Math 3140 – PDEs HW 5

Due: August 1, 2016

1. This questions attempts to display the duality of the Fourier transform. We kow that

$$\mathcal{F}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = -i\omega F(\omega),$$

which means derivatives in x transform to a multiplicative factor in ω, but for this problem, prove that

$$\mathcal{F}\left[xf(x)\right] = -i\frac{\mathrm{d}F}{\mathrm{d}\omega}.$$

In other words: multaplicative factors in x are derivatives in ω.

Solution: This problem wasn't meant to be super deep but rather just illustrate the idea: doing something in one framework (x or ω) and getting a result out in the other can always be reversed. We established in class that derivatives in x lead to multiplication by ω but here we show that multiplication by x leads to derivatives in ω. **This is a useful general principle: if you know a transform property in one domain, you know it in the other**.

Although there are a number of ways of doing this (integration by parts being the most straightforward), I'll show the *cutest* way here. Consider the Fourier transform of f, not xf

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,\mathrm{d}x.$$

However, we want something with $\mathrm{d}F/\mathrm{d}\omega$, so take a derivative on both sides of the equation with respect to ω to yield

$$\frac{\mathrm{d}F}{\mathrm{d}\omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty} ixf(x)e^{i\omega x}\,\mathrm{d}x,$$

since the only ω on the right hand side is in the exponential. Note we're almost there, we just need to move the i term over, which you would normally just divide by since it's a constant, but note that

$$\frac{1}{i} = \frac{1}{i}\cdot\underbrace{\frac{i}{i}}_{=1} = \frac{i}{i^2} = \frac{i}{-1} = -i.$$

Thus, we get our desired result that

$$-i\frac{\mathrm{d}F}{\mathrm{d}\omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty} xf(x)e^{i\omega x}\,\mathrm{d}x = \mathcal{F}\left[xf(x)\right].$$

2. Solve the advection-diffusion equation $u_t = u_{xx} + u_x$ on the real line, given initial concentration $u(x,0) = \delta(x)$. Use the Fourier transform.

Solution: We first Fourier transform both sides of the equation, noting that t derivatives move freely in and out of the transform and each derivative picks up a factor of $(-i\omega)$, which brings us to our standard ODE in t which we can integrate to find

$$\mathcal{F}(u_t = u_{xx} + u_x) = U_t = -\omega^2 U - i\omega U$$
$$U_t = -(\omega^2 + i\omega)U$$
$$U(\omega, t) = A(\omega)e^{-(\omega^2+i\omega)t} \qquad \text{where} \qquad U(\omega, 0) = A(\omega)$$

In other words, $A(\omega)$ is the Fourier transform of our initial condition, which means we must compute this

$$\mathcal{F}[\delta(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x)e^{i\omega x}\,\mathrm{d}\omega = \frac{1}{2\pi}e^{i\omega 0} = \frac{1}{2\pi}.$$

Note that we've used the property of the δ function that

$$\int_{-\infty}^{\infty} \delta(x-a)g(x)\,\mathrm{d}x = g(a),$$

where here the shift was zero so $a = 0$. To summarize, the Fourier transform of the δ function is a constant. This is a nice fact to know. Basically to produce an instanteous pulse, we need all frequencies. We can now write our full solution as

$$U(\omega, t) = e^{-i\omega t}\frac{1}{2\pi}e^{-\omega^2 t}$$

How do we invert this? Note that we can utilize the shift theorem, which says

$$\mathcal{F}^{-1}\left[e^{i\omega x_0}G(\omega)\right] = g(x - x_0).$$

Thus, we just need to shift the inverse Fourier transform of the Gaussian part, which is

$$\mathcal{F}^{-1}\left[\frac{1}{2\pi}e^{-\omega^2 t}\right] = \frac{1}{\sqrt{4\pi t}}e^{-\frac{-(x)^2}{4t}}$$

and now piecing this with the shift theorem, we finally get

$$u(x, t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{-(x+t)^2}{4t}}$$

Think about why this solution makes perfect sense. It's basically a Gaussian with moving mean $\mu = t$. Why is this the mean? We have an advection term with velocity $v = 1$. The diffusion term gives us a bell curve and the advection term pushes it to the right at speed $v = 1$, which is exactly what the solution says.

3. Solve the wave equation

$$u_{tt} = u_{xx}, \qquad u(x, 0) = u_0(x) = 0, \qquad u_t(x, 0) = v_0(x) = \delta(x)$$

where $x \in (-\infty, \infty)$ using **Fourier transforms**, not D'Alembert's formula.

Hint: you will end up with the product of two Fourier transforms, which you must untangle using the convolution theorem.

