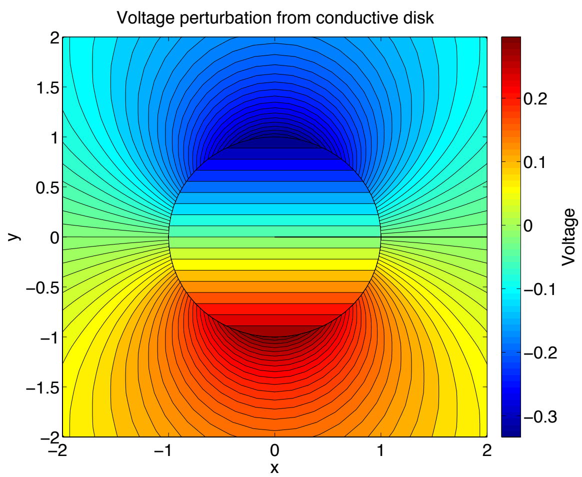
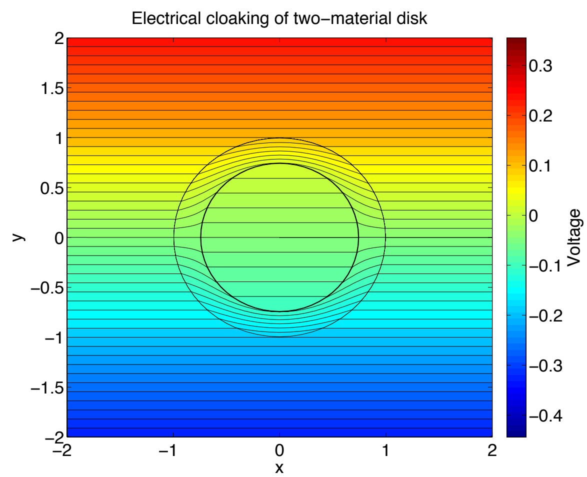
Math 3140 Project - Electronic Cloaking





Introduction

In science fiction, a cloaking device is something that makes an object invisible to a sensor for a physical field, be it radar, vision, sonar, or electric. In this project, you will study the effects of inhomogeneities in conducting materials on the potential due to an applied electric field, and how to make objects "invisible" to electric fields. This project has applications to devices that detect and count blood cells. Such detection problems also arise in nature: some aquatic animals, including sharks and other fish, can sense disturbances in electric fields produced by prey animals for hunting purposes.

Mathematical background

Consider an unbounded 2D homogenous material with an homogenous conductivity σ^* at all points, and no charge density anywhere. The electrical potential $w(r,\theta)$ in this material satisfies the Laplace equation

$$\nabla \cdot (\sigma^* \nabla w) = 0,$$

where the inner factor $\sigma^* \nabla w$ is the electric flux that dictates current flow, where the far-field boundary condition at $r \to \infty$ approaches $\mathbf{E} = \nabla w = (0,1)$, which is a constant upward pointing field. We can express this boundary condition in polar coordinates as $\lim_{r\to\infty} w_r(r,\theta) = \sin(\theta)$. We can consider this far-field condition as a field that we experimentally apply to the medium. Naturally, the solution to this equation is the simple potential $w(x,y) = y$, which does not depend on the conductance σ^*. We can also write this solution in polar coordinates

$$w(r,\theta) = r\sin(\theta). \tag{1}$$

Because everywhere the conductance is uniform, the field simply passes through vertically with no distortion.

Detection of electrical inhomogeneities in the medium

Now consider that the material possesses a disk-shaped inclusion of a higher conductance material. The inclusion is centered at the origin with radius r_1 and conductance $\sigma_1 > \sigma^*$. The Laplace equation in this situation is expressed as

$$\nabla \cdot (\sigma(r)\nabla w) = 0, \tag{2}$$

with the conductance now a function of r:

$$\sigma(r) = \begin{cases} \sigma_1, & 0 \le r < r_1 \\ \sigma^*, & r_1 \le r < \infty \end{cases}. \tag{3}$$

The higher conductance inclusion will admit more electrical current through it than elsewhere, and a voltage sensor positioned nearby the inclusion should detect a perturbation from the background field.

