Math 1321 – Midterm 1: February 19, 2016

This outline is not meant to be an end-all exhaustive study replacement, but rather a structured baseline for your studying. That is, this is a study *companion*.

The topics covered on the exam are roughly described below. I recommend you identify which topics are your own weaknesses and practice those without your notes until you are comfortable with the topic.

I'll be around as much as possible during the week of the exam and will happily answer any questions. I'm particularly likely to be in my office if you see nothing on my schedule, found here: `http://www.math.utah.edu/~miles/#contact`. Joaquin will also review in lab the day before the exam.

Some other general comments: this exam is not meant to test your algebra skills, but being able to do computations at a reasonable pace will be required. The exam will be doable in 50 minutes but only with sufficient preparation. Do not bother memorizing stupid things like the names of the quadratic surfaces. **Do** memorize the conditions required for series convergence tests if you plan on using them (you will have to use them).

Anything covered in lab, quiz, homework (even ungraded), and review practice is fair game, and I'll mostly draw from these sources when constructing the exam.

Chapter 8: Sequences and Series

8.1: Sequences

- A sequence is just a list of numbers in a particular order, typically denoted $\{a_n\}$.
- Given the terms of the sequence, be able to come up with the general formula. This could also be recursive like Fibonacci.
- Understand what it means to be the limit of a sequence or what the terms go to as $n \to \infty$. Understand why this is different than our limit as $x \to \infty$ but why we can compute it the same way.
- Be able to use the squeeze theorem that states $|a_n| \to 0 \implies a_n \to 0$. Why is this true?
- Understand what it means to for a sequence to be bounded above or below, that is $a_n \leq M$ or $M \leq a_n$.
- A sequence can be monotonically (or exclusively) increasing or decreasing. Big theorem: monotonic + bounded = convergent. Why? Be able to argue this geometrically.
- A geometric sequence r^n converges when $|r| < 1$. Why?

8.2: Series

- An (infinite) series is when we add terms of an infinite sequence.
- We often consider the partial sums $s_n = \int_{i=1}^{n} a_i = a_1 + \cdots + a_n$. Note that the partial sums form a sequence.
- If the series converges, it converges to the sum of the series.
- Geometric series: $\sum_{n=1}^{\infty} ar^{n-1} = a/(1-r)$ if $|r| < 1$ and diverges otherwise. Be able to manipulate to this form.
- The harmonic series $\sum 1/n$ diverges.
- Theorem: if $\sum a_n$ converges, $a_n \to 0$. The converse is not always true: see the harmonic series.

- Test for divergence (contrapositive of above statement): if $a_n \not\to 0$, then $\sum a_n$ does not converge.

8.3: Integral Test, Comparison Tests

- The integral of a function describes an area. We can think of the series as an area.
- Integral test: if f is continuous, positive, decreasing function, and $a_n = f(n)$ then $\sum a_n$ either converges or diverges if $\int f(x)\,dx$ does.
- Make sure you can argue and verify that the conditions of the test are true.
- If the sum doesn't start at 1 then just start the integral later. $\sum_{n=5}^{\infty} \implies \int_5^{\infty}$.
- p-series: $\sum 1/n^p$ converges when $p > 1$ and diverges otherwise. Very important.
- Comparison test: if $\sum b_n$ converges and $a_n \leq b_n$ then $\sum a_n$ converges. Also if $\sum b_n$ diverges and $a_n \geq b_n$ then $\sum a_n$ diverges. Why?
- Limit comparison test: if $\lim_{n\to\infty} a_n/b_n = c$ then the series share the same behavior. This is often useful if the other comparison test fails.
- We can estimate the remainder of a series, defined to be $R_n = s - s_n$ by the integrals: $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$. Understand the geometry of this.

