Math 3140 – PDEs HW I

Due: June 27, 2016

1. If $u(x,t)$ is the temperature in a metal rod with variable cross-sectional area $A(x)$ with uniform thermal properties, specifically $u(x,0) = \sqrt{x}(L-x)$, $A(x) = \sqrt{x}$, and $c\rho = 1$.
   (a) Give an expression for the total thermal energy $H_{tot}[0,L](t)$ in the rod $x \in [0,L]$.
   (b) What is the energy at time $t = 0$?
   (c) Derive the heat equation for this case.

Solution: Since we know that the relationship between the heat energy $e(x,t)$ density and the temperature $u(x,t)$ is $e = c\rho u$, where c is the specific heat and ρ is the mass density, we simply need to add up the heat energy density over our whole rode, resulting in

$$H_{tot}(t) = \int_0^L c\rho A(x) u(x,t) dx.$$

For our specific problem, this yields

$$H_{tot}(t) = \int_0^L \sqrt{x} u(x,t) dx$$

and at $t = 0$, we know the initial condition, so the initial total enerfy is therefore

$$H_{tot}(0) = \int_0^L \sqrt{x} \cdot \sqrt{x}(L-x)\, \mathrm{d}x = L^3/6.$$

To derive the general, heat equation, we write down the total heat energy in an arbitrary interval (a,b):

$$H_{tot} = \int_a^b c\rho A(x) u(x,t) dx.$$

The time dynamics of the total energy is

$$dH_{tot}/dt = \int_a^b c\rho(x) A(x) u_t(x,t) dx = A(a)\phi(a) - A(b)\phi(b)$$

Use the fundamental theorem of calculus/divergence theorem in 1D:

$$dH_{tot}/dt = \int_a^b c\rho A(x) u_t(x,t) dx = -\int_a^b \frac{\partial}{\partial x}(A(x)\phi(x,t)) dx$$

Fourier's law: $\phi = -K_0(x) u_x$, so

$$dH_{tot}/dt = \int_a^b c\rho A(x) u_t(x,t) dx = \int_a^b \frac{\partial}{\partial x}[A(x) K_0(x) \frac{\partial}{\partial x} u(x,t))] dx$$

Since the interval (a, b) is arbitrary, we can remove the common integral on each side to get the end result:

$$c\rho(x)A(x)\frac{\partial u}{\partial t}(x,t)dx = [A(x)K_0(x)u_x]_x$$

2. Let $A(x) = 1 + 0.1x$ describe a cross-sectional area of a conical-shaped heat conducting rod on $x \in [0, 10]$. Suppose further that it is possible to make the rod material have a variable heat conductive property along the extent $x \in [0, 10]$ by $K_0(x)$, and you can also vary the mass density $\rho(x)$. Find a specific set of functions $K_0(x)$ and $\rho(x)$ and a such that the resulting heat equation has a constant heat diffusivity coefficient $\kappa = 2$: $u_t = \kappa u_{xx}$. Assume a constant specific heat $c = 1$ for all x.

Solution: Using the result from the previous problem, the more general heat equation form with non-uniform thermal properties is

$$\rho(x)A(x)\frac{\partial u}{\partial t}(x,t)dx = [A(x)K_0(x)u_x]_x \tag{1}$$

. We know $A(x) = 1 + 0.1x$ is fixed, but we want to choose $\rho(x)$ and $K_0(x)$ such that we can simplify this to

$$u_t = \kappa u_{xx},$$

where $\kappa = 2$ is a constant. If we rearrange (1) a bit, we find

$$u_t = \frac{1}{\rho(x)A(x)}\frac{\partial}{\partial x}\left\{\rho(x)A(x)K_0 u_x\right\},$$

so if we chose, say, $1/\rho A = 1$ and $\rho A K_0 = 2$, or, more specifically This implies

$$\rho(x) = 1/A(x), \quad K_0(x) = \frac{2}{A(x)},$$

then our heat equation simplifies to

$$u_t = 2u_{xx}.$$

In other words, we have chosen the non-uniform thermal properties to cancel out the non-uniform cross sectional area and give us an effectively uniform heat equation.

3. Derive PDE for a rod with no heat sources Q whose lateral surface area is not insulated by following the steps below.

(a) Assume that heat energy flowing out of the lateral sides per unit surface area per unite time is $w(x,t)$. Derive the PDE for the temperature $u(x,t)$ that contains $w(x,t)$ in the PDE.

(b) Assume that $w(x,t)$ is proportional to the temperature difference between the rod $u(x,t)$ and a known outside temperature $\gamma(x,t)$ (i.e., Newton's law of cooling). Derive

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{P}{A}[u(x,t) - \gamma(x,t)]h(x)$$

where $h(x)$ is a positive x-dependent proportionality parameter, P is the perimeter length of the rod, and A is the cross-sectional area.

(c) Write down the PDE above for the special case of a rod with constant thermal properties and zero outside temperature.

