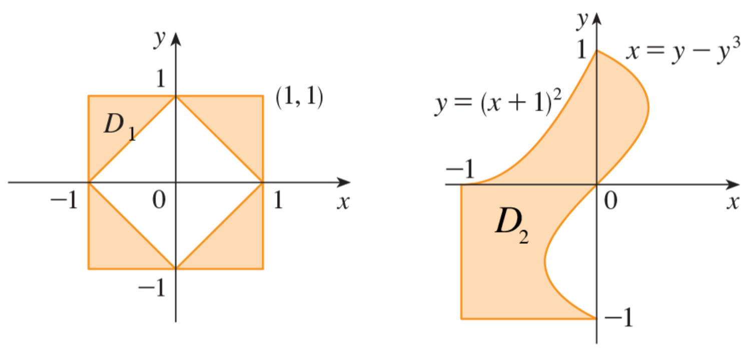
Name:____ Lab Score:_____/40

Answer each question in the area below. This assignment is due **one week** after the distribution of the lab, collected at the beginning of the next lab. Show all work and explain your reasoning. Due to the length of time allowed to complete the assignment, your work is expected to be clear and polished. If the work is at all ambiguous, it is considered incorrect.

1. Consider the regions D_1, D_2 shown below.



For each, set up the integral as a type I or type II integral and then evaluate.

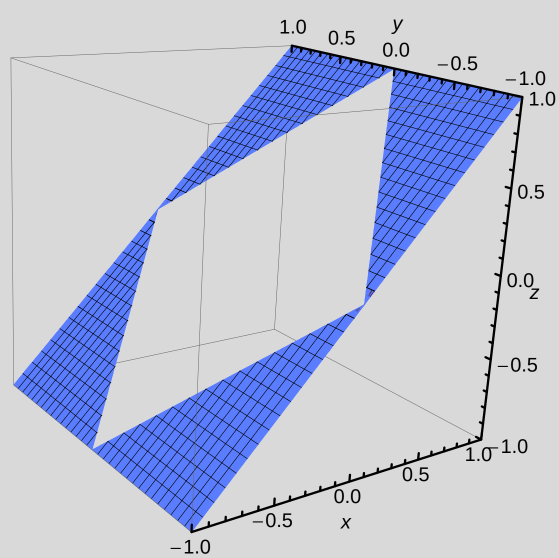
5 (a) $\int_{D_1} x \, \mathrm{d}A$.

5 (b) $\int_{D_2} y \, \mathrm{d}A$.

Solution:

(a) Although D_1 can be split into 4 regions and each integral can be computed individually, we can make a simplifying observation. Note that the quantity (height) of the surface is just x, our integrand. Thus, the right two regions have the same base (shape) and x values, so their integral will produce the same value. The same can be said from the left two.

This hopefully is more apparent if we plot $z = x$ (the height) over the region:



Thus, we need to just compute say, the top left and the top right integral. For the top right, we can set this up as a type 1 or type 2 region:

$$\int_0^1 \int_{1-x}^1 x \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 \int_{1-y}^1 x \, \mathrm{d}x \, \mathrm{d}y.$$

Expanding the first out:

$$\int_0^1 \left[\int_{1-x}^1 x \, \mathrm{d}y \right] \mathrm{d}x = \int_0^1 x^2 \, \mathrm{d}x = \frac{1}{3}.$$

Thus the total right hand side contributes 2/3.

The top left integral can be written as type 1 or type 2:

$$\int_{-1}^0 \int_{x+1}^1 x \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 \int_{-1}^{1-y} x \, \mathrm{d}x \, \mathrm{d}y.$$

Again, choosing the first, we have

$$\int_{-1}^0 \left[\int_{x+1}^1 x \, \mathrm{d}y \right] \mathrm{d}x = \int_{-1}^0 -x^2 \, \mathrm{d}x = -\frac{1}{3}.$$

In other words, the total contribution from the left is $-2/3$ and the total integral is therefore 0. Why is this the case? Well, due to the symmetry, the positive x on the right cancels nicely will all the negative x's on the left. This shouldn't be too big of a surprise.

(b) There's really only one way I can think to do this one sanely. We want to take horizontal slices (specify x, so type 2) and therefore split our integral into two parts, separated by $y = 0$.