8.4: Other Convergence Tests

- Alternating series test: an alternating series of the form $\sum(-1)^{n-1} b_n$ converges if $b_{n+1} \leq b_n$ and $b_n \to 0$. Be able to verify this.
- Often it's helpful to think of b_n as $f(x)$ and prove $f'(x) < 0$.
- A series is absolutely convergent if $\sum |a_n|$ converges.
- Theorem: If $\sum a_n$ is absolutely convergent, then it is convergent.
- Ratio test: $|a_{n+1}/a_n| = L$. If $L > 1$: diverges, $L < 1$: absolutely convergent (and therefore convergent), $L = 1$: inconclusive.
- Ratio test is typically very helpful if series has a factorial or powers of n.

8.5: Power Series

- We can think of series having a variable in it, we call a power series centered at $x = a$: $\sum c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$.
- Almost always use the ratio test to determine convergence. A series converges for either all, one, or some x values.
- If it converges for some, we consider the interval of convergence and the radius of convergence to be the distance from a that it converges.
- We often need to consider endpoints separately since ratio test does not handle well.

8.6: Representing Functions as Power Series

- We can use geometric series to understand when we can write a function as a series.
- For instance, $1/(1-x) = 1 + x + x^2 + \cdots$ when $|x| < 1$.
- We can manipulate functions that don't obviously look of this form, like $2x^2/(2+x^2)$. Know how to do this.
- Theorem: if we can represent a function as a power series, we can compute the derivative and integral by taking the term-wise derivative and integral. Very useful for constructing new series like $\ln(1+x)$.

8.7: Taylor Series, Maclaurin Series

- We make the observation that if $f(x)$ = power series, then we know the coefficients $c_n = f^{(n)}(a)/n!$.

- This actually gives us a recipe for constructing series: we call this the Taylor series. $f(x) = \sum f^{(n)}(a)(x-a)^n/n! = f(a) + f'(a)(x-a)/1! + f''(a)(x-a)^2/2! + \cdots$.
- When $a = 0$, we have a special cased called a Maclaurin series.
- Note the technicality: if the Taylor series exists, it is of this form, but we don't know that every function has a Taylor series.
- $e^x = 1 + x/1! + x^2/2! + x^3/3! + \cdots$ for all x.
- A function is equal to its Taylor series if the remainder $R_n(x) = f(x) - T_n(x) \to 0$.
- Taylor's Inequality: if $|f^{(n+1)}(x)| < M$ then $|R_n| < \frac{M}{(n+1)!}|x-a|^{n+1}$. That is, if the derivatives of a function are bounded, the remainder goes to 0 as we hope.
- Be able to compute a Taylor series and argue (via Taylor's theorem) that the series is equal to the function on some interval.
- Know the Taylor series for $\sin x, \cos x, e^x$. Often helpful starting points.
- Multiplying and dividing series is somewhat tricky but useful. For instance $e^x \sin x$.

8.8: Applications of Taylor Series

- Given some error tolerance, compute how many terms are needed to approximate a function.
- Be familiar with applications of Taylor series in computing limits, integrals, sums, and applications to physics.

Chapter 9: Vectors and the Geometry of Space

9.1: 3 Dimensions

- Cartesian, or rectangular coordinates in 3 dimensions: (x, y, z)
- In 2D, $y = 2$ is a line, in 3D, $y = 2$ is a plane
- Know other common examples, for instance $x^2 + y^2 = 1$ is a cylinder
- Distance between two points $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- Sphere with center (h, k, l) and radius r: $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$.

9.2: Vectors

- New fundamental quantity: vectors have direction and magnitude (think of arrows)
- Two vectors are equal if they have the same direction, magnitude
- 0 vector has 0 magnitude and is the only vector with no direction
- Displacement vector: describing moving from A to B, denoted $\overrightarrow{AB}$.
- We can add vectors by placing them tip (of first) to tail of second and draw resulting total arrow.
- Scalar multiplication: rescales vector magnitude and/or flips if negative.
- We think of vectors as their components: $\mathbf{v} = \langle a, b, c \rangle$.
- From this, displacement vector is easy: $\overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$.
- Magnitude of vector, denoted $|\mathbf{a}|$ or $\|\mathbf{a}\|$: distance formula.
- Unit vector = length 1.
- Standard coordinate vectors: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$ and $\mathbf{k} = \langle 0, 0, 1 \rangle$.
- We can write any vector in $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form (convince yourself of why this is true)
- Be familiar with tension example: forces in different directions

9.3: Dot product

- Motivated definition of dot by notion of work.

