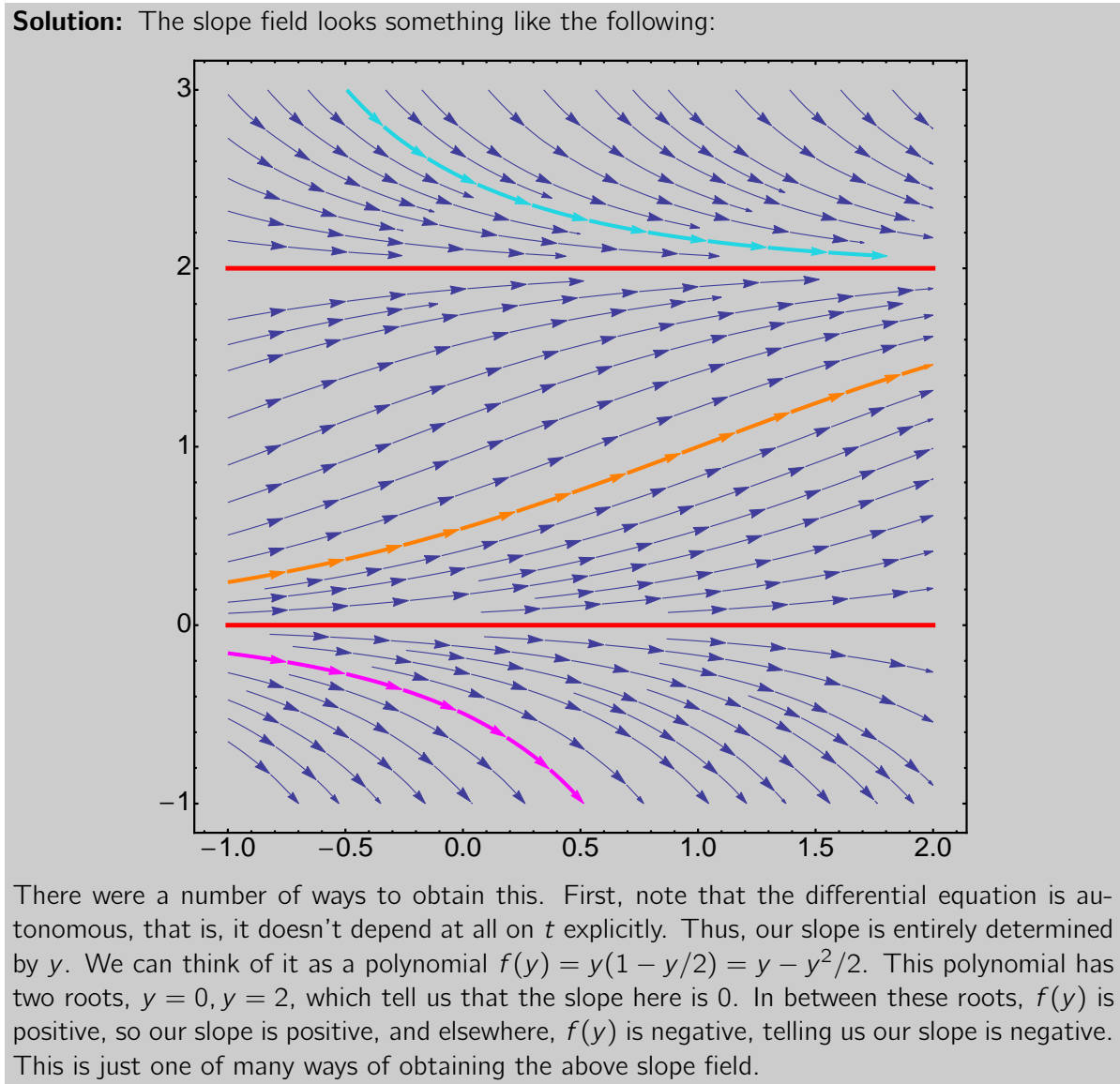
Quiz Score: \_\_\_\_/10

1. Consider the following differential equation:

$$\frac{dy}{dt} = y \left( 1 - \frac{y}{2} \right).$$

- 4 (a) Draw a slope (or direction) field for this differential equation. Make note of important features of this slope field.

**Solution:** The slope field looks something like the following:



There were a number of ways to obtain this. First, note that the differential equation is autonomous, that is, it doesn't depend at all on  $t$  explicitly. Thus, our slope is entirely determined by  $y$ . We can think of it as a polynomial  $f(y) = y(1 - y/2) = y - y^2/2$ . This polynomial has two roots,  $y = 0, y = 2$ , which tell us that the slope here is 0. In between these roots,  $f(y)$  is positive, so our slope is positive, and elsewhere,  $f(y)$  is negative, telling us our slope is negative. This is just one of many ways of obtaining the above slope field.

- 1 (b) This system has two *fixed points* or *equilibria*. What are they?

**Solution:** We can see from the figure (in red) and by the reasoning above that when  $y = 0$  or when  $y = 2$ , then  $y' = 0$ , thus the solution is not changing in time, and we can say that it is an equilibrium.

- 1 (c) We say that a fixed point of a system is *stable* or *attracting* if trajectories tend toward it. Similarly, a fixed point is *unstable* if trajectories tend away from it. By considering trajectories of solutions, make an argument for which fixed point from part (b) is stable and which is unstable.

**Solution:** Although this may have seem like a bit of a curveball at first, it is really just a question about the information we already know in disguise. Examine the cyan, orange, and magenta trajectories above, simply obtained by following the slope field. Notice that both the orange and magenta trajectories tend *away* from the  $y = 0$  equilibrium. Similarly, the orange and cyan trajectories tend *toward* the  $y = 2$  equilibrium. Thus, we can conclude that  $y = 2$  is stable and  $y = 0$  is unstable. *Fun fact:* there's actually a third possibility called a *saddle point*. Can you construct a slope field with an equilibrium that is neither stable nor unstable?

- 4] 2. Solve the following initial value problem:

$$\frac{dy}{dt} = ty^2, \quad y(0) = 1.$$

**Solution:** This is simply a separation of variables question. Our first step is to separate:

$$\frac{dy}{y^2} = t dt.$$

We can now integrate both sides, examining the left hand side first:

$$\text{LHS} = \int \frac{dy}{y^2} = \int y^{-2} dy = \frac{y^{-1}}{-1} + C_1 = -\frac{1}{y} + C_1$$

Similarly, integrating the right hand side:

$$\text{RHS} = \int t dt = \frac{t^2}{2} + C_2.$$

Combining these two results and consolidating  $C_1, C_2$  into a single unknown  $C$  we find:

$$-\frac{1}{y} = \frac{t^2}{2} + C.$$

If we now plug in the initial condition  $t = 0, y = 1$ , we see that:

$$-\frac{1}{1} = \frac{0^2}{2} + C,$$

which clearly suggests that  $C = -1$ . Thus, we have that:

$$-\frac{1}{y} = \frac{t^2}{2} - 1.$$

This is a sufficient answer, but if you wanted to rearrange for  $y(t)$  explicitly, you'd find, by flipping the numerators and denominators:

$$-\frac{y}{1} = \frac{1}{t^2/2 - 1} \implies y(t) = -\frac{1}{t^2/2 - 1}.$$