Thus, on the top, we see that the left boundary is $y = (x+1)^2$ and the right is $x = y - y^3$. However, we need both to be in a form of $x = \cdots$ since that's what we're describing by "left" and "right". Therefore, we see

$$y = (x+1)^2 \implies \pm\sqrt{y} - 1 = x.$$

It's clear here we really want the positive square root, so our integral for the top region becomes:

$$\int_0^1 \left[\int_{\sqrt{y}-1}^{y-y^3} y \, \mathrm{d}x \right] \mathrm{d}y == \int_0^1 y^2 - y^4 - y^{3/2} + y \, \mathrm{d}y = \frac{7}{30}.$$

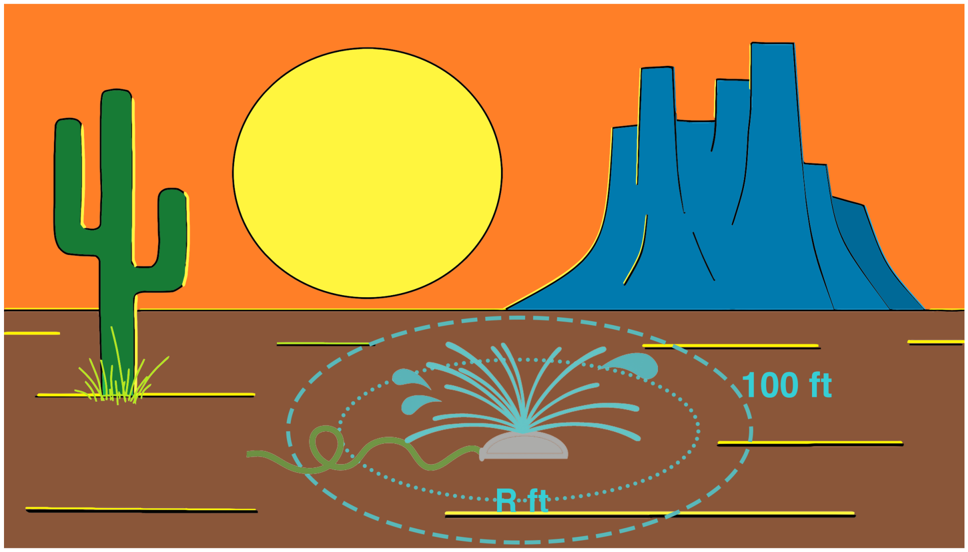
For the bottom region, we see that the right side is still $x = y - y^3$ but the left is now always $x = -1$. Thus, we have:

$$\int_{-1}^0 \left[\int_{-1}^{y-y^3} y \, \mathrm{d}x \right] \mathrm{d}y = \int_{-1}^0 y^2 - y^4 + y \, \mathrm{d}y = -\frac{11}{30}.$$

Combining these, we get:

$$\iint_D y \, \mathrm{d}A = -\frac{11}{30} + \frac{7}{30} = -\frac{2}{15}.$$

2. Since we live in a desert, watering your lawn with a sprinkler is a smart, environmentally friendly thing to do. Suppose you buy the fanciest sprinkler you can find. It distributes water in a radius of 100 feet and supplies e^{-r} feet per hour of water at a distance r away from the sprinkler.



6 (a) For a fixed distance R such that $0 < R \leq 100$, what is the total amount of water supplied per hour to the region inside the circle of radius R centered at the sprinkler?

2 (b) Determine an expression for the *average* amount of water per hour, per square foot supplied to the region inside the circle of radius R.

2 (c) Use part (b) to determine the average amount of water per hour for the entire sprinkler capacity.

Solution:

(a) I put this problem on the lab not because it's difficult or particularly deep, but just to illustrate that the "height" that you're adding up in a 2D integral may not be a physical height at all, but rather just some quantity you want to add up. Here, that quantity is the amount of water.