Cloaking the disk

To cloak the disk, now consider two heterogeneous materials centered at the origin. Just as above, there is an inner disk with radius r_1 of material with the higher conductivity σ_1 than the outside medium σ^*. Additionally, we include an outer annulus with radius $r_2 > r_1$ with conductivity σ_2 that is more insulating than the outside media σ^*. We wish to find a combination of values for r_1, r_2, σ_1 and σ_2 such that the potential outside the inclusion $r > r_2$ is the same as the potential without the inclusion listed above in (1). That is, this disk inclusion, does not disturb the electric field outside the disk, so any sensor that measures the electric field at a distance will detect no perturbation of the field, so it is as if the disk inclusion is not present at all. That is, its cloaked! The conductance function is now

$$\sigma(r) = \begin{cases} \sigma_1, & 0 \le r < r_1 \\ \sigma_2, & r_1 \le r < r_2 \\ \sigma^*, & r_2 \le r < \infty \end{cases}. \tag{4}$$

To solve this problem of making the inclusion invisible, we must first solve for the electric potential for with parameters σ^*, r_1, r_2, σ_1 and σ_2 unspecified, and then find a specific set of these parameters that satisfies the condition that $w(r, \theta) = r\sin(\theta)$ for $r > r_2$.

Composite materials made up of these coated circle inclusions may be useful in applications where electric fields must be unperturbed by materials but also must be lighter or stronger, or less dense, than a uniform conducting medium σ^*.

This project is broken into two phases. In the first phase, you will solve for the for the perturbed voltage of the higher conductance disk and mathematically model a sensor to detect the inclusion. In the second phase, we apply the outer jacket of insulating material to the disk and solve the equations in order to cloak the inclusion.

Phase One Questions

1. Separation of variables and eigenvalue analysis. We will break the work down into parts:

   (a) Separate variables $w^i(r,\theta) = \phi_i(r)\gamma_i(\theta)$ on each domain, where $i = 1$ is the inner domain and $i = 2$ is the outer domain superscript (not an exponent). Solve the γ-equation on the outer domain and list all eigenvalues λ and eigenfunctions.

   (b) For the outer domain, solve the ϕ-equation for all eigenvalues. Which solutions (associated with certain eigenvalues) are consistent with the far-field boundary condition?

   (c) Write out the list of all admissible outer solutions $w_n^2(r,\theta) = \phi_n(r)\gamma_n(\theta)$ consistent with the far field boundary condition. Write them out as a linear combination with undetermined coefficients (i.e., a Fourier series).

   (d) On the inner domain, solve the ϕ-equation for the $\lambda_1 = n^2 = 1$ eigenvalue only. Hint: there should be two linearly independent solutions. We require that the voltage in the inner disk is bounded. Use this condition to exclude one of the solutions. Justify your assertion.

   (e) On the inner domain, solve the γ-equation for the $\lambda = n^2 = 1$ eigenvalue only and write out the admissible $w_1^1(r,\theta) = \phi_1(r)\gamma_1(\theta)$ solutions (there should be two). Write them out as a linear combination with undetermined coefficients.

   (f) Boundary condition part one: At the interface $r = r_1$ we require that the voltage is continuous. For the $n^2 = 1 = \lambda_1$ eigenvalue solutions, express the continuity condition to obtain an equation for the aforementioned undetermined coefficients.

   (g) Boundary condition part two: At the interface $r = r_1$, we require that the electric flux is also continuous. Using the inner and outer flux expressions, obtain another equation for the undetermined coefficients in a similar manner for the $n^2 = 1 = \lambda_1$ eigenvalue solutions. On a circle, the component of the gradient that crosses the interface is normal to the circle, in the radial direction. Hence, radial derivative w_r is magnitude of the gradient crossing the interface.

   (h) Using the two equations obtained above, solve for the undetermined coefficients in terms of the conductances σ^* and σ_1, and r_1. Write down your complete solutions for w^1 and w^2.

