Name:____ Quiz Score:_____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

[2] 1. What were the two fundamental ingredients for the derivation of the heat equation? *Hint:* you could call both of these "laws".

Solution: We started with a statement about the heat energy $e(x, t)$ density (per volume), which means in the interval $[a, b]$, the total heat energy is then

$$H_{tot} = \int_a^b e(x, t) A \, \mathrm{d}x,$$

and therefore, the rate at which it changes in time is

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{tot} = \frac{\mathrm{d}}{\mathrm{d}t} \int_a^b e(x, t) A \, \mathrm{d}x = \int_a^b \frac{\partial e}{\partial t} A \, \mathrm{d}x.$$

At this step we use the **Conservation of Energy** to say the total energy can change in two ways

$$\int_a^b \frac{\partial e}{\partial t} \, \mathrm{d}x = \underbrace{\phi(a, t)A(a) - \phi(b, t)A(b)}_{\text{fluxing through ends}} + \underbrace{Q(x, t)A}_{\text{generated/destroyed}}$$

Using the Fundamental Theorem of Calculus, we can rewrite this as

$$0 = \int_a^b \frac{\partial e}{\partial t} A + \frac{\mathrm{d}\phi A}{\mathrm{d}x} - QA \, \mathrm{d}x,$$

and since this is true for *all* intervals $[a, b]$, the integrand must be zero and we conclude

$$\frac{\partial e}{\partial t} A + \frac{\mathrm{d}\phi A}{\mathrm{d}x} - QA = 0.$$

We note that $e = c\rho u$, which relates the energy e and temperature u through the specific heat c and mass density ρ, but we still need **Fourier's Law**, which tells us about the explicit form of the flux:

$$\phi = -K_0 \frac{\partial u}{\partial x},$$

where K_0 is called the conductivity. Thus, piecing this together, we have

$$c\rho \frac{\partial u}{\partial t} A = \frac{\mathrm{d}}{\mathrm{d}x} \left\{ K_0 \frac{\partial u}{\partial x} \right\} + QA,$$

which if A, c, ρ, K_0 are all constants and $Q = 0$, reduces to

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2},$$

the classical heat equation.

3 2. What are the three possible types of boundary conditions we can specify for PDEs?

Solution: I was hoping you would provide more general answers, but in the specific context of the heat equation (as that's the only PDE we've seen), these were fine. For some PDE for $u(x,t)$ we can specify three types of boundary conditions at $x = \alpha$:

1. Dirichlet, which is of the form $u(\alpha, t) = g(t)$, says you are fixing u to be something at the boundary. In the context of the heat equation, this is saying we are fixing the temperature to be something prescribed $g(t)$.
2. Neumann, which is of the form $\frac{\partial u}{\partial x}(\alpha, t) = h(t)$, which says we are fixing the *derivative* of u to be something at the boundary. In the context of the heat equation, we know the flux at the boundary is $-K_0 u_x(\alpha, t)$, so we can fix this to be say, insulated (flux=0), which means $-K_0 u_x(L, t) = 0$ so $u_x(L, t) = 0$. This is sometimes called a "no-flux" boundary condition.
3. Robin, which is a mixture of the previous two is some function of u and its derivative at the boundary, so $f(u, u_x) = f(t)$. An example for the heat equation would be using Newton's Law of Heating/Cooling for the boundary, saying that the flux is proportional to the difference between the temperature of the rod and some outside temperature, γ, so

$$-K_0 u_x(L, t) = -H\left\{u(L, t) - \gamma(t)\right\}.$$

Not that this relationship includes *both* $u(L)$ and $u_x(L)$, making it a Robin condition.

5 3. Solve for the equilibrium temperature distribution for the driven heat equation

$$u_t = k u_{xx} + Q(x, t), \qquad Q(x, t) = \cos x - 1,$$

with the initial and boundary conditions

$$u(x, 0) = e^{\sin x}, \qquad u(0, t) = 1, \qquad u(2\pi, t) = 2.$$

Solution: Equilibrium means that as $t \to \infty$, $\frac{\partial u}{\partial t} \to 0$, that is, the distribution is not changing with time. Our equilibrium distribution $u(x)$ then becomes, assuming $k = 1$

$$0 = u_{xx} + \cos x - 1 \qquad \Longrightarrow \qquad u_{xx} = -\cos x + 1.$$

Integrating once, we get

$$u_x = -\sin x + x + c_1,$$

and again

$$u = \sin x + \frac{x^2}{2} + c_1 x + c_2.$$

The equilibrium distribution must still satisfy the boundaries, so

$$u(0) = 1 = \sin 0 + 0^2 + c_1 \cdot 0 + c2 \qquad \Longrightarrow \qquad c_2 = 1.$$

And the second boundary condition is then

$$u(2\pi) = 2 = \sin 2\pi + \frac{(2\pi)^2}{2} + c_1(2\pi) + 1,$$

which, after rearranging a bit, yields

$$c_1 = \frac{2}{(2\pi)^2}.$$

Thus, our full equilibrium solution is

$$u(x) = \sin x + \frac{x^2}{2} + \frac{2x}{(2\pi)^2} + 1.$$