Name:______________________________ Lab Score:_____/40

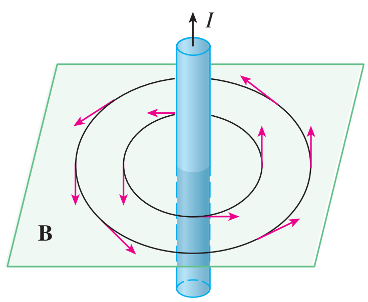
Answer each question in the area below. This assignment is due **one week** after the distribution of the lab, collected at the beginning of the next lab. Show all work and explain your reasoning. Due to the length of time allowed to complete the assignment, your work is expected to be clear and polished. If the work is at all ambiguous, it is considered incorrect.

1. Passing an electric current I through a long wire produces a magnetic field **B** that is tangent to any circle that is perpendicular to the wire with center at the axis of the wire.

   Ampére's Law states that

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I,$$

   where μ_0 is a constant called the *permeability of free space*, which basically measures a magnetic fields ability to propogate through air.



4 (a) Define the magnitude of the magnetic field

$$\mathcal{B} = \|\mathbf{B}\|.$$

From the geometry of the problem, consider the magnetic field at a distance r units away from the wire and find a **unit vector** such **u** that

$$\mathbf{B} = \mathcal{B}\mathbf{u}.$$

6 (b) Using your answer from part (a), show that

$$\mathcal{B} = \frac{\mu_0 I}{2\pi r}.$$

2. Recall that Green's Theorem can be written as

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s = \iint_D \nabla \cdot \mathbf{F} \, \mathrm{d}A.$$

5 (a) Using the formulation of Green's theorem, show *Green's first identity*, a theorem used incredibly commonly in PDEs, which states

$$\iint_D f \nabla^2 g \, \mathrm{d}A = \oint_C f(\nabla g) \cdot \mathbf{n} \, \mathrm{d}s - \iint_D \nabla f \cdot \nabla g \, \mathrm{d}A,$$

where, D, C satisfy the conditions of Green's theorem.

0 (b) **Not a question but a useful fact:**

$$\nabla g \cdot \mathbf{n} = D_n g,$$

which appears in the above expression is often called the **normal derivative** of g.

5 (c) Use Green's first identity to prove *Green's second identity*, which states

$$\iint_D \left(f\nabla^2 g - g\nabla^2 f\right)\, \mathrm{d}A = \oint_C (f\nabla g - g\nabla f)\cdot \mathbf{n}\, \mathrm{d}s,$$

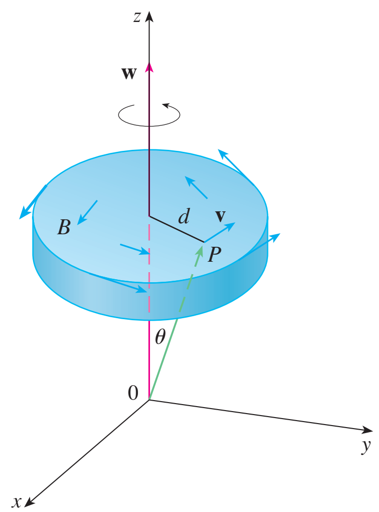
where again, D, C satisfy all the conditions of Green's theorem.

3. In this problem, we'll make more apparent the relationship between curl and rotations. Let B be a blob rotating around the z axis. This rotation can be described by the vector

$$\mathbf{w} = \omega \mathbf{k},$$

where ω is the angular speed of B, that is $\omega = v/d$ for some tangential speed v at a point P a distance d units away. Let the arbitrary point P be described by the vector

$$\mathbf{b} = \langle x, y, z \rangle.$$



3 (a) Considering θ in the figure above, show that the velocity field of our blob B can be described by

$$\mathbf{v} = \mathbf{w} \times \mathbf{b}, \qquad \text{where} \qquad v = \|\mathbf{v}\|.$$

3 (b) Show that

$$\mathbf{v} = \langle -\omega y, \omega x \rangle.$$

3 (c) Finally, show that

$$\operatorname{curl} \mathbf{v} = 2\mathbf{w}.$$

1 (d) Conclude what the curl represents in this physical situation.

4. Consider a ball of radius a whose temperature is inversely proportional to the distance from the center.

1 (a) Write down an expression for the temperature of the ball $u(x, y, z)$.

2 (b) Suppose the ball has conductivity K, then the **heat flow** is defined to be

$$\mathbf{F} = -K\nabla u.$$

That is, heat flows against the gradient of temperature. *Think about why this physically makes sense.* Compute $\mathbf{F}$ for this ball.

3 (c) What is the unit normal vector $\mathbf{n}$? *You should get a nice expression.*

4 (d) Compute the **heat flux** across the surface of the ball. *You could do this in parameterized variables, but I recommend* x, y, z. *You'll see why.*