Solution: We did a problem in class almost identical to this problem. Note that yet again t derivatives move in and out of the transform and derivatives give you a factor of $(-i\omega)$, yielding

$$\tilde{U}_{tt} = (-i\omega)^2\tilde{U} = -\omega^2\tilde{U}.$$

Thus, we have a second order ODE in t

$$\tilde{U}'' + \omega^2\tilde{U} = 0,$$

where prime denotes time derivative. However, we know $\omega^2 > 0$, so we get complex roots, meaning our solutions are of the form

$$\tilde{U}(\omega, t) = A(\omega)\cos\omega t + B(\omega)\sin\omega t. \tag{1}$$

However, here is the step that things differ from the example in class. Note that we can Fourier transform our two initial conditions, which state

$$\tilde{U}(\omega, 0) = 0, \qquad \tilde{U}_t(\omega, 0) = \frac{1}{2\pi},$$

where I'm now using the transform of the δ function we found in the previous problem. Plugging in $t = 0$ to (1), we find

$$\tilde{U}(\omega, t) = A(\omega) = 0 \qquad \Longrightarrow \qquad A(\omega) = 0.$$

Next, we take a derivative of (1), which yields

$$\tilde{U}_t(\omega, t) = \omega B(\omega)\cos\omega t,$$

which at $t = 0$, we find

$$\tilde{U}_t(\omega, 0) = \omega B(\omega) = \frac{1}{2\pi} \qquad \Longrightarrow \qquad B(\omega) = \frac{1}{2\pi\omega}.$$

Thus, the Fourier transform of the solution to the PDE, in total,

$$\tilde{U}(\omega, t) = \frac{1}{2\pi\omega}\sin\omega t. \tag{2}$$

Something we can immediately note: if multiplying by a factor of ω yields a derivative, dividing (the inverse of multiplication) should be the inverse of a derivative: an integral.

Althoughh the hint says to use the convolution theorem, this may not be the easiest way in retrospect. Instead, rewrite $\sin\omega t$ as the exponential expression

$$\sin\omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

which means we can rewrite our Fourier transform as

$$\frac{1}{2\pi\omega}\sin\omega t = \frac{1}{2\pi\omega}\left[\frac{e^{i\omega t} - e^{i\omega t}}{2i}\right] = \frac{1}{4\pi i\omega}e^{i\omega t} - \frac{1}{4\pi i\omega}e^{-i\omega t}.$$

Why is this convenient? We can now use the shift theorem, to say that

$$u(x,t) = \mathcal{F}^{-1}\left[\tilde{U}\right] = g(x+t) - g(x-t),$$

where

$$g(x) = \mathcal{F}^{-1}\left[\frac{1}{4\pi i\omega}\right].$$

This inverse Fourier transform is non-trivial to compute, though. However, I'm going to use a trick. There are tons of ways you could have done this. It's even directly in the table.

We're going to use the derivative trick in reverse of how we typically do.

$$\mathcal{F}^{-1}\left[-i\omega G(\omega)\right] = \frac{\mathrm{d}g}{\mathrm{d}x}.$$

The reason for this is that we can't invert $1/\omega$ easily, but

$$i\omega g(\omega) = \omega\frac{1}{4\pi\omega} = \frac{1}{2}\cdot\frac{1}{2\pi}.$$

However, note this is exactly the Fourier transform of the δ function. Thus, we get that

$$-\frac{1}{2}\delta(x) = \frac{\mathrm{d}g}{\mathrm{d}x}.$$

Thus, we just need to integrate both sides to get the inverse fourier transform directly, $g(x)$. Note that this is exactly just the step function, $\Theta(x)$:

$$g(x) = \frac{1}{2}\Theta(x).$$

Hence, the solution to our PDE is then

$$u(x,t) = g(x+t) - g(x-t) = \frac{1}{2}\left[\Theta(x+t) - \Theta(x-t)\right],$$

which note is exactly

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t}\delta(\xi)\,\mathrm{d}\xi,$$

as D'Alembert's solution agrees with.

I'll also quickly demonstrate an alternative route that uses some ideas from class and the convolution theorem. Reexamine the form (2), which I'll rearrange to

$$\tilde{U}(\omega,t) = \frac{1}{2\pi}\cdot\frac{\sin\omega t}{\omega}.$$

However, we know the inverse Fourier transforms of each of these:

$$\mathcal{F}^{-1}\left[\frac{1}{2\pi}\right] = \delta(x), \qquad \mathcal{F}^{-1}\left[\frac{\sin\omega t}{\omega}\right] = \Theta(x-t).$$

Thus, we use the convolution theorem

$$\mathcal{F}^{-1}\left[F(\omega)\cdot G(\omega)\right] = f(x) * g(x),$$

which, after convolving gives you the same result.