   Solution:

$$w^1(r,\theta) = (1+\beta) r \sin(\theta)$$
$$\beta = \frac{1 - \frac{\sigma^*}{\sigma_1}}{1 + \frac{\sigma^*}{\sigma_1}}$$
$$w^2(r,\theta) = (r + \frac{\beta}{r}) \sin(\theta)$$

(i) Choose values σ^* and σ_1, and r_1 that you like and plot your results. To aid in these plots, you can utilize the following Matlab code template.

```
% Create polar domain points r and theta
[r,th] = meshgrid(0:.01:sqrt(8),0:pi/30:(2*pi));
% Convert to Cartesian
x = r.*cos(th);
y = r.*sin(th);
w = r.*sin(th); %unperturbed solution.  Put your own w-solution
here
figure(1)
clf; %clears figure
contourf(x,y,w,61) %makes colored contour plot with 61 lines
xlim([-2 2]) % limits visible x-range
ylim([-2 2]) % limits visible y-range
set(gca,'fontsize',18) % sets axis fontsize
xlabel('x')
ylabel('y')
title('Title your plot')
t = colorbar('peer',gca); %sets the color bar graphics handle
set(get(t,'ylabel'),'String', ' Voltage'); %labels the color bar
```

2. Using the solution you obtained above, find the perturbation from the undisturbed solution (1) and express the perturbed solution $p(x,y)$ in cartesian coordinates.

Solution:

$$p(r,\theta) = \frac{\sin(\theta)}{r}$$

so the dipole has a squared dependency while the conducting disk has a first-power dependency.

3. Suppose you have placed an array of voltage sensors that can read the value of $p(x, y)$ on a horizontal line at a unknown y-distance from the inclusion. How would you determine the location of the inclusion by using the data from the sensor array?

4. The disk inclusion as a dipole-like potential: Suppose you have a single point mass of positive charge q and a single point mass of negative charge $-q$. You position the positive one at the point $(0, \Delta y)$ and the negative one at $(0, 0)$. Using the law of electrostatics for point charges, determine the aggregate potential V of the two point charges are a small but non-zero distance Δy apart. This is termed the dipole potential. Approximate the dipole potential by approximating the monopole potential difference as a derivative. Such an approximation is valid for points (x, y) that are much farther from the origin than Δy.

Solution: A single potential at the origin is proportional to

$$\frac{q}{r} = \frac{q}{\sqrt{x^2 + y^2}} = f(x, y).$$

We difference two of them in their positions

$$\begin{aligned} V(x, y) &= \frac{q}{\sqrt{x^2 + (y - \Delta y)^2}} - \frac{q}{\sqrt{x^2 + (y)^2}} \\ &\approx -\frac{\partial}{\partial y} f(x, y)\Delta y \\ &= \frac{\Delta y}{2} \frac{q}{(x^2 + y^2)^{\frac{3}{2}}} 2y \\ &= \frac{(\Delta y) q y}{(x^2 + y^2)^{\frac{3}{2}}} \end{aligned}$$

5. Express the dipole potential in polar coordinates and compare it to the perturbed potential solution $p(r, \theta)$ due to the disk material. What type of dependence on r do each have?

Solution:

$$V(r,\theta) \approx \frac{r\sin(\theta)}{r^3} = \frac{\sin(\theta)}{r^2}$$
$$p(r,\theta) = \frac{\sin(\theta)}{r}$$

so the dipole has a squared dependency while the conducting disk has a first-power dependency.

6. Suppose you position a voltage sensor at a distance at a position y below and x horizontal distance the center of the inclusion. The sensor is swept across the x variable in an effort to detect if the inclusion is present. For every distance $y > r$, indicate what error tolerance the sensor requires in order to detect the if inclusion is present or not. That is, what is the difference between dipole potential w and the reference potential as a function of y below the center of the inclusion?

Phase Two Questions

1. Assume the solution can be separated on each domain $w^i(r,\theta) = \phi_i(r)\gamma_i(\theta)$, where $i = 1, 2, 3$ for the innermost disk, middle annulus, and outer media, respectively—note the i is a superscript, not an exponents. Introduce this functional form into the PDE and separate into two DEs for ϕ and γ individually just as in phase one. For the outer media $i = 3$, find the eigenvalue(s) that define the required solution $w^3(r,\theta) = r\sin(\theta)$, by finding solutions to the γ equation.

Solution: On the interior of each domain, σ is constant so it vanishes from the problem.

$$\nabla \cdot [\sigma \nabla w] = \nabla^2 w = 0$$
$$r[r\phi_r\gamma]_r + \gamma_{\theta\theta}\phi = 0$$
$$r(\phi_r + r\phi_{rr})\gamma + \gamma_{\theta\theta}\phi = 0$$
$$\frac{r(\phi_r + r\phi_{rr})}{\phi} = -\frac{\gamma_{\theta\theta}}{\gamma} = \lambda\gamma(\theta) = \sin(\theta) \implies \lambda = 1$$

2. For $i = 3$ and the eigenvalue(s) you found from above, find all linearly independent solutions (there are two) to the ϕ-equation, which is termed the Cauchy-Euler DE,

who's solution is in your textbook. Use the far-field boundary condition to rule out one of them and solve for the exact solution.