- If $\mathbf{F}$ is some force exerted an angle away θ from the displacement $\mathbf{D}$ then we found that $W = |\mathbf{F}||\mathbf{D}|\cos\theta$
- We take this as definition: $\mathbf{a}\cdot\mathbf{b} \overset{\text{def}}{=} |\mathbf{a}||\mathbf{b}|\cos\theta$, where $\theta \in [0,\pi]$.
- Sometimes called the scalar product because you put in two vectors and get a scalar out.
- If $\mathbf{a}\cdot\mathbf{b} = 0$, they are orthogonal, also the converse is true. Why?
- Understand dot product differences when θ is acute, obtuse, right.
- Know how to obtain θ for vectors.
- Component formula: $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.
- Projecting b onto a: $\text{proj}_{\boldsymbol{a}}\,\mathbf{b} = \text{comp}_{\boldsymbol{a}}\mathbf{b}\dfrac{\mathbf{a}}{|\mathbf{a}|}$, where $\text{comp}_{\boldsymbol{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$. Know the picture in different cases and where these formulae come from

9.4: Cross product

- Similar to the dot product, we motivate this by physics: imagine a wrench of some direction described by $\mathbf{r}$ and exerting a force on it in another direction $\mathbf{F}$. We get a *torque* out
- torque: $\tau = |\mathbf{r}|\mathbf{F}|\sin\theta\mathbf{n}$, where $\mathbf{n}$ is a vector orthogonal to the other two.
- $\mathbf{a}\times\mathbf{b} \overset{\text{def}}{=} |\mathbf{a}||\mathbf{b}|\sin\theta\mathbf{n}$, again $\theta \in [0,2\pi]$ and $\mathbf{n}$ is unit orthogonal to both vectors, determined by the right hand rule.
- Sometimes called the vector product because you put in two vectors and get a vector out.
- Angle θ always measured from $\mathbf{a}$ to $\mathbf{b}$. Drawing tails attached helps figure this out often
- If $\mathbf{a}\times\mathbf{b} = 0$, they are parallel, converse is also true.
- Cross product does NOT commute, but $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$.
- Know how to use determinants to obtain the cross product. General result (probably don't want to memorize): $\mathbf{a}\times\mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$.
- $|\mathbf{a}\times\mathbf{b}|$ gives you the area of the parallelogram described by the two vectors
- Useful formula: $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.
- Another useful formula: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})c$.

9.5: Equations of lines, planes

- Vector equation of a line is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0$ is a vector pointing to some point, and $\mathbf{v}$ is a vector pointing in the direction of the line. Any point on the line can be described this way for some t.
- Alternatively, if $\mathbf{v} = \langle a,b,c\rangle$, we can say $\langle x,y,z\rangle = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle$.
- Parametric equations: $x = x_0 + at, \ldots$ and so on. Just take each component of above formula.
- We can eliminate t to obtain the symmetric equations: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$.
- If we want a line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$, we can parameterize by $\mathbf{r} = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$, where $0 \le t \le 1$.
- Two lines are skew if they do not intersect and are not parallel
- To define a plane, we need two things: any point on the plane $P_0 = (x_0, y_0, z_0)$ and the normal vector $\mathbf{n}$.
- Notice that $\mathbf{n}$ is orthogonal to any displacement vector in the plane, so we have $\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0) = 0$, which is the vector equation of a plane.
- More usefully, if $\mathbf{n} = \langle a,b,c\rangle$, we have $\langle a,b,c\rangle\cdot\langle x-x_0, y-y_0, z-z_0\rangle = 0$, which is often called the scalar equation of a plane.
- Angle between two planes: angle between normals.