We know that the water is delivered a distance r at a rate e^{-r} so the total amount of water inside a circle of radius R can just be found by the double integral in polar coordinates

$$\int_0^{2\pi}\int_0^R e^{-r} r \, dr \, d\theta,$$

where we have importantly remembered the factor of r because of our formulation in polar. Computing this via integration by parts:

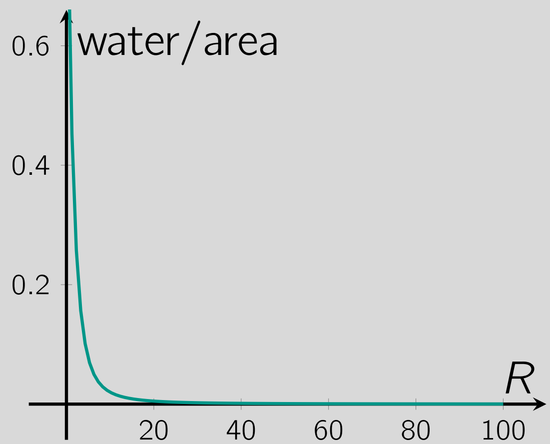
$$\int_0^{2\pi}\int_0^R e^{-r} r \, dr \, d\theta = 2\pi \int_0^R e^{-r} r \, dr = 2\pi \left[1 - e^{-R}(1+R)\right].$$

(b) Although this is really the total amount of water delivered to a circle of radius R, this is a poor measure of how good of a sprinkler this is. We really want to know its efficiency, or the average amount of water distributed per area. So, we observe that the area of this circle is $\pi R^2$and therefore

$$\text{water per area} = \frac{2\pi\left[1 - e^{-R}(1+R)\right]}{\pi R^2} = \frac{2 - 2e^{-R}(1+R)}{R^2}.$$

This is a much more reasonable useful to measure.

(c) In retrospect, I should have had you plot the above function from $R = 0, \ldots, 100$ but regardless I'll show you the result:



In other words, the larger area you are trying to water: the worse this sprinler is. At $R = 100$, you get approximately .0002 feet per hour, per foot. Not a great ratio.

3. The *indefinite* integral of the **Gaussian function**

$$\int e^{-x^2}\,\mathrm{d}x$$

is in general, challenging to evaluate since it has no explicit antiderivative. Despite this, this integral is of interest for many applications, such as:

1. probability/statistics, where this curve describes the standard "bell curve", known as the *normal distribution*
2. PDEs: we'll see this is the solution to the heat/diffusion equation
3. signal processing: there are (noise) filters that use this as the basic building block, known as Gaussian filters

In this problem, we'll evaluate a particular *definite* integral:

$$I := \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x.$$

[2] (a) Show that

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,\mathrm{d}A. \qquad (3.1)$$

[5] (b) By convering to polar coordinates, compute (3.1).

[2] (c) From the result in part (b), conclude the value of I.

[1] (d) By taking a change of variables $z = \sqrt{2}x$, show that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x = 1,$$

which is a fundamental result in probability/statistics.

Solution:

(a) If we take I^2 and realize that it's just the product of two copies of I:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right),$$

we realize that we can really use whichever dummy variable we'd like, so call one of them y, and we get a new insight:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y\right) = \int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}A,$$

the desired result. In other words, we've transformed I^2 into a double integral that we can now study.

(b) Converting this to polar, noting that $x^2+y^2=r^2$ and we need the factor of r, we get:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right) = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Evaluating this by a u substitution $u = r^2$:

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta. = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{r=\infty} = \pi.$$

(c) Since $I^2 = \pi$, we can conclude $I = \sqrt{\pi}$. Notice that we **cannot** evaluate I directly, but we know what the value is through this cute argument.

(d) I only added this calculation because this is a form you'll much more commonly see. If we take

$$I = \int_{-\infty}^{\infty} e^{-z^2}\,\mathrm{d}z = \sqrt{\pi},$$

we can now take $z = \sqrt{2}x$ to pick up the factor:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\frac{1}{\sqrt{2}}\,\mathrm{d}x = \sqrt{\pi},$$

so we can conclude the final result

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x = 1.$$

4. Xavier and Yolanda match on Tinder and immediately hit it off. They decide to meet in person for a date at at 9 PM. Denote the random variable describing Xavier's arrival time as X and the random variable describing Yolanda's arrival time as Y, where X, Y are measured in minutes after 9 PM.