> **Solution:** $\phi(r) = \alpha r + \beta r^{-1}$, but far field requires $\alpha = 1,\ \beta = 0$.

3. For the $\lambda = 1$ eigenvalue, find the two linearly independent solutions on the interior annulus, and the one admissible solution for the interior disk. Express these solutions as linear combinations with unknown coefficients A, B, and C.

4. Now, to find the solution for the interior annulus and inner disk we must derive equations to solve for A, B, and C as a function of the physical parameters r_1, r_2, σ_1, σ_2, and σ^*. There are four such equations, two of which by specifying flux matching conditions on each of the two boundaries.

Solution:

$$\lim_{r\to r_i^+} \sigma(r) w_r(r) = \lim_{r\to r_i^-} \sigma(r) w_r(r)$$
$$r(\phi_r + r\phi_{rr}) - \phi = 0$$
$$\sigma^* \phi'(r_2^+) = \sigma_2 \phi'(r_2^-)$$
$$\sigma_2 \phi'(r_1^+) = \sigma_1 \phi'(r_1^-)$$
$$\sigma^* \phi'(r_2^+) = (r)'\sigma^* = \sigma^*$$

The solution in the middle annulus

$$\sigma^* \phi'(r_2^+) = \sigma^*(r)' = \sigma^*$$
$$r(\phi_r + r\phi_{rr}) - \phi = 0\phi(r) = Ar + Br^{-1} + C$$
$$\phi' = A - B/r^2$$
$$\phi'' = +2B/r^3$$
$$r(A - B/r^2 + r(2B/r^3)) - Ar - Br^{-1} = Ar - Ar - B/r + 2B/r - B/r = 0$$
$$\sigma_2(A - B/r_2^2) = \sigma^*$$

Flux into inner disk from annulus: $\sigma_2(a - b/r_1^2)$. Now go and solve inner disk:

$$\phi(r) = Dr + F$$
$$\phi' = D$$
$$\sigma_1 D = \sigma_2(A - B/r_1^2)$$

5. Find the remaining two equations for the coefficients A, B, and C by requiring that the solutions on the three domains are continuous.

6. Using the four equations you obtained in the previous quesiotns, solve for the three coefficients A, B, and C as a function of the physical parameters. Obtain an expression for the squared ratio of the radii in terms of the three conductances:

$$\left(\frac{r_2}{r_1}\right)^2 = f(\sigma_1, \sigma_2, \sigma^*) = ?$$

   The equations can be quite large, so its recommended to use a computer algebra system like Maple or Mathematica so you do the algebra exactly without mistakes. This solution and the resulting coefficients guarantee a solution that cloaks the disk. To achieve these results in Maple, you can use the "solve" command, which we demonstrate as follows for a simple example.

```
restart;
exp1:= 2A+B+1=0 #a simple linear expression for A and B
exp2:= B-A +2=0 # another equation
B:= solve(exp1,B) #solves for B in terms of A
A:=solve(exp2,A) #solves for A by using B defined above
```

7. Choose a σ^* value and a inner conductance $\sigma_1 > \sigma^*$. Plot a graph of the $\left(\frac{r_2}{r_1}\right)^2$ value as a function of different σ_2 values. What must be true about σ_2 to guarantee a realistic solution?

8. In the above graph that you created, is there a specific size of disk r_2 that is required for the solution to be cloaked? Explain your reasoning why or why not.

9. Choose values for all parameters and find the required r_1 and r_2 values that enable the disk to be cloaked. Plot your the potential w by adapting the Matlab code used in phase one. Also, describe how current will flow through the disk by drawing on a print out of the color-coded potential plot.

10. Suppose a material was made of many of these coated inclusions with radii r_1 and r_2 found in the previous problem. The inclusions and the surrounding medium form a composite material, with the inclusions packed in the medium like stacked oranges in the supermarket. Specify what the potential w will be everywhere outside the circle inclusions.