

Name: _____

Quiz Score: ____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

- 5 1. Determine whether the following series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Hint: use the integral test with $u = \ln x$ and make sure to state every requirement of the test even if it is obviously satisfied!

Solution: The integral test considers, in general, a series

$$\sum_{j=k}^{\infty} a_j$$

and says that we can relate this to the integral of $f(x)$, where $f(j) = a_j$ *only* if $f(x)$ satisfies three requirements:

1. $f(x)$ is continuous.
2. $f(x)$ is positive valued.
3. $f(x)$ is decreasing.

Here, we see that our choice of $f(x)$ is $f(x) = \frac{1}{x(\ln x)^2}$. We must check the three conditions listed above.

1. For $x \geq 2$, the whole thing is continuous (as $x = 0$ is the only problematic point).
2. Since $(\ln x)^2$ is always positive and $x \geq 2$, the whole quantity is always positive.
3. For this quiz, it's perfectly okay to just say the function is decreasing because the numerator stays 1 and the denominator grows, however we could show this a little more precisely by considering

$$f'(x) = -\frac{\ln x + 2}{x^2(\ln x)^3},$$

which we see, for $x \geq 1/e^2$ is always negative, meaning the function is decreasing.

Thus, $f(x)$ satisfies all of the requirements of the integral test meaning that if

$$\int_2^{\infty} f(x) dx = \int_2^{\infty} \frac{1}{x(\ln x)^2} dx$$

converges or diverges, so does $\sum_{j=2}^{\infty} 1/(n \ln^2 n)$. To see whether this integral converges or diverges, consider, as in the hint, $u = \ln x$, so $du = dx/x$, meaning we have

$$\int_2^{\infty} f(x) dx = \int_{\ln 2}^{\infty} \frac{1}{u^2} du = \left[-\frac{1}{u} \right]_{u=\ln 2}^{u=\infty} = \lim_{t \rightarrow \infty} \left\{ -\frac{1}{t} \right\} + \frac{1}{\ln 2} = \frac{1}{\ln 2},$$

so this integral indeed converges and therefore, by the integral test, as does the series.

- 5] 2. Determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1}.$$

Hint: use the alternating series test and again, make sure to state and show that this indeed satisfies all of the requirements of the test. As a hint, it may be useful to consider $f'(x)$ where $f(x) = x/(x^2 + 1)$.

Solution: The alternating series test considers series of the form

$$\sum_{n=1}^{\infty} (-1)^n b_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n-1} b_n,$$

where the terms b_n must satisfy the following conditions:

1. b_n is decreasing, that is, $b_n \geq b_{n+1}$
2. b_n decays to 0 or $\lim_{n \rightarrow \infty} b_n = 0$.

Thus, in our case, $b_n = \frac{n}{n^2+1}$ and we must check these two properties.

1. It's pretty clear to see that n^2 grows much faster than n so $\lim_{n \rightarrow \infty} b_n = 0$. We could do this more precisely by taking

$$\lim_{n \rightarrow \infty} \frac{n}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1 + \frac{1}{n}} = 0.$$

2. Again, the above basically already shows its decreasing, but to see this more rigorously, we can consider $f(x) = x/(x^2 + 1)$ and

$$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2},$$

which means that so long as $x > 1$ (which it is), $f'(x) < 0$ meaning we have a decreasing function (and therefore terms of the series.)

Thus, the requirements of the alternating series test are satisfied, which means that this series **converges**. A very important nuance here: **the alternating series test cannot prove divergence, only convergence**, however the integral test can prove either.