- Parallel planes: same normal.
- Distance D from a point P_1 to a plane is $D = |\text{comp}_{\mathbf{n}}\,\mathbf{b}| = \frac{|\mathbf{n}\cdot\mathbf{b}|}{\mathbf{n}}$, where $\mathbf{b} = \langle x_1, x_0, y_1 - y_0, z_1 - z_0$. Know geometrically why this is true.
- Distance between two planes: pick a point on one plane, repeat above.
- Distance between two lines: can be thought of as distance between two planes. Find the normal and pick points.

9.6: Functions and surfaces

- We can think of $f(x, y) = z$, where we put in two variables and get a height z out.
- Know how to find domain/range of these functions. Pretty much the same as previous.
- Be familiar with basic shapes. For instance, $z = ax + by + c$ or any linear function of x, y, z is a plane.
- Key idea for identifying shapes: take one coordinate to be constant, say $x = k$. This is like slicing the picture. Piece together from this.
- Be able to identify shapes given equations or the reverse.

Chapter 10: Vector Functions

10.1: Vector functions, Space curves

- We now think of vector functions. Input: some variable, typically t output: a vector. This is new.
- Typically: $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$.
- Limit of r: just take limit of each component.
- Continuity of r: $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.
- We can think of $\mathbf{r}t$ as an "arm" pointing to different points as t changes. We can alternatively (and often more usefully) think of the space curve described by $x = f(t), y = g(t), z = h(t)$, which describes the path of the tip of the arm. Understand the difference between these two.
- Helix: $\mathbf{r} = \langle \cos t, \sin t, t\rangle$.
- Know how to find intersection of two surfaces: space curve.

10.2: Derivatives and the integrals of vector functions

- Derivative of vector function: just component-wise derivative
- Geometrically, derivative of vector function gives us a vector tangent at the point
- We can normalize to length 1, called unit tangent vector: $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$.
- Important theorem: if $|\mathbf{r}| = c$ then $\mathbf{r}' \perp \mathbf{r}$, that is $\mathbf{r}' \cdot \mathbf{r} = 0$. Know the proof. Is the converse true? Try to prove it.
- Integrals of vector functions: just integral of each component.

10.3: Arc length and curvature

- Arc length formula the same as before: $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$, but notice that $|\mathbf{r}'(t)| = \sqrt{f'^2 + g'^2 + h'^2}$
- Nicer formula: $L = \int_a^b |\mathbf{r}'(t)|\,dt$.
- We can consider $s(t)$, the arc length function from some point $t = a$ as $s(t) = \int_a^t |\mathbf{r}'(t)|\,dt$, which means that $ds/dt = |\mathbf{r}'(t)|$.
- Issue: same curves have different parameterizations. How do describe the curve itself, independent of the actual parameterization? Parameterize by arc length, s.

- As curve gets curvier, T changes rapdily, and when curve is flatter, T changes slowly, we can give this quantity a name, curvature: $\kappa = |d\mathbf{T}/ds|$.
- Not very useful formula, but we can recognize that $\kappa = |\mathbf{T}'(t)|/|\mathbf{r}'(t)|$. Understand the proof of this.
- Harder to prove, but worth studying the proof: $\kappa = |\mathbf{r}' \times \mathbf{r}''|/|\mathbf{r}'|^3$. Most useful formula if I give you $\mathbf{r}$ and ask you to compute κ.
- Special case: if $y = f(x)$, then $\kappa = |f''|/[1 + f'^2]^{3/2}$.
- $\mathbf{T}$ is the tangent vector, so what is tangent to the tangent? The unit normal vector, $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$. Also, there is a third vector orthogonal to both called the binormal $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Be able to calculate these.