Name:________________________________ Quiz Score:_____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

[3] 1. Recall the standard inner product on the interval $[-1, 1]$ by

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x,$$

Find a value of n such that $a(x) := x^n$ is orthogonal to $b(x) = 1$ in this inner product.

Solution: For two vectors to be orthogonal, their inner product must be zero, so we have

$$\langle a(x), b(x)\rangle = \int_{-1}^{1} 1 \cdot x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1}\bigg|_{x=-1}^{x=1}.$$

It's clear any odd n satisfies this, so let's take the easiest choice: $n = 1$, so $a(x) = x$.

[3] 2. Is the collection $\{a(x), b(x)\}$ an *orthonormal* set? *Why or why not*? And if not, can you make it one?

Solution: To answer this we recall that orthonormal vectors are not only orthogonal but also have magnitude 1. What is the magnitude of a function? We know the inner product defines this. That is,

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_{-1}^{1} f^2\,\mathrm{d}x}.$$

We can test each of our functions:

$$\|b\| = \sqrt{\int_{-1}^{1} 1\,\mathrm{d}x} = \sqrt{2}.$$

Thus, the new vector $\hat{b} = 1/\sqrt{2}$ is indeed a unit vector. Similarly, for $a(x) = x$, we have

$$\|a\| = \sqrt{\int_{-1}^{1} x^2\,\mathrm{d}x} = \sqrt{2/3},$$

so $\hat{a} = \sqrt{3/2}x$ is a unit vector, and the collection $\{\hat{a}, \hat{b}\}$ is orthonormal.

[4] 3. Using $a(x), b(x)$ from question 1. find the values of β, γ such that

$$\|e^x - \beta a(x) - \gamma b(x)\|^2$$

is minimized. That is, find the best approximation to $f(x) = e^x$ using $a(x), b(x)$. *Why can you do this*?

Solution: Because we showed that a, b are orthogonal in the previous problem (and only because of this fact!) can we then say that the coefficients that make the approximation most accurate are the orthogonal projection coefficients

$$\beta = \frac{\langle a, f\rangle}{\langle a, a\rangle}, \qquad \gamma = \frac{\langle b, f\rangle}{\langle b, b\rangle}.$$

We just found the magnitudes of a, b from the previous problems, so we need now only need to compute the numerators, which, for the first, after integration by parts is

$$\langle a, f\rangle = \int_{-1}^{1} xe^x \,\mathrm{d}x = [xe^x]_{x=-1}^{x=1} - \int_{-1}^{1} e^x \,\mathrm{d}x = \frac{2}{e}.$$

and the second inner product

$$\langle b, f\rangle = \int_{-1}^{1} 1 \cdot e^x \,\mathrm{d}x = e - \frac{1}{e}.$$

Thus, piecing this all together, we get

$$\beta = \frac{\langle a, f\rangle}{\langle a, a\rangle} = \frac{2/e}{2/3}, \qquad \gamma = \frac{\langle b, f\rangle}{\langle b, b\rangle} = \frac{e - \frac{1}{e}}{2}.$$