   Xavier arrives sometime after 9 PM and is more likely to arrive promptly than late. Yolanda will more likely be late than prompt, but will only arrive a maximum of 10 minutes late. You can assume their arrivals are independent and have probability density functions

$$f_1(x) = \begin{cases} e^{-x} & \text{if } x \geq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad f_2(y) = \begin{cases} \frac{1}{50}y & \text{if } 0 \leq y \leq 10 \\ 0 & \text{otherwise.} \end{cases}$$

   After Yolanda arrives, she'll wait up to half an hour for Xavier, but he won't wait at all for her.

3 (a) Verify that f_1, f_2 are indeed probability density functions.

7 (b) Find the probability that Xavier and Yolanda successfully meet.

Solution:

(a) As we established in class, there are really two requirements for a function $f(x)$ to be a probability density function. For one, we don't want negative probabilities (since this makes no sense), but here, it's very obvious that $f_1(x)$ and $f_2(y)$ are non-negative in their domains. The other slightly more interesting thign to check is that they integrate to 1. Checking that explicitly:

$$\int_{-\infty}^{\infty} f_1(x)\,\mathrm{d}x = \int_0^{\infty} e^{-x}\,\mathrm{d}x = \left[-e^{-x}\right]_{x=0}^{x=\infty} = 1.$$

And for the second density:

$$\int_{-\infty}^{\infty} f_2(y)\,\mathrm{d}y = \int_0^{10} \frac{1}{50}y\,\mathrm{d}y = \left[\frac{y^2}{25}\right]_{y=0}^{y=10} = 1.$$

Thus, these are indeed probability density functions.

(b) Since the two events are *independent*, their joint probability density is then

$$f(x,y) = f_1(x)f_2(y) = \frac{1}{50}e^{-x}y.$$

To compute the probability of some range of x, y occuring, we need to consider how to describe those set of events. Here, we're given a few pieces of information. The first is that after Yolanda arrives, she'll wait up half an hour for Xavier. In other words, he must arrive BEFORE 30 minutes after she arrives, or

$$X \leq Y + 30.$$

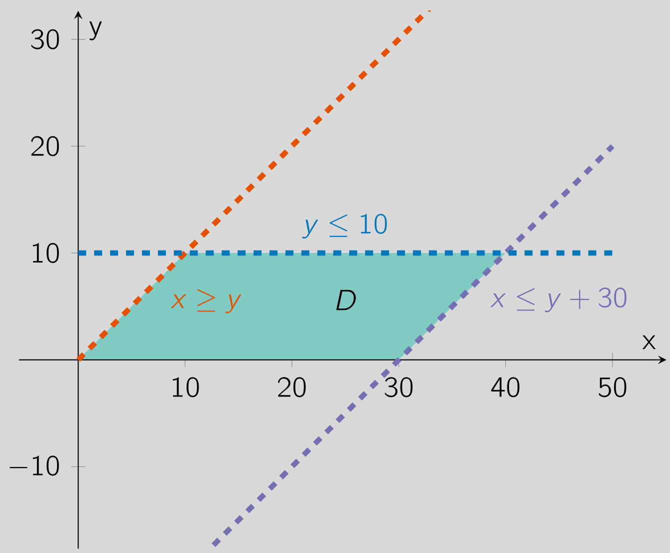
Secndly, he won't wait for her, meaning that she must arrive BEFORE he does, or

$$Y \leq X.$$

We also know that

$$Y \leq 10$$

since that is the latest she can be. Piecing these all together gives us our region D that describes the set of events where they meet, seen below.



Now, to compute the actual probability of this event occuring, we just integrate the joint density

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_D \frac{y}{50}e^{-x}\,\mathrm{d}A.$$

Here, we can set this up as either type 1 or type 2, but I think it's easiest to specify the range of x, so we get

$$\int_0^{10}\int_y^{y+30} \frac{y}{50}e^{-x}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{10} \frac{1}{50}e^{-30-y}(-1+e^{30})y\,\mathrm{d}y = \frac{1}{50e^{40}}\left(-11+e^{10}\right)\left(-1+e^{30}\right),$$

which is approximately equal to 0.01999 or a 2% chance of this meeting occuring.