(d) Using the above PDE, suppose we include the additional assumption that the rod has uniform temperature: $u(x,t) = u(t)$. Solve the resulting ODE assuming $u(t = 0) = -7$.

Solution: (a)

$$H_{tot}(t) = \int_a^b c\rho A u(x,t)dx$$

The time dynamics of the total energy is determined by flux through the cross sections A and out through the lateral surface by the flux $w(x,t)$

$$dH/dt = \int_a^b c\rho A u_t(x,t)dx = A(\phi(a) - \phi(b)) + (b-a)Pw(x,t).$$

(b) The outside temp $\gamma(x,t)$ and newton's law of cooling:

$$dH/dt = \int_a^b c\rho A u_t(x,t)dx = A(\phi(a) - \phi(b)) + (b-a)Ph(\gamma(x,t) - u(x,t)).$$

Make sure to note that the u has the negative sign! The resulting PDE:

$$c\rho A\frac{\partial u}{\partial t} = AK_0\frac{\partial^2 u}{\partial x^2} + Ph[\gamma(x,t) - u(x,t)]$$

(c) $\gamma = 0$:

$$c\rho A\frac{\partial u}{\partial t} = AK_0\frac{\partial^2 u}{\partial x^2} - Phu(x,t)$$

(d) Uniform spatial temperature u_0 implies $u_{xx} = 0$, so

$$\frac{\partial u}{\partial t} = -\frac{Ph}{Ac\rho}u(x,t)$$

which has solution

$$u(t) = u_0 e^{-\frac{Ph}{Ac\rho}t}$$

4. Two rods of the same constant cross sectional area A, but of different material conductivity K_1 and K_2 are joined together at an interface x_0. The rods are in perfect thermal contact, meaning that (1), the temperature is continuous at $x = x_0$:

$$\lim_{x\to x_0^-} u(x,t) = \lim_{x\to x_0^+} u(x,t),$$

and (2), no heat energy is lost at the interface x_0. That is, heat energy flowing (i.e. fluxing) out of one rod at x_0 flows into the other and vice versa, meaning the flux on both sides is equal. Using the equal flux condition, find the value of the constant α in terms of K_1 and K_2 such that

$$\frac{\lim_{x\to x_0^+} u_x(x,t)}{\lim_{x\to x_0^-} u_x(x,t)} = \alpha$$

by equating the flux on each side of the interface.

Under what special condition on the two materials is the spatial derivative u_x continuous at x_0?

Solution: The PDE equation is

$$\begin{aligned} u_t &= k_1 u_{xx} \quad x \in [0, x_0] \\ u_t &= k_2 u_{xx} \quad x \in (x_0, L] \\ \text{Temperature continuity:} \quad \lim_{x\to x_0^+} u(x,t) &= \lim_{x\to x_0^-} u(x,t) \\ \text{Heat flux balance:} \quad \lim_{x\to x_0^+} \phi(x,t) &= \lim_{x\to x_0^-} \phi(x,t) \\ \implies \phi(x_0^+, t) = -K_2 u_x(x_0^+, t) &= -K_1 u_x(x_0^-, t) = \phi(x_0^-, t) \\ \implies \frac{u_x(x_0^+, t)}{u_x(x_0^-, t)} &= \frac{K_1}{K_2} = \alpha. \end{aligned}$$

So $\alpha = K_1/K_2$. Thus, for $\alpha \neq 1$, the flux is not continuous. This makes sense, as if the materials are two different properties, they would have to allow for different amounts of heat flow between in them. The $\alpha = 1$ case is that they are effectively the same material.

5. The rod $x \in [0,1]$ made of two materials joined at $x_0 = 1/4$, we specify $K_1 = 1$ and $K_2 = 2$. Suppose $u(x,0) = x(1-x)$ on the interval $x \in [0, x_0]$. Using flux balance as in the above problem, specify the value of the limit

$$\lim_{x\to x_0^+} u_x(x_0, 0) = ?$$

6. Solve the following heat equations $u_t = ku_{xx} + Q$ for $u(x)$ in equilibrium ($u_t = 0$) on the domain $x \in [0, 2]$, given the following:

   (a) $Q = 0,\ u(0) = -1,\ u(2) = 1.$

   (b) $Q = -x(2 - x),\ u(0) = 0,\ u(2) = 0.$

Solution: (a)

$$u(x) = -1 + \frac{x}{2}$$

(b)

$$\begin{aligned}
u(x) &= Ax^4 + Bx^3 + Cx^2 + Dx + E \\
u''(x) = 12Ax^2 + 6Bx + C = -x^2 + 2x \quad &\implies A = -1/12,\ B = 1/3\ C = 0 \\
u(x) &= \frac{-1}{12}x^4 + \frac{1}{3}x^3 + Dx + E \\
u(0) &= E = 0 \\
u(2) = \frac{-16}{12} + \frac{8}{3} + D2 = 0 \implies &D = \frac{\frac{16}{12} - \frac{8}{3}}{2}
\end{aligned}$$