

Name: \_\_\_\_\_

Quiz Score: \_\_\_\_\_/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

- 7 1. Compute the Taylor series of

$$f(x) = \frac{1}{x} \quad \text{around } a = 1.$$

**Solution:** The definition of a Taylor series around  $x = a$  (the expansion point), is defined to be

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!} (x - a)^j = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \dots,$$

where  $f^{(n)}$  denotes the  $n$ th derivative of the function  $f(x)$ . Thus, we need to compute the derivatives of our function. Let's see a pattern:

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad f'''(x) = -\frac{3 \cdot 2}{x^4}, \quad f^{(4)}(x) = \frac{4 \cdot 3 \cdot 2 \cdot 1}{x^5},$$

and hopefully from this, the pattern is clear:

$$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}.$$

Using  $a = 1$  for this problem, we have

$$f^{(n)}(1) = (-1)^n n!,$$

and using this in the definition of the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n = \sum_{j=0}^{\infty} \frac{(-1)^j j!}{j!} (x - 1)^j,$$

and noting that the factorials cancel, our final answer is then

$$\sum_{n=0}^{\infty} (-1)^n (x - 1)^n = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \dots$$

- 3 2. Compute the radius of convergence of the series you computed in question 1.

**Solution:** The solution we got to the previous problem is

$$\sum_{n=0}^{\infty} (-1)^n (x-1)^n = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots$$

We now invoke the ratio test to determine for which  $x$  values this series converges. The ratio test says that if you have a series

$$\sum_{n=0}^{\infty} a_n \quad \text{and} \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| := L$$

then the series converges for  $L < 1$ , (and as a reminder, diverges for  $L > 1$  and if  $L = 1$ , we have no clue). Thus, in this case,  $a_n = (-1)^n (x-1)^n$ , so we have

$$L = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (x-1)^{n+1}}{(-1)^n (x-1)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{(x-1)^n} \right| = \lim_{n \rightarrow \infty} |x-1|,$$

thus our condition for convergence is

$$|x-1| < 1 \implies R = 1,$$

where  $R$  is the radius of convergence. Note that we don't know if this series converges for the end points, that is at  $x = 0, x = 2$  but only the radius was asked for this problem. A separate test could be used for each of these cases if the interval of convergence was asked for.