

Name: _____

Quiz Score: _____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

- 5 1. Determine whether the following series converges or diverges. **If it converges, find the sum.**

$$\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2}$$

Solution: At first glance, this series looks fairly horrible. Because I'm not super evil (I think), this should signal that the convergence test intended for this problem is a relatively easy one to compute. In fact, the convergence test for this problem is likely the first you should check for all series: **the test for divergence**. This test states that if you have

$$\sum_{k=1}^{\infty} a_k, \quad \text{where} \quad \lim_{k \rightarrow \infty} a_k \neq 0,$$

then $\sum_k a_k$ diverges. In other words, unless your terms get smaller (to the point of basically being 0), adding up infinitely many of them has no shot of converging. For this particular problem:

$$\lim_{k \rightarrow \infty} \frac{k(k+2)}{(k+3)^2} = \lim_{k \rightarrow \infty} \frac{k^2 + 2k}{k^2 + 6k + 9} = 1 \neq 0,$$

so **by the test for divergence, this series diverges.**

- 5 2. Determine whether the following series converges or diverges. **If it converges, find the sum.**

$$\sum_{k=1}^{\infty} \frac{1}{\pi^{k-1}}$$

Solution: Again, this series maybe looks a bit messy at first, but let's try writing out a few terms:

$$\sum_{k=1}^{\infty} \frac{1}{\pi^{k-1}} = 1 + \frac{1}{\pi} + \left(\frac{1}{\pi}\right)^2 + \left(\frac{1}{\pi}\right)^3 + \dots$$

What characterizes the terms in this series? Each term differs from the previous by a factor of $1/\pi$. In other words, the ratio of the terms is *constant*, meaning we have a **geometric series**. The canonical form for our geometric series is:

$$\sum_{k=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots,$$

which means in our case, we have $a = 1$ and $r = 1/\pi$. When does a geometric series converge?

$$|r| < 1 \quad \implies \quad \frac{1}{\pi} \approx \frac{1}{3.1415} < 1,$$

so **this geometric series converges** because $|r| < 1$. We know that **if** the geometric series converges, it converges to

$$\sum_{k=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad \text{for } |r| < 1,$$

which means we ultimately have that

$$\sum_{k=1}^{\infty} \frac{1}{\pi^{k-1}} = \frac{1}{1 - \frac{1}{\pi}} = \frac{\pi}{\pi - 1}.$$